Soc–L-Small –T–ABSO Submodules and Related Concepts

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Abstract— Let V be a ring with identity and W be a unitary left-module. The Soc-L-small-T-ABSO submodule, a generalization of the small-T-ABSO, is a new class of submodules that we introduce in this study, and we also outline some of its key characteristics and examples that are achieved and not achieved in it and its relationship to other submodules

Keywords— Small − T− ABSO submodules; small submodule ;Large- small submodule; Socle of module.

1 Introduction

Assume that V be a ring with identity and W be a unitary left-module. A. Yousefian and F. Soheilnia in 2011 introduce the concept of T-Absorbing and weakly T-Absorbing submodules where " A proper submodule A of an V − module W is said to be T− ABSO submodule of W if whenever $a, b \in V, x \in W$ such that $abx \in A$, then either $ax \in A$ or $bx \in A$ or $ab \in (A:_V W)$ [3] [4]", and " A proper submodule A of V-module W, is called small two absorbing submodule (small-T-ABSO) submodule of M denoted if $a, b \in V$ and $x \in W$ with $< x > \ll W$ and $abx \in A$ then $ax \in A$ or $bx \in A$ or $ab \in (A:_V W)$[2]",In [1] A.A.Abduljaleel present the idea of large-small submodule as an expansion of a small submodule so that "A proper submodule A of V-module W, is called Large small (L-small) submodule of M denoted by ($A \ll_L W$), if A+H=W where H a submodule of W,then $H \leq_e W$".[1] provided the notion of a Socle-Two-Absorbing submodule, in which " A proper submodule A of an V-module W is called Socle –Two– Absorbing (in short is Soc− T− ABSO submodule of W if wheneve $a, b \in V, x \in W$ such that $abx \in A$, then either $ax \in A + Soc(W)$ or $bx \in A + Soc(W)$ or $ab \in (A + Soc(W) :_V W)$[5]".

These ideas prompt us to introduce the following novel idea, which we will name is Soc–L-Small–T–ABSO Submodules. There are two parts to this research. We introduce various concepts in the first section, along with some of their fundamental characteristics. The second section examines the connections, characteristics, and fundamental findings of the Soc–L-Small–T–ABSO Submodule.

2 Preliminaries

There are numerous fundamental notions, and this section lists their characteristics

Lemma (MODULAR LAW) 2.1 [5]: Let A, B and C be submodules of an V- module W and $B \leq C$ then $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$.

Proposition 2.2 [1]: Let A, B are submodules of V-module W such that $B \leq A \leq W$, if $B \ll_L A$ and $A \leq_e W$, then $B \ll_L W$.

Proposition 2.3 [8]: Let W is faithful, finitely generated and multiplication V-module and let I be an ideal of V then $I \ll_L V$ if and only if $IW \ll_L W$

Lemma 2.4 [9]: Let Wbe a faithful multiplication V- module then $Soc(V)W = Soc(W)$.

Definition.2.6[7]: Let A be a proper submodule of an V – module W then A is called socle-Essential-T-ABSO submodule (for short Soc–E−T–ABSO) if whenever $a, b \in V, x \in W, abx \in A$ and $< x > \leq_e W$. Then either $ax \in A + Soc(W)$ or $bx \in A + Soc(W)$ or $ab \in (A + Soc(W) :_V W)$.

3 Main Results

Definition 3.1 : A proper submodule A of a V-module W is called socle large small two absorbing submodule (for short Soc-L-small-T-ABSO) if $a, b \in v$ and $x \in W$ with $< x > \ll_L W$ and $abx \in A$ then $ax \in A + Soc(w)$ or $bx \in A + Soc(w)$ or $ab \in (A + Soc(w) :_v w)$,A proper ideal I of a a ring V is Soc-L-small-T-ABSO) if it Soc-L-small-T-ABSO submodule of the V-module V

For example $A = (\bar{0})$ in Z_{32} as Z-module $Soc(Z_{32}) = (16)$ is Soc-L-small-T-ABSO since $A + Soc(Z_{32}) = (\bar{0}) + (\overline{16}) = (\overline{16})$, and $(\bar{0}), (\bar{2}), (\bar{4}), (8), (\overline{16})$ are only L-small submodule in Z_{32} and $2.1.(\overline{16}) \in (\bar{0})$and $1.(\overline{16}) \in A + Soc(Z_{32}) = (\overline{16})$

Then A is Soc-L-small-T-ABSO.

