New Kind of Banach Algebra Via Proximit Structure

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Abstract— This article introduces the concept of Banach proximit algebra and examines several results in this new context. We give new definitions for the terms "proximit algebra," "commutative proximit algebra," "identity proximit algebra," and "normed proximit algebra." However, we introduce the concepts of subproximit algebra, proximit homomorphism, and proximit isomorphism of Banach proximit algebras in a natural way. Finally, we offer a few intriguing possibilities to support our arguments.

Keywords— proximit algebra , subproximit algebra, identity proximit algebra, commutative proximit algebra.

1 Introduction and Preliminaries

The use of functional analysis, topology, probability theory, hyperspaces, and domain theory is known as approach theory. Approach theory is frequently concealed under other ideas, as shown by Lowen [19]. In 1989 approach space, a topological and metric space extension have certain fundamental concepts and theorems developed by Lowen [18]. Approach vector spaces are a new class of topological vector spaces that [2] has invented. The normed approach space was also created by Abbas and Hussein [1]. To analyze an approach completion space, the definition of an approach normed space was applied in [3]. The discovery that some Banach spaces exhibit intriguing behaviors when they are given an additional multiplication operation led to the beginning of the study of Banach algebras in the 20th century. The space of bounded linear operators on a Banach space was a common example, but function spaces are also crucial examples (of continuous, bounded, vanishing at infinity, etc. functions as well as functions with absolutely convergent Fourier series). Banach algebra is a broad field with many subfields of study and applications. In the general theory of Banach algebras, algebras over the complex field have received the majority of the attention.

Surjective isometries between sets of invertible elements in unital Jordan-Banach algebras were introduced by Peralta [23]. [4] provides a description of the key techniques and findings in the theory of commutative and noncommutative Banach algebras.

The Banach algebras produced by an invertible isometry of an L_p-space and its inverse were fully described in [10]. The reader is directed to [17] for a thorough discussion of the theory of commutative Banach algebras. [11] algebras of operators on Banach spaces and homomorphisms thereof provided some properties of linear systems defined over a commutative Banach algebra. The sum of two group invertible elements in a Banach algebra has a group inverse when certain new necessary and sufficient requirements [5] are met.

Normed algebraic extensions are presented in [9]. [13] provided theory algebra of Banach algebra operators as axiomatics, examples, and invertibility in Banach algebra. Real normed algebras studied in [12]. In three different subfields of Banach algebra theory, each of which has some connection to homomorphisms, derivations, or both, issues and findings are presented in [6]. Concerning operators on Banach spaces, see [8]. [21] investigated the fundamental properties of Hilbert space operator algebras and Banach algebras. It is expected that the reader has already completed a first-level functional analysis course. We specifically assume that the reader is familiar with the information found in an introduction to functional analysis [20]. Both [7] and [22] are references for Banach algebras.

After introducing and studying the Banach proximit algebra structure, we go on to a proximit algebra. We introduce key concepts in proximit space, including sub proximit algebra, normed proximit algebra, proximit algebra, unity, and communality. Using proximit algebra, Banach algebra homomorphism, and isometry with proximit structure, we examine new problems and arrive at novel conclusions for these ideas in this study.

Definition 1.1:[14]

A collection of functions $t_\varepsilon: 2^X \to 2^X$, $\varepsilon \in R^+$ is called a tower on X if the following conditions are satisfied :

(T1) $\forall A \in 2^X, \forall \varepsilon \in R^+ : A \subset t_\varepsilon(A)$,

(T2) $\forall \varepsilon \in R^+ : t_\varepsilon(\emptyset) = \emptyset$

(T3) $\forall A, B \in 2^X, \forall \varepsilon \in R^+ : t_\varepsilon(A \cup B) = t_\varepsilon(A) \cup t_\varepsilon(B)$,

(T4) $\forall A \in 2^X, \forall \varepsilon, \gamma \in R^+ : t_\varepsilon(t_\gamma(A)) \subset t_{\varepsilon+\gamma}(A)$,

(T5) $\forall A \in 2^X, \forall \varepsilon \in R^+ : t_\varepsilon(A) = \bigcap_{\varepsilon<\gamma} t_\gamma(A)$,

Take note that with (T3) and (T5), we get $\forall A \subset B \subset X, \forall \alpha \leq \beta \in R^+ : t_\alpha(A) \subset t_\beta(B)$.

Definition 1.2:**[14]**

Given a set X, a functional $\rho: 2^X \times 2^X \to [0, \infty]$ is said to be an *Dh-functional* on X if and only if it satisfies the following requirements:

(G1) $\forall\, A, B \in 2^X:\ \rho(A, B) = \rho(B, A)$,

(G2) $\forall\, A, B \in 2^X,\ A = \emptyset \vee B = \emptyset \Rightarrow \rho(A, B) = \infty$,

$(G3)\ \forall\, A, B \in 2^X, \rho(A, B) = 0 \Rightarrow A \cap B \neq \emptyset$,

$(G4)\ \forall A, B, C \in 2^X : \rho(A, B \cup C) = \{\rho(A, B), \rho(A, C)\}$,

(G5) $\forall A, B \in 2^X, \forall \varepsilon, \eta \in [0, \infty] : \rho(A, B) \leq \rho(A^\varepsilon, B^\eta) + \varepsilon + \eta$.

For every $A \in 2^X, \varepsilon \in [0, \infty]$, we write $t_\varepsilon(A) := \{x \in X \mid \rho(\{x\}, A) \leq \varepsilon\}$. Therefore the triple (X, ρ, t_ε) is called proximit space.

Example 1.3: [14]

Let $X = [0, \infty]$, define $\rho: 2^{[0,\infty]} \times 2^{[0,\infty]} \to [0, \infty]$ by

$\rho(A, B) = \{ \quad 0 \qquad n, m$

$= \infty, A, B\ unbounded \qquad \infty \qquad n, m$

$= \infty, A, B\ bounded \qquad inf_{\ m \in B} |n - m| \qquad n, m$

$< \infty$

Then (X, ρ, t_ε) is proximit space.

Definition 1.4: [15]

A function $f:(X,\rho_X,t_\varepsilon)\to(Y,\rho_Y,t_\varepsilon)$ such that (X,ρ_X,t_ε) and (Y,ρ_Y,t_ε) are proximit space is called Dh-contraction if $f\left(t_{\varepsilon_X}(A)\right)\subseteq t_{\varepsilon_Y}(f(A))\ \forall A\subseteq X,\forall\varepsilon\in R^+$.

Proposition 1.5: [15]

Suppose that (X,ρ_X,t_ε) is proximit spaces. Then identity map $I_x:(X,\rho_X,t_\varepsilon)\to(X,\rho_X,t_\varepsilon)\ \forall A\subseteq X,\ \forall\varepsilon\in R^+$ is Dh-contraction.

