

Higher Reverse Left (Respectively Right) Centralizerhomo on Prime Ring

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ABSTRACT: Assuming K is a ring, the present study introduces original algebraic structures termed higher reverse left (resp. right) centralizerhomo and Jordan higher reverse left (resp. right) centralizerhomo. In this work investigates the interconnection between these concepts, as well as their fundamental properties.

Keywords: Prime Ring, Jordan Triple, Higher Reverse, Centralizerhomo



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1. INTRODUCTION

Throughout this paper K is a ring, K is called prime if $rKd = 0$ implies $r = 0$ or $d = 0$ with $r, d \in K$ and K is called simeprime if $rKr = 0$ implies $r = 0$, K is called 2-torsion free if $2r = 0$ then $r = 0$ with $r \in K$ [1]. A left (resp. right) centralizer of a ring K is an additive mapping $f: K \rightarrow K$ which satisfies $f(rd) = f(r)d$ (resp. $f(rd) = rf(d)$), for each $r, d \in K$, f is called left (resp. right) Jordan centralizer of K if $f(r^2) = f(r)r$ (resp. $f(r^2) = rf(r)$) [2] and f is called higher left centralizer of K if $\forall n \in N, f_n(rd) = \sum_{i=1}^n f_i(r)f_{i-1}(d)$, for all $r, d \in K$ and f is called Jordan higher left centralizer of K if for $\forall n \in N, f_n(r^2) = \sum_{i=1}^n f_i(r)f_{i-1}(r)$, for each $r \in K$, and f is called Jordan triple left centralizer of K if $\forall n \in N, f_n(rdr) = \sum_{i=1}^n f_i(r)f_{i-1}(d)f_{i-1}(r)$, for all $r, d \in K$ [3] and f is called higher reverse left (resp. right) centralizer of K if $f_n(rd) = \sum_{i=1}^n f_i(d)f_{i-1}(r)$ (resp. $f_n(rd) = \sum_{i=1}^n f_{i-1}(d)f_i(r)$), for all $r, d \in K$ and $n \in N$, and f is called Jordan higher reverse left (resp. right) centralizer of K if $f_n(r^2) = \sum_{i=1}^n f_i(r)f_{i-1}(r)$ (resp. $f_n(r^2) = \sum_{i=1}^n f_{i-1}(r)f_i(r)$), for all $r \in K$ and $n \in N$, and f is called homomorphism if satisfied the following $f(rd) = f(r)f(d)$ for each $r, d \in K$, and f is called Jordan homomorphism if $f(r^2) = f(r)f(r)$ [3]. Then f is called an anti-homomorphism, we have that $f(rd) = f(d)f(r)$ for each $r, d \in K$ [4], and f is called higher homomorphism of K $f_n(rd) = \sum_{i=1}^n f_i(r)f_i(d)$, for all $r, d \in K$ and $n \in N$, and f is called Jordan higher homomorphism of K if $f_n(r^2) = \sum_{i=1}^n f_i(r)f_i(r)$, for all $r \in K$ and $n \in N$ [6]. In this paper whether there is a relation between a concept of higher reverse left (resp. right) centralizerhomo and Jordan of this.

2. HIGHER REVERSE LEFT (RESPECTIVELY RIGHT) CENTRALIZERHOMO ON PRIME RING

Definition (2.1): Let $f = (f_i)_{i \in N}$ be a family of additive mappings of a ring K into itself. Then f is called a higherreverse left (resp. right) centralizerhomo for shortcut (Hig. Rev. L. (resp. right) CH.) for all $r, d \in K$ and $n \in N$ $f_n(rd) = \sum_{i=1}^n f_i(d)f_{i-1}(r) + f_i(d)f_i(r)$ (resp. right) $f_n(rd) = \sum_{i=1}^n f_{i-1}(d)f_i(r) + f_i(d)f_i(r) \dots (1)$
 f is called a Jordan higher reverse left (resp. right) centralizerhomo for shortcut (Jor. Hig. Rev. L. (resp. right) CH.)
 $f_n(r^2) = \sum_{i=1}^n f_i(r)f_{i-1}(r) + f_i(r)f_i(r)$ (resp. right) $f_n(r^2) = \sum_{i=1}^n f_{i-1}(r)f_i(r) + f_i(r)f_i(r) \dots (2)$
 f is called a Jordan triple higher reverse left (resp. right) centralizerhomo for shortcut (Jor. Tri. Hig. Rev. L. (resp. right) CH.)