Remarks and examples 3.2:

1. It is clear that every small-T-ABSO submodule is Soc-L-small-T-ABSO but the converse is not true in general as the following example $A = (\bar{0})$ in Z_{32} as Z-module,is Soc-L-small-T-ABSO However is not small-T-ABSO in Z_{32}, since $2.2.(\bar{8}) = (\bar{0})$ and $(\bar{8}) \ll Z_{32}$, so $2.\bar{8} = \overline{16} \notin (\bar{0})$ and $2.2 = 4 \notin \left((\bar{0}):_{v} Z_{32}\right) = 32Z$.
2. Let W be a semisimple module and $Soc(W) \subseteq A$. Then A is small-T-ABSO submodule of W if and only if A is Soc-L-small-T-ABSO

Proof:$\Longrightarrow$) it's clear

$\Longleftarrow$) let $ab(x) \in A$ such that $a, b \in V$ and $(x) \ll_{L} W$ since W is semi-simple, then $(x) \ll W$ and since A is Soc-L-small-T-ABSO. Then either $ax \in A + Soc(W)$ or $bx \in A + Soc(W)$or $ab \in (A + Soc(W):_{V} W)$, since $Soc(W) \subseteq A$, then either $a(x) \in A$ or $b(x) \in A$ or $ab \in (A:_{V} W)$. Then A is small-T-ABSO

3. Every Soc-T-ABSO submodule is Soc-L-small-T-ABSO but the converse is not true in general for example in the Z-module Z A is Soc-L-small-T-ABSO, for each proper submodule A of Z.

However if we take $A = 30Z$, then it is clear that A is not Soc-T-ABSO since $2.3.5 = 30 \in 30Z$ and $Soc(Z) = 0$,so $2.5 \notin 30z$ and $3.5 \notin 30Z$ and $2.3 = 6 \notin (30Z: Z) = 30Z$

4. If W is Hollow then every Soc-L-small-T-ABSO is Soc-E-T-ABSO submodule of W.
5. Every simple module is hollow, then every Soc-L-small-T-ABSO is Soc-E-T-ABso submodule.
6. If W is an V-module in which every cyclic submodule is an L-small submodule then every Soc-L-small-T-ABSO is Soc-T-ABSO submodule of W.
7. Soc-L-small-T-ABSO submodule need not be Soc-E-T-ABSO submodule, as the following example: In 3 from remark and example $A = 30Z$ is not Soc-E-T-ABSO submodule since $3.5.(2) = 30 \in 30Z$ and $(2) \leq_e Z$ but $5.(2) \notin 30$ and $3.(2) \notin 30Z$ and $5.3 \notin (30z:z) = 30Z$

Proposition 3.3: Let W be an R-module over aring V, and A is Soc-L-small-T-ABSO submodule of W and let $B \leq_e W$ such that $B \nsubseteq A$, then $B \cap A$ is Soc-L-small-T-ABSO submodule of B.

Proof; since $B \nsubseteq A$ then $B \cap A$ is a proper submodule of B. Now let $abx \in B \cap A$, where $a, b \in V$ and $x \in B$ with $< x > \ll_L B$.

Implies that $abx \in B$ and $abx \in A$ but A is Soc-L-small-T-ABSO and since $< x > \leq B \leq W$ and $B \leq_e W$ then $< x > \ll_L W$ by proposition2.2 then either $ax \in A + Soc(W)$ or $bx \in A + Soc(W)$ or $abW \subseteq A + Soc(W)$, since $x \in B$ implis that $ax \in \big(A + Soc(W)\big) \cap B$ or $bx \in \big(A + Soc(W)\big) \cap B$ or $abW \subseteq \Big(\big(A + Soc(W)\big) \cap B\Big)$ since $B \leq_e W$ then $Soc(W) \subseteq B$ then by lemma 2.1 $ax \in (A \cap B) + (Soc(w) \cap B)$ or $bx \in (A \cap B) + (Soc(W) \cap B)$ or $abW \subseteq (A \cap B) + (Soc(W) \cap B)$ since $\big(Soc(W) \cap B = Soc(B)\big)$ then implies that $ax \in (A \cap B) +$

$\mathrm{Soc(B)}$ or $\mathrm{bx} \in (\mathrm{A} \cap \mathrm{B}) + \mathrm{Soc(B)}$ or $\mathrm{abB} \subseteq \mathrm{abw} \subseteq (\mathrm{A} \cap \mathrm{B}) + \mathrm{Soc(B)}$. Then $\mathrm{A} \cap \mathrm{B}$ is Soc-L-small-T-ABSO submodule of B.