Definition 1.6: [15]

If $(\Upsilon,\rho_\Upsilon,t_\varepsilon)$ and (W,ρ_W,t_ε) are proximit spaces. A function $\varphi:\Upsilon\longrightarrow W$ is said to be sequentially Dh-contraction if $lim_{\ n\to\infty}\ \varphi\left(t_{\varepsilon_\Upsilon}(\{u_n\})\right)=0$ whenever $lim_{\ n\to\infty}\ t_{\varepsilon_W}(\varphi(\{u_n\}))=0$.

Theorem 1.7: [15]

If $(\Upsilon,\rho_\Upsilon,t_\varepsilon)$ and (W,ρ_W,t_ε) are proximit spaces. Then a function $\varphi:\Upsilon\longrightarrow W$ is Dh-contraction if and only if its sequentially Dh-contraction.

Definition 1.8: [15]

A quadruple $(X,\rho,t_\varepsilon,+)$ is said to be proximit group if and only if

1. (X,ρ,t_ε) is proximit space.
2. $(X,+)$ is group.
3. $+:X\otimes X\to X$ such that $(x,y)\longmapsto x+y$ is Dh-contraction.
4. $-:X\longrightarrow X$ such that $x\longmapsto -x$ is Dh-contraction.

Definition 1.9: [15]

A quintuple $(X,\rho,t_\varepsilon,+,.)$ is called proximit vector space such that X is a non-empty set with two binary operations (addition and scalar multiplication) and triple (X,ρ,t_ε) is Proximit space if satisfy the following conditions $\forall g,g^*\in X,\vartheta,\zeta\in field\ E$.

1. $(X,\rho,t_\varepsilon,+)$ is proximit group.
2. $\zeta.g\in X$
3. $\zeta.(g+g^*)=\zeta g+\zeta g^*$
4. $(g+g^*)\zeta=g.\zeta+g^*.\zeta$
5. $(\vartheta.\zeta).g=\vartheta(\zeta g)$
6. $1.g=g$

Definition 1.10:[16]

Let X be proximit vector space. A quadrilateral $(X,\|.\|_p,\rho,t_\varepsilon)$ said to be normed proximit space if satisfy the following :

1. $\|u\|_p\geq 0$ for all $u\in X$
2. If there exists $c\geq 1$, then $\|u+v\|_p\leq c(\|u\|_p+\|v\|_p)$
3. $\lim_{\varsigma\to 0}\|\varsigma u\|_p=0\ \forall\ u\in X$
4. $\rho(A,B)=inf_{u\in A}\ inf_{v\in B}\|u-v\|_p$
5. $\|\ (f\circ t_\varepsilon)(A)\|_p=\|u\|_p$ where $f:2^X\longrightarrow X$ is choice function defined by $f(A)=u\ \ \forall u\in A$.

Normed proximit space is called quasi-normed proximit space if satisfy the following condition $\|\varsigma u\|_p=|\varsigma|\|u\|_p\quad\forall\ \varsigma\in F$

2 Normed proximit algebra and New Results

we introduce a new definition of proximit algebra, commutivite , unital and normed proximit algebra also we discuss new properties about of them.

Definition 2.1

Let B be a non-empty set. We say B is a proximit algebra if

(1) $(B,\rho,t_\varepsilon,+,.)$ is a proximit vector space over a field F

(2) A multiplication operators of proximit space X satisfy the conditions: for all $U,V,W\subseteq X\,,\zeta\in F$.

a) $\zeta(U.V)=(\zeta U)V=U(\zeta V)$

b) $(UV)W=U(VW)$

c) $(U+V)W=UW+VW \quad and \quad U(V+W)=U.V+UW$

(3) $.:B\times B\longrightarrow B$ such that $(U,V)\mapsto U.V \;\; for\ every\ U,V\subseteq B$ is Dh-contraction.

Proposition 2.2:

Given proximit vector spaces X,Y throughout the same field F. Let $L(X,Y)$ be the set of all linear mappings of X into Y with the pointwise addition, scalar multiplication and the product defined by composition

$$(T_1T_2)(s)=T_1\big(T_2(s)\big)\ \forall s\in X$$

is proximit algebra.

Proof

1) We proof $L(X,Y)$ is a proximit vector space, to prove this, we must

a) $(L(X,Y),\rho,t_\varepsilon,+)$ is proximit group.

Define $\rho(U,V)=\{\ \infty \quad , \quad U=\emptyset \ \ or \ \ V=\emptyset \ \ inf_{\ T_1\in U}\ inf_{\ T_2\in V}|T_1(s)-T_2(s)|, \quad U\neq\emptyset \ \ and \ \ V\neq\emptyset$

(h1) If $U\neq\emptyset\ and\ V\neq\emptyset$, then

$$\rho(U,V)=inf_{\ T_1\in U}\ inf_{\ T_2\in V}|T_1(s)-T_2(s)|=inf_{\ T_2\in V}\ inf_{\ T_1\in U}\ |T_2(s)-T_1(s)| =\rho(V,U)$$

(h2) If $U=\emptyset$, then $\rho(U,V)=\infty$ and if $V=\emptyset$, then $\rho(U,V)=\infty$.

(h3) Let $\rho(U,V)=0$, then $inf_{\ T_1\in U}\ inf_{\ T_2\in V}|T_1(s)-T_2(s)|=0$. Hence $T_1(s)=T_2(s)$ implies that there exists $x\in T_1(s)\cap T_2(s)$. Thus If $T_1(s)\neq\emptyset\ and\ T_2(s)\neq\emptyset$, then $T_1(s)\cap T_2(s)\neq\emptyset$

(h4)

$$\rho(U,V\cup W)=inf_{\ T_1\in U}\ inf_{\ T_2\in V\cup W}|T_1(s)-T_2(s)| =\{inf_{\ T_1\in U}\ inf_{\ T_2\in V}|T_1(s)-T_2(s)|,inf_{\ T_1\in U}\ inf_{\ T_2\in W}|T_1(s)-T_2(s)|\}\ =\{\rho(U,V),\rho(U,W)\}$$

(h5) If $\varepsilon,\eta\in[0,\infty]$, then

$$\rho(U,V)=inf_{\ T_1\in U}\ inf_{\ T_2\in V}|T_1(s)-T_2(s)|\leq inf_{\ T_1\in U^\varepsilon}\ inf_{\ T_2\in V^\eta}|T_1(s)-T_2(s)|+\varepsilon+\eta=\rho(U^\varepsilon,V^\eta)+\varepsilon+\eta.$$

b) we prove the function $f:L(X,Y)\times L(X,Y)\ \longrightarrow L(X,Y)$ defined as $(T_1,T_2)\mapsto T_1+T_2$ is Dh-contraction: let $x\in f(t_\varepsilon(U),t_\varepsilon(V))$, we get $x\in f(inf_{\ T_1\in U}\ |T_1(s)|,inf_{\ T_2\in V}|T_2(s)|)$. Hence $x\in\{inf_{\ T_1\in U}\ |T_1(s)|+inf_{\ T_2\in V}|T_2(s)|\}$ from this we have $x\in\{inf_{\ T_1\in U}\ inf_{\ T_2\in V}|T_1(s)+T_2(s)|\}$ implies $x\in\{inf_{\ T_1\in U}\ inf_{\ T_2\in V}|(T_1+T_2)(s)|\}$. Thus $x\in t_\varepsilon(U+V)$, we obtain $x\in t_\varepsilon(f(U,V))$

c) we prove the function $f: L(X,Y) \longrightarrow L(X,Y)$ defined as $T_1 \longmapsto -T_1$ is Dh-contraction: let $x \in f(t_\varepsilon(U))$, we get $x \in (-inf\ _{T_1 \in U} |T_1(s)|)$. Hence $x \in inf\ _{-T_1 \in U} |T_1(s)|$ from this we have $x \in t_\varepsilon(-U)$ implies $x \in t_\varepsilon(f(U))$. d) It is reminder of proximit group conditions are satisfied.