$$\mathfrak{f}_n(rdr) = \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(r) \quad (\text{resp. right}) \quad \mathfrak{f}_n(rdr) = \sum_{i=1}^n \mathfrak{f}_{i-1}(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(r) \quad (3)$$

Example (2.2): Let K be a ring of 3×3 matrices s.t $K = \left\{ \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & 0 \\ 0 & r & 0 \end{pmatrix} : \forall r \in Z \right\}$ and $\mathfrak{f}_n: K \rightarrow K$ be an additive

mapping of ring K into itself, defined by $\mathfrak{f}_n \left(\begin{pmatrix} 0 & r & 0 \\ 0 & 0 & 0 \\ 0 & r & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & nr & 0 \\ 0 & 0 & 0 \\ 0 & nr & 0 \end{pmatrix}$, for all $\begin{pmatrix} 0 & r & 0 \\ 0 & 0 & 0 \\ 0 & r & 0 \end{pmatrix} \in K, n \in N$,

therefore $\mathfrak{f}_n$ is Hig. Rev. L. (resp. right) CH. of K .

Lemma (2.3): Let $\mathfrak{f} = (\mathfrak{f}_i)_{i \in N}$ be a Jor. Hig. Rev. L. (resp. right) CH. of a ring K . Then for all $r, d, p \in K$ and $n \in N$

- i. $\mathfrak{f}_n(rd + dr) = \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(d)$
- ii. $\mathfrak{f}_n(rdp + pdr) = \sum_{i=1}^n \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(p) \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(p) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(p)$
- iii. In particular, if K is a 2-torsion free commutative ring then
 $\mathfrak{f}_n(rdp) = \sum_{i=1}^n \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(p) \mathfrak{f}_i(d) \mathfrak{f}_i(r)$

Poof:

- i. $\mathfrak{f}_n((r+d)(r+d)) = \sum_{i=1}^n \mathfrak{f}_i(r+d) \mathfrak{f}_{i-1}(r+d) + \mathfrak{f}_i(r+d) \mathfrak{f}_i(r+d)$
 $= \sum_{i=1}^n (\mathfrak{f}_i(r) + \mathfrak{f}_i(d)) (\mathfrak{f}_{i-1}(r) + \mathfrak{f}_{i-1}(d)) + (\mathfrak{f}_i(r) + \mathfrak{f}_i(d)) (\mathfrak{f}_i(r) + \mathfrak{f}_i(d))$
 $= \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(d)$ (4)

Alternatively:

$$\mathfrak{f}_n((r+d)(r+d)) = \mathfrak{f}_n(r^2 + rd + dr + d^2)$$

$$= \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(r) + \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(d) \mathfrak{f}_i(d) + \mathfrak{f}_n(rd + dr) \quad (5)$$

From (4) and (5), we obtain:

$$\mathfrak{f}_n(rd + dr) = \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(d)$$

- ii. Substituting $r + p$ for r in Definition (2.1) (3) we obtain:

$$\mathfrak{f}_n((r+p)d(r+p)) = \sum_{i=1}^n \mathfrak{f}_i(r+p) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r+p) + \mathfrak{f}_i(r+p) \mathfrak{f}_i(d) \mathfrak{f}_i(r+p)$$

$$= \sum_{i=1}^n (\mathfrak{f}_i(r) + \mathfrak{f}_i(p)) \mathfrak{f}_{i-1}(d) (\mathfrak{f}_{i-1}(r) + \mathfrak{f}_{i-1}(p)) + (\mathfrak{f}_i(r) + \mathfrak{f}_i(p)) \mathfrak{f}_i(d) (\mathfrak{f}_i(r) + \mathfrak{f}_i(p))$$

$$= \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(p) + \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(p) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(p) + \mathfrak{f}_i(p) \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(p) \mathfrak{f}_i(d) \mathfrak{f}_i(p)$$
 (6)