Proposition 3.4: Let W be an V-module, and A be a proper submodule of W with $(\mathrm{B}:\mathrm{W}) \nsubseteq (\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$ and A+Soc(W) proper sub-module of B for each submodule B of W such that $(\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$ is prime ideal of V. Then A is a Soc-L-small-T-ABSO of W.

Proof : Let $\mathrm{abx} \in \mathrm{A}$, where $\mathrm{a}, \mathrm{b} \in \mathrm{V}$ and $\mathrm{x} \in \mathrm{W}$ with $< \mathrm{x} > \ll_{\mathrm{L}} \mathrm{W}$ with $\mathrm{ax} \notin \mathrm{A} + \mathrm{soc(W)}$ and $\mathrm{bx} \notin \mathrm{A} + \mathrm{Soc(W)}$. Then $\mathrm{A} + \mathrm{Soc(W)} \subsetneq \mathrm{A} + \mathrm{Soc(W)} + < \mathrm{ax} >$ and $\mathrm{A} + \mathrm{Soc(W)} \subsetneq \mathrm{A} + \mathrm{Soc(W)} + < \mathrm{bx} >$

When $\mathrm{A} + \mathrm{Soc(W)} \subsetneq \mathrm{A} + \mathrm{Soc(W)} + < \mathrm{ax} > = \mathrm{B}$ and so $(\mathrm{B}:\mathrm{W}) \nsubseteq (\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$ then there exists $\mathrm{c} \in (\mathrm{B}:\mathrm{W})$ and $\mathrm{c} \notin (\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$. That is $\mathrm{cW} \subseteq \mathrm{B}$ and $\mathrm{cW} \nsubseteq \mathrm{A} + \mathrm{Soc(W)}$ implies, that $\mathrm{abcW} \subseteq \mathrm{ab}(\mathrm{A} + \mathrm{Soc(W)} + < \mathrm{ax} >) \subseteq \mathrm{A} + \mathrm{Soc(W)}$ it follows that $(\mathrm{ab})\mathrm{c} \in (\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$. But $(\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$ is prime ideal of V.

Then $\mathrm{ab} \in (\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$.similarly if $\mathrm{A} + \mathrm{Soc(W)} \subsetneq \mathrm{A} + \mathrm{Soc(W)} + < \mathrm{bx} > = \mathrm{B}$

Thuse A is Soc-L-small-T-ABSO submodule of W.

Proposition3.5: Let W be an uniform V-module. a proper submodule A of W is Soc-L-small-T-ABSO submodule of W if and only if when-ever $\mathrm{a}, \mathrm{b} \in \mathrm{V}$ with $\mathrm{H} \ll_{\mathrm{L}} \mathrm{W}, \mathrm{abH} \subseteq \mathrm{A}$ implice that either $\mathrm{aH} \subseteq \mathrm{A} + \mathrm{Soc(W)}$ or $\mathrm{bH} \subseteq \mathrm{A} + \mathrm{Soc(W)}$ or $\mathrm{ab} \in (\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$.

Proof:$\Longrightarrow$) suppose that $\mathrm{abH} \subseteq \mathrm{A}$ but $\mathrm{aH} \nsubseteq \mathrm{A} + \mathrm{Soc(W)}, \mathrm{bH} \nsubseteq \mathrm{A} + \mathrm{Soc(W)}, \mathrm{ab} \notin (\mathrm{A} + \mathrm{Soc(W)}:\mathrm{W})$ then there exist $\mathrm{h}_1\mathrm{h}_2 \in \mathrm{H}$ such that $\mathrm{ah}_1 \notin \mathrm{A} + \mathrm{Soc(W)}$ and $\mathrm{bh}_2 \in \mathrm{A} + \mathrm{Soc(W)}$. Hence $< \mathrm{h}_1 >, < \mathrm{h}_2 >$