$$\big((\alpha T_1)T_2\big)(s) = \alpha T_1\big(T_2(s)\big) = \alpha(T_1T_2)(s) = \big(\alpha(T_1T_2)\big)(s) = \big((\alpha T_1T_2)\big)(s)$$
$$= \big(T_1(\alpha T_2)\big)(s)\ \ \forall \alpha \in F, \forall T_1T_2 \in L(X,Y)$$

2) a) $\zeta(T_1(T_2(s))) = \zeta(T_1T_2)(s)) = (T_1\zeta T_2)(s)) = T_1(\zeta(T_2)(s))$

b) $(T_1(T_2))(T_3))(s) = (T_1(T_2)(s))(T_3)(s)) = (T_1T_2)(s))(T_3)(s) = T_1(T_2(T_3(s))) = (T_1(T_2(T_3)))(s)$

c) $\big((T_1+T_2)T_3\big)(s) = \big(T_1(s)+T_2(s)\big)T_3(s) = T_1(s)T_3(s) + T_2(s)T_3(s) = (T_1T_3 + T_2T_3)(s))\ and\ \big(T_1(T_2+T_3)\big)(s) = T_1(s)\big(T_2(s)+T_3(s)\big) = T_1(s)T_2(s) + T_1(s)T_3(s) = (T_1T_2 + T_1T_3)(s))$

3) We prove the mapping $f: L(X,Y) \times L(X,Y) \to L(X,Y)$ defined by $(T_1,T_2)(s) \longmapsto (T_1T_2)(s) = T_1(T_2(s))$ is Dh-contraction.

Let $a \in f(t_\varepsilon(U), t_\varepsilon(V))$, we get $a \in f(inf\ _{T_1 \in U} |T_1(s)|, inf\ _{T_2 \in V}|T_2(s)|)$. Hence $a \in \{inf\ _{T_1 \in U} |T_1(s)|.\ inf\ _{T_2 \in V}|T_2(s)|\}$ from this we have $x \in \{inf\ _{T_1 \in U}\ inf\ _{T_2 \in V}|T_1(s).T_2(s)|\}$ implies $x \in \{inf\ _{T_1 \in U}\ inf\ _{T_2 \in V}|(T_1T_2)(s)|\}$ and so that $x \in \{inf\ _{T_1 \in U}\ inf\ _{T_2 \in V}|(T_1(T_2(s))|\}$. Thus $x \in t_\varepsilon(U.V)$, we obtain $x \in t_\varepsilon(f(U,V))$ Then its Dh-contraction, $L(X,Y)$ is proximit algebra.

Definition 2.3

An proximit algebra B is said to be

(1) real proximit algebra if $F = R$ and if, and complex proximit algebra if $F = C$.

(2) commutative if $(B,+,.)$ is commutative, that is $t_\varepsilon(U.V) = t_\varepsilon(V.U)\ \ such\ that$

$$U.V = \{n.m : n \in U, m \in V\}\ \ \forall\ U.V \subseteq B$$

Definition 2.4

A proximit algebra B is said to be unital if $(B,+,.)$ has a unit denoted by I, that is $t_\varepsilon(U.I) = t_\varepsilon(I.U) = t_\varepsilon(U)\ \ \forall\ U \subseteq B$ where $U.I = \{n.e : n \in U, e \in I\}$.

Proposition 2.5

A proximit algebra can only have one unit element.

Proof. Let U, U^* be unit sets in B, we get

$$t_\varepsilon(U.U^*) = t_\varepsilon(U^*.U) = t_\varepsilon(U)$$
$$t_\varepsilon(U^*.U) = t_\varepsilon(U.U^*) = t_\varepsilon(U^*)$$

$t_\varepsilon(U) = t_\varepsilon(U^*)$. Then $U = U^*$, so the unity is unique.

Definition 2.6

A set of B that has both left $t_\varepsilon(VU) = I$ and right inverse $t_\varepsilon(U.V) = I$ is known as an inverse. A set is said to be invertible if an inverse exists for it. The representation of the collection of invertible sets of B is $G(B)$.

Proposition 2.7

If U has a right V and a left inverse W, then $V = W$, so that U is invertible.

Proof. Let $U.V, W \subseteq B$,

$$t_\varepsilon(V) = t_\varepsilon(V.I) = t_\varepsilon(V.U.W) = t_\varepsilon(I.W) = t_\varepsilon(W)$$

Then $V = W$.

Definition 2.8

Let B be a proximit algebra and $N \subseteq B$. We say N is a subproximit algebra if N itself is proximit algebra under the operations of B.

Definition 2.9

If B is proximit algebra and $\|.\|_p$ is proximit norm on B satisfying

$$\|UV\| \leq sup_{A,S\subseteq B}\,\{\|U\|.\|V\|\}$$

$\|.\|_P$ is called proximit algebra norm and $(B;\ \rho, t_\varepsilon, \|.\|_P)$ is called a normed proximit algebra.

A complete normed proximit algebra is called a Banach proximit algebra.

Proposition 2.10

$B = C(K)$ with the pointwise multiplication $(FH)(x) = F(x)H(x)$ is a normed proximit algebra with proximit norm $\|F\|_p = \max_{x\in K}|F(x)|$.

Proof

we prove $C(K)$ is proximit algebra.

1) We prove $(C(K), \rho, t_\varepsilon, +, .)$ is proximit vector space that is we show that $a)$ $(C(K), \rho, t_\varepsilon, +)$ is proximit group. Define

$$\rho(U,V) = \{\ \infty \qquad , \quad U = \emptyset \ \ or \quad V = \emptyset\ |F(x) - H(x)|\ , \quad U \neq \emptyset \ \ and\ V \neq \emptyset$$

(h1) If $U \neq \emptyset\ and\ V \neq \emptyset$, then $\rho(U,V) = \max_{x\in K, F\in U, H\in V} |F(x) - H(x)|\} = \max_{x\in K, F\in U, H\in V} |F(x) - H(x)| = \rho(U,V)$.

(h2) If $U = \emptyset$, then $\rho(U,V) = \infty$ and if $V = \emptyset$, then $\rho(U,V) = \infty$.