Alternatively:

$$\mathfrak{f}_n((r+p)d(r+p)) = \mathfrak{f}_n(rdr + rdp + pdr + pdp)$$

$$= \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(p) + \mathfrak{f}_i(p) \mathfrak{f}_i(d) \mathfrak{f}_i(p) + \mathfrak{f}_n(rdp + pdr)$$
 (7)

From (6) and (7), we obtain:

$$\mathfrak{f}_n(rdp + pdr) = \sum_{i=1}^n \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(p) \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(p) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(p)$$

- iii. By (ii) and since K is commutative 2-torsion free ring we have the require result.

Remark (2.4): Let $\mathfrak{f} = (\mathfrak{f}_i)_{i \in N}$ be Jor. Hig. Rev. L. (resp. right) CH. of a ring K . Then for $r, d \in K$ and $n \in N$, we define $\Omega_n: K \times K \rightarrow K$ $\Omega_n(r, d) = \mathfrak{f}_n(rd) - \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r)$ (resp. right) $\Omega_n(r, d) = \mathfrak{f}_n(rd) - \sum_{i=1}^n \mathfrak{f}_{i-1}(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r)$

Lemma (2.5): Let $\mathfrak{f} = (\mathfrak{f}_i)_{i \in N}$ be a Jor. Hig. Rev. L. (resp. right) CH. of a ring K . Then for all $r, d, p \in K$ and $n \in N$

- i. $\Omega_n(r, d) = -\Omega_n(d, r)$
- ii. $\Omega_n(r + d, p) = \Omega_n(r, p) + \Omega_n(d, p)$
- iii. $\Omega_n(r, d + p) = \Omega_n(r, d) + \Omega_n(r, p)$

Proof

- i. By lemma (2.3) (i). we obtain:

$$\mathfrak{f}_n(rd + dr) = \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \quad (8)$$

Alternatively

$$\mathfrak{f}_n(rd) + \mathfrak{f}_n(dr) = \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \quad (9)$$

From (8) and (9), we obtain:

$$\begin{aligned} \mathfrak{f}_n(rd) - \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) &= -\mathfrak{f}_n(dr) + \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \\ \mathfrak{f}_n(rd) - \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) &= -(\mathfrak{f}_n(dr) - \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(d)) \\ \Omega_n(r, d) &= -\Omega_n(d, r) \end{aligned}$$

$$\text{ii. } \Omega_n(r + d, p) = \mathfrak{f}_n((r + d)p) - \sum_{i=1}^n \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(r + d) + \mathfrak{f}_i(p) \mathfrak{f}_i(r + d)$$

$$\begin{aligned} &= \mathfrak{f}_n(rp + dp) - \sum_{i=1}^n \mathfrak{f}_i(p) (\mathfrak{f}_{i-1}(r) + \mathfrak{f}_{i-1}(d)) + \mathfrak{f}_i(p) (\mathfrak{f}_i(r) + \mathfrak{f}_i(d)) \\ &= \mathfrak{f}_n(rp) - \sum_{i=1}^n \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(p) \mathfrak{f}_i(r) + \mathfrak{f}_n(dp) - \sum_{i=1}^n \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(p) \mathfrak{f}_i(d) \\ &= \Omega_n(r, p) + \Omega_n(d, p) \end{aligned}$$

$$\text{iii. } \Omega_n(r, d + p) = \mathfrak{f}_n(r(d + p)) - \sum_{i=1}^n \mathfrak{f}_i(d + p) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d + p) \mathfrak{f}_i(r)$$

$$\begin{aligned} &= \mathfrak{f}_n(rd + rp) - \sum_{i=1}^n (\mathfrak{f}_i(d) + \mathfrak{f}_i(p)) \mathfrak{f}_{i-1}(r) + (\mathfrak{f}_i(d) + \mathfrak{f}_i(p)) \mathfrak{f}_i(r) \\ &= \mathfrak{f}_n(rd) - \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \mathfrak{f}_n(rp) - \sum_{i=1}^n \mathfrak{f}_i(p) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(p) \mathfrak{f}_i(r) \\ &= \Omega_n(r, d) + \Omega_n(r, p) \end{aligned}$$