$\ll_L$ W since $< h_1 >, < h_2 > \ll_L$ H and W is uniform V-module and hence every submodule of W is L-small. Now, $abh_1 \in A$ and $ab \notin (A + Soc(W): W), ah_1 \notin A + Soc(W)$ we get $bh_1 \in A + Soc(W)$.also since $abh_2 \in A$ and $ab \notin (A + Soc(W): W)$, $bh_2 \notin A + Soc(W)$. We get $ah_2 \in A + Soc(W)$. Now since $ab(h_1 + h_2) \in A$ and $ab \notin (A + Soc(W): W)$ we have $a(h_1 + h_2) \in A + Soc(W)$ or $b(h_1 + h_2) \in A + Soc(W)$ if $a(h_1 + h_2) \in A + Soc(W)$ i.e $ah_1 + ah_2 \in A + Soc(W)$ and since $ah_2 \in A + Soc(W)$ we get $ah_1 \in A + Soc(W)$ which is contradiction !

If $b(h_1 + h_2) \in A + Soc(W)$ then $bh_1 + bh_2 \in A + Soc(W)$ since $bh_1 \in A + Soc(W)$ then $bh_2 \in A + Soc(W)$ which is contradiction !

then either $aH \subseteq A + Soc(W)$ or $bH \subseteq A + Soc(W)$ or $ab \in (A + Soc(W): W)$

$\Leftarrow$) it's clear

Proposition 3.6: Let W be a uniform V-module and A be a proper submodule of W. Then the following are equivalent:

1. A is Soc-L-small-T-ABSO
2. $(A + Soc(W):_W I)$ is Soc-L-small-T-ABSO submodule of W for each $I \leq V, IW \nsubseteq A + Soc(W)$
3. $(A + Soc(W): a)$ Is Soc-L-small-T-ABSO submodule for each $a \in V, aW \nsubseteq A + Soc(W)$.

Proof: $(1) \Longrightarrow (2)$ let I be an ideal of V with $IW \nsubseteq A + Soc(W)$ then $(A + Soc(W): I)$ be a proper submodule of W let $abx \in (A + Soc(W):_W I)$, where $a, b \in V$, $< x > \ll_L$ W that is $abIx \subseteq A +$

$\mathrm{Soc}(W)$, But $< x > \ll_L W$ implies that $< Ix > \ll_L W$ because W is uniform. Now since A is Soc-L-small-T-ABSO then $aIx \subseteq A + \mathrm{Soc}(W)$ or $bIx \subseteq A + \mathrm{Soc}(W)$ or $ab \in (A + \mathrm{Soc}(W): W)$by proposition3.5

So $ax \in (A + \mathrm{Soc}(W): I)$ or $bx \in (A + \mathrm{Soc}(W): I)$ or $ab \in (A + \mathrm{Soc}(W): W) \subseteq \left((A + \mathrm{Soc}(W): I): W\right)$ and hence $ax \in (A + \mathrm{Soc}(W): I) + \mathrm{Soc}(W)$ or $bx \in (A + \mathrm{Soc}(W): I) + \mathrm{Soc}(W)$ or $ab \in \left((A + \mathrm{Soc}(W): I) + \mathrm{Soc}(W): W\right)$ then $(A + \mathrm{Soc}(W): I)$ is Soc-L-small-T-ABSO subnodule of W.

$(2) \longrightarrow (3)$ it's clear

$(3) \rightarrow (1)$ since A is an essential submodule of W then $\mathrm{Soc}(W) \subseteq A$ and hence $A = A + \mathrm{Soc}(W) = (A + \mathrm{Soc}(W): 1)$ and so A is Soc-L-small-T-ABSO submodule of W .

Proposition 3.7: Let W be a faithful finitely generated and multiplication in which every submodule is an essential, if I is a Soc-L-small-T-ABSO ideal of V. Then IW is a Soc-L-small-T-ABSO submodule of W.