(h3) Let $\rho(U,V) = 0$, then $\max_{x\in K, F\in U, H\in V} |F(x) - H(x)| = 0$. Hence $F(x) = H(x)$ implies that $F(x) \in U \cap V$, then $U \cap V \neq \emptyset$

(h4) $\rho(U, V \cup W) = \max_{x\in K, F\in U, H\in V\cup W} |F(x) - H(x)| = \{\max_{x\in K, F\in U, H\in V} |F(x) - H(x)|, \max_{x\in K, F\in U, H\in W} |F(x) - H(x)|\} = \{\rho(U,V), \rho(U,W)\}$

(h5) If $\varepsilon, \eta \in [0, \infty]$, then

$$\rho(U,V) = \max_{x\in K, F\in U, H\in V} |F(x) - H(x)| \leq \max_{x\in K, F\in U^\varepsilon, H\in V^\eta} |F(x) - H(x)| + \varepsilon + \eta = \rho(U^\varepsilon, V^\eta) + \varepsilon + \eta.$$

b) we prove the function $f: C(K) \times C(K) \longrightarrow C(K)$ defined as $(F,H) \longmapsto F + H$ is Dh-contraction: let $x \in f(t_\varepsilon(U), t_\varepsilon(V))$, we get $x \in f(max_{\ x\in KF\in U}\ |F(x)|, max_{\ x\in K,H\in V}|H(x)|)$. Hence $x \in \{max_{\ x\in KF\in U}\ |F(x)| + max_{\ x\in K,H\in V}|H(x)|\}$ it follows $x \in \{max_{\ x\in KF\in U}\ max_{\ x\in K,H\in V}|F(x) + H(x)|\}$. So that $x \in \{max_{\ x\in KF\in U}\ max_{\ x\in K,H\in V}\ |(F + H)(x)|\}$. Thus $x \in t_\varepsilon(U + V)$, we obtain $x \in t_\varepsilon(f(U,V))$.

c) we prove the function $f: C(K) \longrightarrow C(K)$ defined as $F \longmapsto -F$ is Dh-contraction: let $x \in f(t_\varepsilon(U))$, we get $x \in \{-\max_{x\in K, F\in U} |F(x)|\}$. Hence $x \in \max_{x\in K, -F\in U} |F(x)|$ from this we have $x \in t_\varepsilon(U)$ implies $x \in t_\varepsilon(f(U))$.

d) It is reminder of proximit group conditions are satisfied.

$$(\alpha FH)(x) = \alpha F(x)H(x) = F(x)\alpha H(x) = \big(F(\alpha H)\big)(x) \forall \alpha \in F, \forall F, H \in C(K)$$

2) a) $\zeta(F.H)(x)) = \zeta(F(x)H(x)) = F(x)\zeta H(x)) = (F(\zeta(H)(x))$

b) $((F.H)G)(x) = \big(F(x).H(x)\big)G(x) = F(x)(H(x)G(x)) = (F(H.G))(x)$

c) $(F + H)G)(x) = (F(x) + H(x))G(x) = F(x)H(x) + F(x)G(x) = (F.H + F.G)(x)\ and\ \big(F(H + G)\big)(x) = F(x)\big(H(x) + G(x)\big) = F(x)H(x) + F(x)G(x) = (FH + FH)(x)$

3) We prove the mapping $f: C(K) \times C(K) \to C(K)$ defined by $(F, H)(x) \mapsto (F.h)(x) = F(x).H(x)$ is Dh-contraction.

Let $a \in f(t_\varepsilon(U), t_\varepsilon(V))$, we get $a \in f(\max_{x\in K, F\in U} |F(x)|, \max_{x\in K, H\in V} |H(x)|)$. Hence $a \in \{\max_{x\in K, F\in U} |F(x)|. \max_{x\in K, H\in V} |H(x)|\}$ from this we get $x \in \{\max_{x\in K, F\in U} \max_{x\in K, H\in V} |F(x)|.|H(x)|\}$ implies $x \in \{\max_{x\in K, F\in U, H\in V} |F(x)H(x)|\}$ and so that $x \in \{\max_{x\in K, F\in U, H\in V} |(F.H)(x)|\}$. Thus $x \in t_\varepsilon(U.V)$, we obtain $x \in t_\varepsilon(f(U, V))$ Then its Dh-contraction. Then its Dh-contraction, $C(K)$ is proximit algebra.

Now, we show that its normed algebra.

$$\|U.V\|_P = \max_{x\in K, F\in U, H\in V} |F(x)H(x)| \leq \max_{x\in K, F\in U, H\in V} \left\{\{|F(x)|.|H(x)|\}\right\}$$
$$= \left\{\max_{x\in K, F\in U, H\in V} |F(x)|. \max_{x\in K, F\in U, H\in V} |H(x)|\right\} = sup_{U,V\in C(K)} \{\|U\|_p \|V\|_p\}$$

Then $C(K)$ is normed algebra.

3 New Properties of Banach proximit algebra

In this part, we introduce proximit homomorphism, proximit isometry, and a joint proximit topological divisor of zero. We also examine new homomorphism features that are invertible with regard to Banach algebra via proximit structure.

Proposition 3.1

The space $C'[a, b]$ of continuously differentiable functions $F: [a, b] \to R$ is proximit Banach space and define $\|F\|_p = \|F\| + \|F'\|$

Then $(C'[a, b], \rho, t_\varepsilon, \|.\|_p)$ is a Banach proximit algebra.

Proof

1) $(C'[a, b], \rho, t_\varepsilon, +, \cdot)$ is proximit vector space.

a) To prove is proximit algebra

Define $\rho(U.V) = \{ \quad \infty \quad , \quad U = \emptyset \;\; or \;\; V = \emptyset \;\; sup_{x\in K, F\in U, H\in V} \|F - G\|_p \quad , \quad U \neq \emptyset \;\; and \;\; V \neq \emptyset$

(h1) If $U \neq \emptyset \; and \; V \neq \emptyset$, then $\rho(U, V) = sup_{F\in U, G\in V} \|F - G\|_p = sup_{F\in U, G\in V} \|G - F\|_p = \rho(V, U)$.

(h2) If $U = \emptyset$, then $\rho(U, V) = \infty$ and if $V = \emptyset$, then $\rho(U, V) = \infty$.

(h3) Let $\rho(U, V) = 0$, then $sup_{F\in U, G\in V} \|F - G\|_p = 0$. Hence $\|F - G\|_p = 0$ implies that $F = G \;\; for \; all \; F \in U, \; G \in V$, then $U \cap V \neq \emptyset$

(h4) $\rho(U, V \cup W) = sup_{F\in U, G\in V\cup W} \|F - G\|_p = \{sup_{F\in U, G\in V} \|F - G\|_p, sup_{F\in U, G\in W} \|F - G\|_p\} = \{\rho(U, V), \rho(U, W)\}$

(h5) If $\varepsilon, \eta \in [0, \infty]$, then $\rho(U, V) = sup_{F\in U, G\in V} \|F - G\|_p \leq sup_{F\in U, G\in V} \|F - G\|_p + \varepsilon + \eta = \rho(U^\varepsilon, V^\eta) + \varepsilon + \eta$.

Then $(C'[a, b], \rho, t_\varepsilon)$ is proximit space. The remaining conditions of definition a proximit vector space are similar to the above Proposition.