Remark (2.6): Note that $\mathfrak{f} = (\mathfrak{f}_i)_{i \in \mathbb{N}}$ be a Hig. Rev. L. (resp. right) CH. of a ring K iff $\Omega_n(r, d) = 0$, for all $r, d \in K$ and $n \in \mathbb{N}$

Lemma (2.7): Let $\mathfrak{f} = (\mathfrak{f}_i)_{i \in \mathbb{N}}$ be a Jor. Hig. Rev. L. (resp. right) CH. of a 2-torsion free ring K . Then for all $r, d, p \in K$ and $n \in \mathbb{N}$

$$\Omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] = 0$$

Proof: Let $w = r(dpd)r + d(rpr)d$

$$\begin{aligned} \mathfrak{f}_n(w) &= \mathfrak{f}_n(r(dpd)r + d(rpr)d) \\ &= \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(dpd) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(dpd) \mathfrak{f}_i(r) + \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(rpr) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(d) \mathfrak{f}_i(rpr) \mathfrak{f}_i(d) \\ &= \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(p) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(p) \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \\ &\quad \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) \mathfrak{f}_{i-1}(p) \mathfrak{f}_{i-1}(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) \mathfrak{f}_i(p) \mathfrak{f}_i(r) \mathfrak{f}_i(d) \end{aligned} \quad (10)$$

Alternatively:

$$\begin{aligned} \mathfrak{f}_n(w) &= \mathfrak{f}_n((rd)p(dr) + (dr)p(rd)) \\ &= \mathfrak{f}_n(dr) \mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(rd) + \mathfrak{f}_n(dr) \mathfrak{f}_n(p) \mathfrak{f}_n(rd) + \mathfrak{f}_n(rd) \mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(dr) + \mathfrak{f}_n(rd) \mathfrak{f}_n(p) \mathfrak{f}_n(dr) \\ &= \mathfrak{f}_n(dr) \mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(d) \mathfrak{f}_{n-1}(r) + \mathfrak{f}_n(dr) \mathfrak{f}_n(p) \mathfrak{f}_n(d) \mathfrak{f}_n(r) + \mathfrak{f}_n(rd) \mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(r) \mathfrak{f}_{n-1}(d) + \\ &\quad \mathfrak{f}_n(rd) \mathfrak{f}_n(p) \mathfrak{f}_n(r) \mathfrak{f}_n(d) \end{aligned} \quad \dots (11)$$

From (10) and (11), we obtain:

$$(\mathfrak{f}_n(rd) - \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r)) (\mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(r) \mathfrak{f}_{n-1}(d) + \mathfrak{f}_n(p) \mathfrak{f}_n(r) \mathfrak{f}_n(d)) + (\mathfrak{f}_n(dr) - \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(d)) (\mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(d) \mathfrak{f}_{n-1}(r) + \mathfrak{f}_n(p) \mathfrak{f}_n(d) \mathfrak{f}_n(r)) = 0$$

$$\Omega_n(r, d) (\mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(r) \mathfrak{f}_{n-1}(d) + \mathfrak{f}_n(p) \mathfrak{f}_n(r) \mathfrak{f}_n(d)) + \Omega_n(d, r) (\mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(d) \mathfrak{f}_{n-1}(r) + \mathfrak{f}_n(p) \mathfrak{f}_n(d) \mathfrak{f}_n(r)) = 0$$

$$\Omega_n(r, d) (\mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(r) \mathfrak{f}_{n-1}(d) + \mathfrak{f}_n(p) \mathfrak{f}_n(r) \mathfrak{f}_n(d)) - \Omega_n(r, d) (\mathfrak{f}_{n-1}(p) \mathfrak{f}_{n-1}(d) \mathfrak{f}_{n-1}(r) + \mathfrak{f}_n(p) \mathfrak{f}_n(d) \mathfrak{f}_n(r)) = 0$$

$$\Omega_n(r, d) (\mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] + \mathfrak{f}_n(p) [\mathfrak{f}_n(r), \mathfrak{f}_n(d)])$$

$$\Omega_n(r, d) (\mathfrak{f}_{n-1}(p) + \mathfrak{f}_n(p)) ([\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] + [\mathfrak{f}_n(r), \mathfrak{f}_n(d)]) = 0$$