Proof: Let $abx \in IW$ where $a, b \in V$, $x \in W$ with $< x > \ll_L W$ then $ab < x > \subseteq IW$. But W is multiplication, then $< x > = JW$ for some ideal J of V

It follows that $abJW \subseteq IW$ and so $abJ \subseteq I + \mathrm{ann}(W) = I$. But $I \ll_L V$ then by proposition 2.3 $IW \ll_L W$ and by proposition 3.5 either $aJ \subseteq I + \mathrm{Soc}(V)$ or $bJ \subseteq I + \mathrm{Soc}(V)$ or $ab \subseteq I + \mathrm{Soc}(V)$.Then either $aJW \subseteq IW + \mathrm{Soc}(V)W$ or $bJW \subseteq IW + \mathrm{Soc}(V)W$ or $ab \subseteq IW + \mathrm{Soc}(V)W$. But W be a fathful multiplication module then by corollary 2.4 we have $\mathrm{Soc(V)W = Soc(W)}$. Thus either $\mathrm{abW \subseteq IW + Soc(W)}$ or $\mathrm{aJW \subseteq IW + Soc(W)}$ or $\mathrm{bJW \subseteq IW + Soc(W)}$. Then either $\mathrm{ab \in (IW + Soc(W){:}W)}$ or $\mathrm{ax \in IW + Soc(W)}$ or $\mathrm{bx \in IW + SOC(W)}$. Hence IW is an Soc-L-small-T-ABSO submodule of W.

Proposition 3.8: Let A proper submodule of an V-module W and S is multiplcation subset of V such that $\mathrm{S^{-1}A \neq S^{-1}W}$. Thus $\mathrm{S^{-1}A}$ is Soc-L-small-T-ABSO $\mathrm{S^{-1}V}$ submodule of $\mathrm{S^{-1}W}$ if A is a Soc-L-small-T-ABSO submodule of W.

Proof: Let $\frac{a}{s_1}$, $\frac{b}{s_2}$, for some $\mathrm{a, b \in V}$, $s_1, s_2 \in \mathrm{S}$, let $\frac{x}{s_3} \in \mathrm{S^{-1}W}$, $\mathrm{x} \in \mathrm{W}$, $s_3 \in \mathrm{S}$, such that $< \frac{x}{s_3} > \ll_L \mathrm{S^{-1}W}$

If $\frac{a}{s_1} \cdot \frac{b}{s_2} \cdot \frac{x}{s_3} \in \mathrm{S^{-1}W}$, then $\frac{abx}{s_1 s_2 s_3} \in s^{-1}\mathrm{A}$ so there exists $\mathrm{t \in S}$ such that $\mathrm{tabx \in A}$. But it is easy to cheek that $< \frac{x}{s_3} > \ll_L \mathrm{S^{-1}W}$ if and only if $< \mathrm{x} > \ll_L \mathrm{W}$ so that $< \mathrm{tx} > \ll_L \mathrm{W}$.

Now $\mathrm{tabx \in A}$ and $< \mathrm{tx} > \ll \mathrm{W}$ and A is Soc-L-small-T-ABSO implies that either $\mathrm{atx \in A + Soc(W)}$ or $\mathrm{btx \in A + Soc(W)}$ or $\mathrm{ab \in (A + Soc(W){:}W)}$. It follows that $\frac{atx}{s_1 t s_3} \in \mathrm{S^{-1}A + Soc(S^{-1}W)}$ or $\frac{btx}{s_1 t s_2} \in \mathrm{S^{-1}A + Soc(S^{-1}W)}$ or $\mathrm{abS^{-1}W \subseteq S^{-1}N + Soc(S^{-1}W)}$

Then $\mathrm{S^{-1}N}$ Soc-L-small-T-ABSO submodule of $\mathrm{S^{-1}W}$

Proposition 3.9: Let $\mathrm{f{:}\, W \to W'}$ be an V-epimorphism, let B be an Soc-L-small-T-ABSO submodule of W' then $\mathrm{f^{-1}(B)}$ is a Soc-L-small-T-ABSO submodule of W.

Proof: Since $\mathrm{B < W'}$, then $\mathrm{f^{-1}(B) < W}$, since f is epimorphisim

Let $abx \in f^{-1}(B)$ for some $a, b \in V$, $x \in W$ with $< x > \ll_L W$ hence $abf(x) \in B$. But $< f(x) > \ll_L W'$, and B is Soc-L-small-T-ABSO then $af(x) \in B + Soc(W')$ or $bf(x) \in B + Soc(W')$or $abW' \subseteq B + Soc(W')$ Thus $ax \in f^{-1}(B) + f^{-1}(Soc(W')) \subseteq f^{-1}(B) + Soc(W)$ or $bx \in f^{-1}(B) + f^{-1}(Soc(W')) \subseteq f^{-1}(B) + Soc(W)$ or $abf^{-1}(W') \subseteq f^{-1}(B) + Soc(W)$. So $abW \subseteq f^{-1}(B) + Soc(W)$

Then $f^{-1}(B)$ is Soc-L-small-T-ABSO submodule of W.