2) Define $f: C'[a, b] \times C'[a, b] \to C'[a, b]$ by

$(F, G)(x) = (F.G)(x) = F(x).G(x) \;\forall F, G \in C'[a, b]$ is Dh-contraction.

Let $a \in f(t_\varepsilon(U), t_\varepsilon(V))$, we get $a \in f(sup_{F\in U} \|F\|_p, sup_{G\in V} \|G\|_p)$. Hence $a \in \{sup_{F\in U} \|F\|_p . sup_{G\in V} \|G\|_p\}$ from this we get $x \in \{sup_{x\in K, F\in U, H\in V} \|F.G\|_p\}$ implies

$x \in \{sup_{x\in K,F\in U,G\in V} \|F(x)G(x)\|_p \}$ and so that $x \in \{sup_{x\in K,F\in U,H\in V} \|(F.G)(x)\|_p\}$. Thus $x \in t_\varepsilon(U.V)$, we obtain $x \in t_\varepsilon(f(U,V))$ Then its Dh-contraction. Then its Dh-contraction, so that $C'[a,b]$ is proximit algebra.

We show that $C'[a,b]$ is Dh-complete. Let $\{F_n\}$ be an Dh-Cauchy sequence in $C'[a,b]$, then $lim_{\ n\to\infty} inf\ inf\ \rho(F_n, F) = 0\ and\ lim_{\ n\to\infty} inf\ inf\ t_\varepsilon(F_n) \subseteq F$, implies $\forall\ F_n, F \in C'[a,b]$ such that $lim_{\ n\to\infty} \|F_n - F\|_p = 0$ and $lim_{\ n\to\infty}$ $inf\ inf\ \rho(F_n, \{x\}) \le \varepsilon\ \ that\ is\ lim_{\ n\to\infty}\ inf_{\ \{x\}\subseteq F} \|F_n - \{x\}\|_p \le \varepsilon$. We obtain $\|F_n - F\|_p = 0$, hence $\{F_n\}$ is Dh-convergent. So that $C'[a,b]$ is Dh-complete.

Now , its proximit norm . Let $F, G \in C'[a,b]$

$$\|F.G\|_p = \|FG\| + \|(FG)'\|$$
$$= \|FG\| + \|FG' + GF'\|$$
$$\le sup\ sup\ \{\|FG\| + \|FG'\| + \|GF'\| + \|F'G'\|\}$$
$$\le sup\ sup\ \{\|F\|.\|G\| + \|F\|.\|G'\| + \|G\|.\|F'\| + \|F'\|,\|G'\|\}$$
$$= sup\ sup\ \{\|F\|(\|G\| + \|G'\|) + \|F'\|(\|G\|. + \|G'\|)\}$$
$$= sup\ sup\ \left\{\|F\|_p.\|G\|_p\right\}$$

Thus, $C'[a,b]$ is Banach proximit algebra.

Proposition 3.2

Let $S \neq \emptyset$; and $B(S) = \{\ f: S \to R\ :\ f\ is\ bounded\}$. For $\{f\},\{g\} \in 2^{B(S)}$, define

$$(\{f\} + \{g\})(s) = \{f(s)\} + \{g(s)\}$$
$$\{\alpha f\}(s) = \alpha\{f(s)\}\ \ for\ all\ \{f\},\{g\} \in 2^{B(S)}, \alpha \in R$$
$$\{fg\}(s) = \{f(s)\}\{g(s)\}$$

Then $B(S)$ is proximit algebra with unit $\{f(s)\} = 1$ for all $s \in S$ with proximit norm

$$\|U\|_\infty = \{|f(s)|: s \in S\}$$

B(S) is a commutative Banach proximit algebra. We define the function as

Proof

$B(S)$ is proximit algebra Its similar of above Proposition

$\rho(U,V) = \{\ \ \infty\ \ \ \ ,\ \ U = \emptyset\ \ or\ \ V = \emptyset\ |f(s) - g(s)|\ \ \ ,\ \ U \neq \emptyset\ \ and\ V \neq \emptyset$

4) We prove is proximit normed space, define $j: B(S) \to B(S)\ such\ that\ j(U) = U\ \ , U \subseteq B(S)$

1) $t_\varepsilon(U) = \{\emptyset\ \ \ ,\ \ \ x \notin U\ |f(s)|\ ,\ x \in U$

   a) $\|U\|_\infty \ge 0$

   b) $if\ there\ exists\ c \ge 1, then$

2) $\|U + V\|_\infty = \{|f(s) + g(s)|: s \in S\}$

3) $\le \{|f(s)| + |g(s)|: s \in S\}$

4) $\le c\left\{\{|f(s)|: s \in S\} + \{|g(s)|: s \in S\}\right\}$

5) $= c\{\|U\|_\infty + \|V\|_\infty\}$

   c) $lim_{\ \alpha\to 0} \|\alpha U\|_\infty = lim_{\ \alpha\to 0}\{|(\alpha f)(s)|: s \in S\} = lim_{\ \alpha\to 0}\{|\alpha||f(s)|: s \in S\} = \ 0.$

   d) If $U = \emptyset\ then\ \rho(U,V) = \infty$, if $U \neq V$, we have

   e) $\rho(U,V) = sup_{\ s\in S, f\in\ U,\ \ g\in V} |f(s) - g(s)| = \|U - V\|_p$

   f) Since $j(U) = U\ \forall U \subseteq B(S)$ and if $x \notin U, then\ t_\varepsilon(U) = \emptyset$ and so

6) $\|(j \circ t_\varepsilon)(U)\|_\infty = \|j(t_\varepsilon(U))\|_\infty = \|j(\emptyset)\|_\infty = \emptyset.$

If $x \in U, then\ t_\varepsilon(U) = |\{f(s)\}|$ and so

7) $\|(j \circ t_\varepsilon)(U)\|_\infty = \|j(t_\varepsilon(U))\|_\infty = \|j(|f(s)|\ )\|_\infty = \||f(s)|\ \|_\infty = \|U\|_\infty.$

g) Let $U, V \in 2^{B(S)}$

$$\|U.V\|_\infty = \{|(fg)(s)|: s \in S\}$$
$$\leq sup\ supsup_{\ f\in U, g\in V}\{|f(s)||g(s)|: s \in S\}$$
$$\leq sup\ sup\left\{\{|f(s)|: s \in S\}.\{|g(s)|: s \in S\}\right\}$$
$$\leq sup\ sup\{\|U.\|_\infty.\|V\|_\infty\}$$

Then $B(S)$ is proximit algebra norm.