Replacing $\mathfrak{f}_{n-1}(p) + \mathfrak{f}_n(p)$ by $\mathfrak{f}_{n-1}(p)$

$$\Omega_n(r, d) \mathfrak{f}_{n-1}(p) ([\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] + [\mathfrak{f}_n(r), \mathfrak{f}_n(d)]) = 0$$

Replacing $[\mathfrak{f}_n(r), \mathfrak{f}_n(d)]$ by $[\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)]$

$$2\Omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] = 0$$

Since K is 2-torsion free

$$\Omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] = 0 \text{ for all } r, d, p \in K, n \in \mathbb{N}$$

Lemma (2.8): Let $\mathfrak{f} = (\mathfrak{f}_i)_{i \in \mathbb{N}}$ be a Jor. Hig. Rev. L. (resp. right) CH. of a prime ring K . Then for all $r, d, p, l, f \in K$ and $n \in \mathbb{N}$

$$\Omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(f)] = 0$$

Proof: Substituting $r + l$ for r in lemma (2.7), we obtain:

$$\Omega_n(r + l, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r + l), \mathfrak{f}_{n-1}(d)] = 0$$

$$\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] + \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(d)] + \omega_n(l, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] + \omega_n(l, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(d)] = 0$$

By lemma (2.7), we obtain:

$$\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(d)] + \omega_n(l, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] = 0$$

Therefore, we obtain:

$$\begin{aligned} & \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(d)] \mathfrak{f}_{n-1}(p) \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(d)] = 0 \\ 0 &= -\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(d)] \mathfrak{f}_{n-1}(p) \omega_n(l, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] \\ & [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] \end{aligned}$$

Since K is prime ring, we obtain:

$$\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(d)] = 0 \quad (12)$$

Now, substituting $d + f$ for d in lemma (2.7), we obtain:

$$\begin{aligned} & \omega_n(r, d + f) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d + f)] = 0 \\ & \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] + \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(f)] + \omega_n(r, f) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] + \\ & \omega_n(r, f) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(f)] = 0 \end{aligned}$$

By lemma (2.7), we obtain:

$$\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(f)] + \omega_n(r, f) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] = 0$$

Therefore, we obtain:

$$\begin{aligned} & \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(f)] \mathfrak{f}_{n-1}(p) \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(f)] = 0 \\ 0 &= -\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(f)] \mathfrak{f}_{n-1}(p) \omega_n(r, f) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] \\ & [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] \end{aligned}$$

Since K is prime ring, we obtain:

$$\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(f)] = 0 \quad (13)$$

Finally, $\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r + l), \mathfrak{f}_{n-1}(d + f)] = 0$

$$\begin{aligned} & \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(d)] + \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(r), \mathfrak{f}_{n-1}(f)] + \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(d)] + \\ & \omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(f)] = 0 \end{aligned}$$

By (12), (13) and lemma (2.7), we obtain:

$$\omega_n(r, d) \mathfrak{f}_{n-1}(p) [\mathfrak{f}_{n-1}(l), \mathfrak{f}_{n-1}(f)] = 0$$

Theorem (2.9): Every Jor. Hig. Rev. L. (r) CH. of a 2-torsion free prime ring is a Hig. Rev. L. (resp. right) CH. of K.