Proposition 3.10: Let W,W' be two V-module , $f: W \to W'$ be an isomorphism if H is Soc-L-small-T-ABSO submodule of W, then f(H) is Soc-L-small-T-ABSO submodule of W.

Proof: Let $aby \in f(H)$ where $y \in W'$ with $< y > \ll_L W$, $ab \in V$ and $y = f(x)$ for some $x \in W$, since f is epimorphism then $abf(x) \in f(H)$, so that $abf(x) = f(h)$ for some $h \in H$, then we have $abf(x) = f(h) = 0$, hence $abx - h \in \ker f \subseteq H$ so that $abx \in H$ and $x \in W$, $xy =< f(x) > \ll_L W$ becouse f is isomorphisim. Now H is Soc-L-small-T-ABSO submodule of W, then $ax \in H + Soc(W)$ or $bx \in H + Soc(W)$ or $abW \subseteq H + Soc(W)$ so $af(x) = f(ax) \in f(H + Soc(W)) \subseteq f(H) + Soc(W')$ or $bf(x) = f(ax) \in f(H + Soc(W)) \subseteq f(H) + Soc(W')$ or $abW' \subseteq abf(W) \subseteq f(H) + Soc(W')$

Then f(H) is Soc-L-small-T-ABSO submodule of W'

Remark 3.11: The condition epimorphism in the proposition 3.10 is not delete by the following example: $f: Z_{32} \to Z_{32}$ defined by

$f(x) = 4x$ for each $x \in Z_{32}$, Let $A =< 2 >$, so A is Soc-L-small-T-ABSO. But $f(A) =< 8 >$ and $Soc(Z_3) =< 16 >$ and hence $2.6. < 2 > \in < 8 >$ such that $< 2 > \ll_L Z_{32}$ but $4 = 2. < 2 > \notin < 8 > + < 16 > = < 8 >$ and $6. < 2 > = 12 \notin < 8 >$ and $2.6 = 12 \notin (< 8 >: Z_{32}) = 8Z$.

Then f(A) is not Soc-L-small-T-ABSO submodule of Z_{32}.

References

[1] A. A. Abduljaleel and S.M.Yaseen,"On Large-Small submodule and Large-Hollow module", Journal of physics Conference series,Vol. 1818, (2021). DOI 10.1088/1742-6596/1818/1/012214

[2] A.A. Harfash, 2-Absorbing submodules(Modules)and Some of Their Generalizations, M. Sc. Thesis, University of Baghdad, (2015).

[3] Al-Hachami, A. K. (2017), Orthogonal Left Derivations of Semi-Prime Rings. Journal of Generalized Lie Theory and Applications. 11.(2)

[4] A.Y. Darani and F. Soheilnia, 2-Absorbing and Weakly 2-AbsorbingSubmodules, Thai Journal of Mathematics, 9(3), 577-584, (2011).

[5] F.Kash,"Modules and Rings" London. Mathematical Society Monograph, Vol. 17, New York, Academic Press, (1982).

[6] M. W. Allami and W. H. Hanoon,"Soc–T–ABSO Submodules and Related Concepts" , accepted for publication in the Journal of Interdisciplinary Mathematics, 2022.

[7] M. W. Allami and W. H. Hanoon,"Soc–E-T–ABSO Submodules and Related Concepts", accepted for publication in the Journal of Discrete Mathematical Sciences &Cryptography, 2023.

[8] P. F. Smith,"Some Remarks On Multiplication Module", Arch. Math.So pp 223-225(1988). https://doi.org/10.1007/BF01187738

[9] Z.A.El-Bast and P.F.Smith," Multiplication Modules"comm.In Algebra 16(4),pp755-779,(1988). . https://doi.org/10.1080/00927878808823601

Article submitted 2 March 2023. Accepted at 31 March

Published at 30 Jun 2023.