Now, we show that $B(S)$ is Dh-complete. Let $\langle U_n \rangle$ be an Dh-Cauchy sequence in $2^{B(S)}$, then $lim_{\ n\to\infty} inf\ inf\ \rho(U_n, U) = 0\ and\ lim_{\ n\to\infty} inf\ inf\ t_\varepsilon(U_n) \subseteq U_n$, implies $\forall\ U_n, U \in 2^{C'[a,b]}$ such that $lim_{\ n\to\infty} \|U_n - U\|_\infty = 0$ and

$lim_{\ n\to\infty} inf\ inf\ \rho(U_n, U) \leq \varepsilon$ $(that\ is\ \ lim_{\ n\to\infty}\ inf_{\ \{x\}\subseteq U} \|U_n - \{x\}\|_p \leq \varepsilon.$

We obtain $lim_{\ n\to\infty}\{|f_n(s) - f(s)|: s \in S\} = 0$, so $|f_n(s) - f(s)| = 0$ for all $f_n \in U_n, f \in U$. Hence U_n is Dh-Cauchy in field R, but R is complete. So that U_n is Dh-convergent in R, hence $\rho(U_n, U) = 0$. So that $B(S)$ is Dh-complete. Therefore, $B(s)$ is Banach proximit algebra.

Definition 3.3

Let G, H be proximit algebras over the same scalar field F. A mapping $\Omega: G \to H$ is called proximit homomorphism if

1) $\Omega(VW) = \Omega(V)\Omega(W) \quad \forall V, W \in G.$
2) $\Omega\ is\ Dh - contarction$

A bijective proximit homomorphism of G into H is said to be proximit isomorphism of G onto H is, and an injective proximit homomorphism of G into H is called a proximit monomorphism of G into H is. Proximit algebras G and H are proximit isomorphic if a proximit isomorphism of G onto H exists.

A subset U is said to be subproximit semi-group of G if $V, W \subseteq U$ such that $VW \in U$. A subproximit algebra of G is proximit vector subspace of G that is also sub proximit semi-group of G.

It is obvious that a sub proximit algebra D of proximit algebra B is itself proximit algebra with the same scalar field and the product in D the restriction to $D \times D$ of the product in B.

Definition 3.4

Suppose that B_1, B_2 are normed proximit algebras. A topological proximit isomorphism of B_1 onto B_2 is proximit isomorphism of B_1 onto B_2 is also proximit homeomorphism of topological proximit space B_1 onto topological proximit space B_2. A proximit isometric isomorphism of B_1 onto B_2 is proximit isomorphism T of proximit algebra B_1 onto proximit algebra B_2 in other words, also proximit isometric mapping of the metric space B_1 onto the metric space B_2. According to the final condition, $t_\varepsilon(T(V) - T(W)) = t_\varepsilon(V - W) \ \ \forall V, W \subseteq B_1$.

However, by the linearity of T, this is equivalent to

$$t_\varepsilon(T(V)) = t_\varepsilon(V) \qquad (V \subseteq B_1)$$

Similarly, for normed proximit vector spaces X, Y proximit isometric linear isomorphism of X onto Y is a linear mapping T from X to Y such that $t_\varepsilon(T(V)) = t_\varepsilon(V) \ \ (V \in X)$.

Notation

Given two normed proximit vector spaces X and Y over a single scalar field F, we describe $BL(X, Y)$ as the vector proximit subspace of $L(X, Y)$, which contains all bounded and continuous linear mappings from X to Y.

$BL(X,Y)$ is commonly considered to be a normed proximit vector space with a proximit norm defined by

$$\|T\|_p = sup\ sup\ \{\|T(V)\| : V \in X\ \ \&\|V\| \leq 1\}$$

Proposition 3.5

Assume that B is a normed proximit algebra. A dense sub proximit algebra of a Banach proximit algebra A, then has proximit isometric isomorphism of B onto it. A cannot be compared up to isometric isomorphism.

Proof.

There is an isometric linear proximit isomorphism T of B onto a dense proximit vector subspace of a proximit Banach space A. Given $V, W \in A$, there exist $V_n, W_n \in A$ such that $V = lim_{\ n\to\infty} T(V_n), W = lim_{\ n\to\infty} T(W_n)$. Since T is proximit isometry, $\{V_n\}$ and $\{W_n\}$ are Dh-Cauchy sequences in B. Since

$$\|V_pW_p - V_qW_q\|_p \leq sup\ sup\ \{\|V_p\|_p . \|W_p - W_q\|_p\}\ + sup\ sup\ \{\|V_p - V_q\|_p . \|W_q\|_p\}$$

$\{V_nW_n\}$ is Dh-Cauchy sequence in B, $\{T(V_nW_n)\}$ is Dh-Cauchy sequence in A, also $V = lim_{\ n\to\infty} T(V_nW_n) = U \in A$.

Moreover, U U can be used to define a product in A by choosing $VW = U$ because it is independent of the sequences $\{V_n\}$ and $\{W_n\}$ that are chosen.

Theorem 3.6

If $\{V_n\}$ and $\{W_n\}$ are an Dh-convergence of Banach proximit algebra B, then the multiplication is Dh-convergence.

Proof.

Let $V, W \in 2^B$ and $\|V_n - V\|_p = 0, \|W_n - W\|_p = 0$ and $c \geq 1$

$$\begin{aligned}\|V_nW_n - VW\|_p &\leq c\{\|(V_n - V)W\|_p + \|V(W_n - W)\|_p\}\\ &\leq c\{sup\ sup\ \{\|V_n - V\|_p\|W\|_p\} + sup\ sup\ \{\|V\|_p . \|W_n - W\|_p\}\ \}\\ &= c\ sup\ sup\ \{\|V_n - V\|_p\|W\|_p + \|V\|_p . \|W_n - W\|_p\}\ = 0\end{aligned}$$

Lemma 3.7

Let B be a normed proximit algebra with unit. If $V, W \subseteq G(B)$ and $\|t_\varepsilon(W) - t_\varepsilon(V)\|_p \leq \frac{1}{2}\|t_\varepsilon(V^{-1})\|_p^{-1}$, then

$$\|t_\varepsilon(W) - t_\varepsilon(V)\|_p \leq \{\|t_\varepsilon(V)\|_p^2\|W - V\|_p\}$$

Proof.

For such V, W we have

$$\begin{aligned}\big|\|t_\varepsilon(W^{-1})\|_p - \|t_\varepsilon(V^{-1})\|_p\big| &\leq \|t_\varepsilon(W^{-1}) - t_\varepsilon(V^{-1})\|_p\\ &= \|t_\varepsilon(W^{-1})[t_\varepsilon(V) - t_\varepsilon(W)]t_\varepsilon(V^{-1})\|_p \leq \frac{1}{2}\, sup\ sup\ \|t_\varepsilon(W^{-1})\|_p\end{aligned}$$

Thus,

$$\|t_\varepsilon(W^{-1})\|_p \leq 2\ sup\ sup\ \|t_\varepsilon(V^{-1})\|_p$$

and so

$$\begin{aligned}\|t_\varepsilon(W^{-1}) - t_\varepsilon(V^{-1})\|_p &\leq sup\ sup\ \{\|t_\varepsilon(W^{-1})\|_p\|t_\varepsilon(V) - t_\varepsilon(W)\|_p\|t_\varepsilon(V^{-1})\|_p\}\\ &\leq 2\ sup\ sup\ \{\|t_\varepsilon(V^{-1})\|_p^2\|t_\varepsilon(V) - t_\varepsilon(W)\|_p\}\end{aligned}$$

Definition 3.8

Suppose V is a set of normed proximit algebra. The formula for the spectral radius of V is $r(t_\varepsilon(V)) = inf\ inf\ \left\{\|t_\varepsilon(V^n)\|^{\frac{1}{n}} :\right\}$, where $n = 1,2, \dots$.