Proof: Let $\mathfrak{f} = (\mathfrak{f}_i)_{i \in \mathbb{N}}$ be a Jor. Hig. Rev. L. (resp. right) CH. of a 2-torsion free prime ring K

Then by lemma (2.8), K is a prime ring, therefore either

$$\omega_n(r, d) = 0 \text{ or } [\mathfrak{f}_{i-1}(l), \mathfrak{f}_{i-1}(f)] = 0 \text{ for all } r, d, l, f \in K \text{ and } n \in \mathbb{N}$$

If $[\mathfrak{f}_{i-1}(l), \mathfrak{f}_{i-1}(f)] \neq 0$ for all $l, f \in K$

Then $\omega_n(r, d) = 0$ for all $r, d \in K, n \in \mathbb{N}$

Hence by Remark (2.6), we get $\mathfrak{f}$ is a Hig. Rev. L. (resp. right) CH. of K

But if $[\mathfrak{f}_{i-1}(l), \mathfrak{f}_{i-1}(f)] = 0$ of all $l, f \in K$ and $n \in \mathbb{N}$

Then K is a commutative ring

By lemma (2.3) (i), we obtain

$$2 \mathfrak{f}_n(rd) = 2 \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r)$$

Since K is a 2-torsion free ring

We get $\mathfrak{f}$ is a Hig. Rev. L. (r) CH. of K

Proposition (2.10): Let $\mathfrak{f} = (\mathfrak{f}_i)_{i \in \mathbb{N}}$ be a Jor. Hig. Rev. L. (resp. right) CH. of a 2-torsion free ring K. Then $\mathfrak{f}$ is a Jor. Tri. Hig. Rev. L. (resp. right) CH. of K.

Proof: Substituting $rd + dr$ for d in lemma (2.3) (i), we obtain:

$$\begin{aligned} & \mathfrak{f}_n(r(rd + dr) + (rd + dr)r) = \sum_{i=1}^n \mathfrak{f}_i(rd + dr) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(rd + dr) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(rd + dr) + \\ & \mathfrak{f}_i(r) \mathfrak{f}_i(rd + dr) \\ &= \sum_{i=1}^n \mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) \mathfrak{f}_{i-1}(r) + 2 \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) \mathfrak{f}_i(r) + 2 \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(r) + \\ & \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(r) \mathfrak{f}_i(d) \end{aligned} \quad \dots (14)$$

Alternatively:

$$\begin{aligned} & \mathfrak{f}_n(r(rd + dr) + (rd + dr)r) = \mathfrak{f}_n(r^2d + rdr + rdr + dr^2) \\ &= \sum_{i=1}^n (\mathfrak{f}_i(d) \mathfrak{f}_{i-1}(r) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(d) \mathfrak{f}_i(r) \mathfrak{f}_i(r) + \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(r) \mathfrak{f}_{i-1}(d) + \mathfrak{f}_i(r) \mathfrak{f}_i(r) \mathfrak{f}_i(d)) + 2 \mathfrak{f}_i(rdr) \end{aligned} \quad \dots (15)$$

From (14) and (15), we obtain:

$$2 \mathfrak{f}_n(rdr) = 2 \sum_{i=1}^n \mathfrak{f}_i(r) \mathfrak{f}_{i-1}(d) \mathfrak{f}_{i-1}(r) + \mathfrak{f}_i(r) \mathfrak{f}_i(d) \mathfrak{f}_i(r)$$

Since a K is a 2-torsion free ring, we got the require result.

Corollary (2.11): Every Hig. Rev. L. (resp. right) CH. of a 2-torsion free ring K is a Jor. Tri. Hig. Rev. L. (resp. right) CH. of K .

Proof: Suppose that $f = (f_i)_{i \in N}$ be Hig. Rev. L. (resp. right) CH., then by definition (2.1), we have f is Jor. Hig. Rev. L. (resp. right) CH., hance by Proposition (2.10), we obtain f is Jor. Tri. Hig. Rev. L. (resp. right) CH.

3. CONCLUSION

We have investigated the properties of Hig. Rev. L. (resp. right) CH. on 2-torsion free prime rings. The main theorems of this paper two key results: first, that every Jor. Hig. Rev. L. (resp. right) CH. is a Hig. Rev. L. (resp. right) CH., and second, that is also a Jor. Tri. Hig. Rev. L. (resp. right) CH. These results establish a clear equivalence between these three concepts in the 2-torsion free prime ring.

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