Proposition 3.9

Let V be subset of normed proximit algebra. Then

$$r(t_\varepsilon(V)) = \lim_{n\to\infty} \|t_\varepsilon(V^n)\|^{\frac{1}{n}}$$

Proof.

Let $\theta = r(t_\varepsilon(V))$ and $\varepsilon > 0$, and select k such that $\|t_\varepsilon(V^k)\|^{\frac{1}{k}} < \theta + \varepsilon$.
The formula $n = t(n)k + q(n)$ can be used to represent any positive integer n in a unique way, where $t(n)$ and $q(n)$ are non-negative integers and $q(n) \leq k - 1$.
Because $\frac{1}{n}q(n) \to 0$, we have $\frac{1}{n}t(n)k \to 1$ as $n \to \infty$, and as a result,

$$\|t_\varepsilon(V^n)\|^{\frac{1}{n}} \leq sup\ sup \left\{\|t_\varepsilon(V^k)\|_p^{\frac{t(n)}{n}} . \|t_\varepsilon(V)\|_p^{\frac{q(n)}{n}}\right\} \to \|t_\varepsilon(V^k)\|_p^{\frac{1}{k}} < \theta + \varepsilon.$$

Thus $\|t_\varepsilon(V^n)\|_p^{\frac{1}{n}} < \theta + \varepsilon$ for each sufficiently large n. Also $\theta \leq \|t_\varepsilon(V^n)\|_p^{\frac{1}{n}}$ for each n

Theorem 3.10

Consider B is Banach proximit algebra with unit, if $V \subseteq B$, and $r(t_\varepsilon(V)) < 1$. Then $1 - t_\varepsilon(V)$ is invertible, and

$$(1 - t_\varepsilon(V))^{-1} = 1 + \sum_{n=1}^{\infty} t_\varepsilon(V^n)$$

Proof.

Choose η with $r(t_\varepsilon(V)) < \eta < 1$. By Proposition 3.9, we have $\|t_\varepsilon(V^n)\|_p < \eta^n$ for every sufficiently large n, and hence $\|1\| + \sum_{n=1}{}^{\infty}\|t_\varepsilon(V^n)\|_p$ is Dh-converges. Since B is a proximit Banach space, implies that the series $1 + \sum_{n=1}{}^{\infty} t_\varepsilon(V^n)_p$ is Dh-converges, with sum $S \in B$, say. Let $S_n = 1 + V + \cdots + V^{n-1}$. Then $S_n \to S$ and $\|V^n\|_p \to 0$ as $n \to \infty$, and we have

$$t_\varepsilon\big((1 - V)S_n\big) = t_\varepsilon\big(S_n(1 - V)\big) = t_\varepsilon(1 - V^n)$$

Therefore, by Dh-contraction of multiplication , we obtain

$$t_\varepsilon((1 - V)S) = t_\varepsilon(S(1 - V)) = 1$$

Corollary 3.11

If B is Banach proximit algebra with unit. Then $\|t_\varepsilon(1 - V)\|_p < 1$ is invertible for all V of B.

Proof

$t_\varepsilon(r(1 - V)W) = \|t_\varepsilon(r(1 - V))W\|_p \leq sup\ sup \left\{\|1\|_p \|t_\varepsilon(1 - V)\|_p + \|W\|_p\right\} < 1$ and so

$t_\varepsilon(V) = t_\varepsilon(1 - (1 - V))$ is invertible.

Theorem 3.12

If B is Banach proximit algebra with unit. Then $G(B)$ is an open subset of B.

Proof

Let $V \subseteq G(B)$. Suppose that $E \in B$ such that $\|t_\varepsilon(E)\|_p < \|t_\varepsilon(V^{-1})\|_p^{-1}$, we have $t_\varepsilon(V) - t_\varepsilon(E) = t_\varepsilon(V(1 - V^{-1}E))$, and $\|t_\varepsilon(V^{-1}E)\|_p \leq sup\ sup \left\{\|t_\varepsilon(V^{-1})\|_p . \|t_\varepsilon(E)\|_p\right\}$. Therefore, by Theorem 3.10, $1 - t_\varepsilon(V^{-1}E) \in G(B)$, and $G(B)$ being a proximit group, $t_\varepsilon(V - E) \in G(B)$. So that , the open ball with a center V and radius $\|t_\varepsilon(V^{-1})\|^{-1}$ is contained in $G(B)$. Therefore $G(B)$ is an open set.

Definition 3.13

If B is a normed proximit algebra, and $S(B)$ be unit sphere in B.

$S(B) = \{V \in B : \|t_\varepsilon(V)\|_p = 1\}$.

An subset V of B is said to be

(i) a left proximit topological divisor of zero if $inf\ inf \{\|t_\varepsilon(VE)\|_p : E \in S(B)\} = 0$

(ii) a right proximit topological divisor of zero if $inf\ inf \{\|t_\varepsilon(EV)\|_p : E \in S(B)\} = 0$

(iii) ajoint proximit topological divisor of zero if $inf\ inf \{\|t_\varepsilon(VE)\|_p + \|t_\varepsilon(EV)\|_p : E \in S(B)\} = 0$.

A subset V of B is proximit joint topological divisor of zero if and only if there exist a sequence $\{E_n\}$ of sets of $S(B)$ such that

$$\|t_\varepsilon(VE_n)\|_p = \|t_\varepsilon(E_nV)\|_p = 0$$

A joint proximit topological divisor of zero is obviously one that has both a left and a right proximit topological divisor.

Notation

We identify by ∂H the topological boundary of a proximit topological space given by a subset H of that space (that is , τ is a proximit topological space and H is a subset of τ then a boundary of H is a set $\partial H = cl(H) \cap cl(\tau \backslash H)$)

Theorem 3.14

Consider Banach proximit algebra B with unit, and $V \in G(B)$. Then V is a joint proximit topological divisor of zero.

Proof

Let $V \in \partial G(B)$, there is $V_n \in G(B)$ with $\lim_{n\to\infty} t_\varepsilon(V_n) = t_\varepsilon(V)$.

We prove first that $\{\|t_\varepsilon(V_n^{-1})\|_p\}$ is unbounded. Suppose on the contrary that $\|t_\varepsilon(V_n^{-1})\|_p \le M$. Then

$$\|t_\varepsilon(V_m^{-1}) - t_\varepsilon(V_n^{-1})\|_p = \|t_\varepsilon(V_m^{-1})(t_\varepsilon(V_n) - t_\varepsilon(V_m))t_\varepsilon(V_n^{-1})\|_p$$
$$\le M^2\ sup\ sup\ \|t_\varepsilon(V_n) - t_\varepsilon(V_m)\|_p$$

This shows that $\{t_\varepsilon(V_n^{-1})\}$ is an Dh-Cauchy sequence.

Let $W = \lim_{n\to\infty} t_\varepsilon(V_n^{-1})$. Then, by Dh-contraction of multiplication, $t_\varepsilon(VW) = t_\varepsilon(WV) = 1$, $V \in G(B)$. But since $G(B)$ is an open set, this contradicts with assumption that $V \in \partial G(B)$. We can now assume, by keeping only a suitable subsequence, that

$\|t_\varepsilon(V_n^{-1})\|_p \ge n$ $(n = 1,2,\dots)$. Let $t_\varepsilon(W_n) = \|t_\varepsilon(V_n^{-1})\|_p^{-1} t_\varepsilon(V_n^{-1})$. Then $t_\varepsilon(W_n) \in S(B)$, and $t_\varepsilon(VW_n) = t_\varepsilon((V - V_n)W_n + V_nW_n) = t_\varepsilon((V - V_n)W_n) + \|t_\varepsilon(V_n^{-1})\|_p^{-1}$

Thus $\lim_{n\to\infty} t_\varepsilon(VW_n^{-1}) = 0$, and similarly $\lim_{n\to\infty} t_\varepsilon(W_n^{-1}V) = 0$.

Proposition 3.15

Let K be a closed ideal in a Banach proximit algebra B. Then the quotient space $\frac{B}{K}$ is a Banach proximit algebra with respect to the quotient norm.

Proof

That $\frac{B}{K}$ is a proximit Banach space. Let π denote the canonical map from B to $\frac{B}{K}$. We must show that

$\|\pi(V)\pi(W)\| \leq sup\ sup\ \{\|\pi(V)\|.\ \|\pi(W)\|\}$ for all $V, W \subseteq B$. By definition of the quotient norm, then

$$\begin{aligned}\|\pi(V)\pi(W)\|_p &= \|(V+M)(W+N)\|_p = \|V.W+K\|_p = inf_{M\subseteq K}\|V.W+M\|_p \\ &= inf_{M\subseteq K}\|(V+M)(W+M)\|_p \\ &\leq sup_{V,W\subseteq B}\left\{inf_{M\subseteq K}\|V+M\|_p.\ inf_{M\subseteq K}\|W+M\|_p\right\} \\ &= sup_{V,W\subseteq B}\left\{\|V+K\|_p.\ \|W+K\|_p\right\} \\ &= sup_{V,W\subseteq B}\left\{\|\pi(V)\|_p.\ \|\pi(W)\|_p\right\}\end{aligned}$$

We obtain the desired result.

Theorem 3.16

Let B be a Banach proximit algebra without a unit. Then B can be embedded into B unital Banach proximit algebra B_1 as an ideal of co-dimension one.

Proof.

Consider the proximit vector space $B_1 = B \oplus R$, and define addition, scalar multiplication and product in $B_1$by

$$(V,\zeta_1)+(W,\zeta_2)=(V+W,\zeta_1+\zeta_2)$$
$$\zeta_2(V,\zeta_1)=(\zeta_2 V,\zeta_2\zeta_1)$$
$$(V,\zeta_1).(W,\zeta_2)=(VW+V\zeta_2+W\zeta_1,\zeta_1\zeta_2),$$

respectively for all $V, W \subseteq B, \zeta_1, \zeta_2 \in F$. It is easily checked that this is associative and distributive.

Moreover, the element (0,1) is a unit for this multiplication.

$$(V,\zeta).(0{,}1)=(V0+V.1+0.\zeta,\zeta.0)=(V,\zeta)=(0{,}1)(V,\zeta)$$

We define a proximit norm, Dh-contraction and tower on B_1 via $\|(V,\zeta_1)\|_p = \|V\|_p + |\zeta_1|\ \ for\ all\ V \subseteq B, \zeta_1 \in R$, $\rho(V,\zeta_1),(W,\zeta_2)) = inf_{s\in V}\ inf_{t\in W}\|s-t\|_p + |\zeta_1| + |\zeta_2|$ and $t_\varepsilon(V) = inf_{s\in V}\|s\|_p + |\zeta_1|$, respectively. Then B_1 is a Banach proximit space when equipped with these functions. Furthermore,

$$\begin{aligned}\|(V,\zeta_1).(W,\zeta_2)\|_p &= \|VW+V\zeta_2+W\zeta_1\|_p + |\zeta_1\zeta_2| \\ &\leq sup\ sup\left\{\|V\|_p\|W\|_p + \|V\|_p|\zeta_2| + \|W\|_p|\zeta_1| + |\zeta_1||\zeta_2|\right\}\end{aligned}$$
$$= sup\ sup\ \{(\|V\|_p+|\zeta_1|)(\|W\|_p+|\zeta_2|)\}\ = sup\ sup\ \{(\|(V,\zeta_1)\|_p)(\|(W,\zeta_2)\|_p)\}$$

Hence $B_1$is a Banach proximit algebra with unit. We may identify B with the ideal $\{(V,0): V \subseteq\ B\}$ in B_1 via the isometric isomorphism $V \longmapsto (V,0)$. Hence proved.

Remark 3.17

Let X and Y be Banach proximit spaces over the field F. Then the set of contraction proximit linear transformations $P(X,Y)$ from B into E is a Banach proximit space under the operator proximit norm $\|T\|_p = sup_{\|V\|\leq 1}\|T(V)\|_p$. When $X = Y$, we also write $P(X)$ for $P(X,X)$.

Proposition 3.18

Every Banach proximit algebra B embeds proximit isometrically into $P(X)$ for some proximit Banach space X. Here, B need not have a unit.

Proof

Consider the map

$$\varphi: B \to P(B_1)$$
$$V \longmapsto L_V$$

where $L_V(Z,\alpha) = (V,0)(Z,\alpha)$ is the left regular representation of B. That

$$\varphi(\alpha V + W) = L_{\alpha V+W} = \alpha L_V + L_W = \alpha\varphi(V) + \varphi(W)$$

and that

$$\varphi(VW) = L_{VW} = L_V L_W = \varphi(V)\varphi(W)$$

For every $V, W \subseteq B$ and $\alpha \in C$ Then

$$\|\varphi(V)\|_p = \|L_V\|_p = \frac{\|(V,0)(Z,\alpha)\|_p}{\|(Z,\alpha)\|_p} \le \|(V,0)\|_p = \|V\|_p$$

and

$$\|\varphi(V)\|_p = \|L_V\|_p \ge \|(V,0)(0,1)\|_p = \|V\|_p$$

So that $\|\varphi(V)\|_p = \|L_V\|_p = \|V\|_p$. We show that is Dh-contraction, define $t_\varepsilon(V) = V$ and let $x \in \varphi(t_\varepsilon(V))$, then $x \in \varphi(V)$ So that $x \in L_V(Z,\alpha) = (V,0)(Z,\alpha)$, $x \in t_\varepsilon(L_V) = t_\varepsilon(\varphi(V))$ this show that φ is Dh-contraction. In particular, the map is proximit isometric.

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