

On Structure of Soft Relative Homotopy and Soft Weak Equivalence in Soft Topological Spaces

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ABSTRACT: In soft topology, the concept of soft relative homotopy (So.R.H) extends classical relative homotopy by studying deformations of soft maps with respect to soft subspaces. Complementarily, soft weak equivalence (So.W.E) generalizes weak homotopy equivalence, identifying soft mappings that preserve homotopy classes without necessarily developing soft homotopy theory and establishing equivalences between soft topological spaces under intervention of this work. We explore soft homotopy theory and soft relative homotopy (So.R.H) and also, we define some important and useful theories on this topic.

Keywords: Deformation in soft spaces, homotopy lifting property, soft fibrations, soft homotopy classes, soft relative homotopy, soft set theory, soft topology, soft weak equivalence, weak homotopy equivalence (analogue)



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1. INTRODUCTION AND MOTIVATION

The theory of soft sets, introduced by Molodtsov in 1999 [1], has emerged as a powerful mathematical tool to handle uncertainty in various branches of science. By generalizing classical set-theoretic concepts, soft set theory provides a flexible framework that has been extended to diverse mathematical structures, including topology. The concept On Mixed soft Serre Fibration work by Jabr and Al Baydli [2]. A study was presented in 2023 by Kadhim and Al Baydli [3] about "Some Result of Triple Fiber Bundle". Also in 2024, Mohsin and Al Baydli [4] give study "soft cofibration and soft Serre cofibration" and soft topological groups. In 2024 Al Baydli and Noor [5] introduce "The mixed slicing structure property". A study was presented in 2025 by Seemaa Mohammed Ali and Majeed [6] about "The action of tensor product for topological groupoids". Although in 2023 by Aziz and Majeed [7] introduce about "On some results of winning strategy of topological games. The development of soft topology (Shabir and Naz, 2011) [8] opened the way for adapting fundamental topological notions - such as continuity, connectedness, and compactness - into the soft setting, thereby offering new perspectives for analyzing spaces under incomplete or imprecise information. A study was presented in 2023 by Sadeq and Majeed [9] about "The action of tensor product for topological groupoids. Although in 2023 Khalaf and Al Baydli [10, 11] introduce "soft homotopy theory with some result" with some result in 2024 introduce "On Structure soft Fibration". Also, in 2024 Tuaimah and AL-Baydli [12], introduce "On soft ideal Homotopy Lifting Property". Homotopy theory, on the other hand, plays a central role in algebraic topology by studying when two continuous maps can be deformed into one another [13, 14]. Within this framework, the concepts of relative homotopy and weak equivalence are fundamental in characterizing the relationships between topological spaces and in establishing equivalences up to homotopy. Motivated by these classical notions, researchers have recently begun to explore their analogues in the context of soft topology. In this regard, the notion of soft relative homotopy has been introduced as a natural extension of classical relative homotopy. It investigates deformations of soft maps between soft topological spaces relative to soft subspaces, thereby adapting the idea of homotopic deformation to uncertain environments. Similarly, the concept of soft weak equivalence generalizes the classical weak homotopy equivalence by characterizing soft maps that preserve homotopy classes without requiring strict isomorphism [15, 16]. The study of these two notions not only enriches

the foundation of soft homotopy theory but also provides essential tools for further developments, such as soft fibrations, lifting properties, and approximation theorems. Thus, the integration of soft relative homotopy and soft weak equivalence contributes to establishing a systematic framework for exploring equivalences and deformations in soft topological spaces, bridging the gap between classical topology and its soft counterpart. While some of the definitions used in this paper are adapted from previous works (such as Thivagar & Richard, 2015; Nazmul & Biswas, 2024), they have been reformulated and extended to the framework of soft topological spaces. The main novelty lies in introducing soft relative versions of these classical notions, where the dependence on parameters and soft mappings yields new behaviors not captured in earlier frameworks. The novelty of this paper lies in extending the concept of relative homotopy to the framework of soft topological spaces and introducing the notion of soft weak equivalence as a new analogue of weak homotopy equivalence in classical topology. Unlike the previous studies by Thivagar & Richard (2015) and Nazmul & Biswas (2024), which mainly discussed soft homotopy and soft fibrations in isolated contexts, this work develops a unified and systematic approach connecting soft relative homotopy and soft weak equivalence. The results obtained here enrich the existing theory by providing new lifting properties, equivalence relations, and deformation results that are not covered in earlier works.

Definition 1.1 [18] (Soft set)

Let a non-empty set E and W the parameters set, the mapping F from W to the power set $P(E)$ is called a soft set over E indicate by (F, W) , also by form $F_w = \{(e, F_w(e)): e \in E, F_w(x) \in P(E)\}$, such that $F_w: W \rightarrow P(E)$ where $F_w(e) = \emptyset$ if $e \notin W$

Definition 1.2 [19] (Soft topology)

Let $S(E, W)$ be the collection of all soft sets, a soft topology on $S(E, W)$ written as τ^E to be the collection of soft subsets of (F, W) , when the following conditions hold:

- $(\tilde{\phi}, W), (\tilde{E}, W) \in \tau^E$;
- The intersection of any two elements of τ^E belongs to τ^E ;
- To a union of arbitrary elements of τ^E is an element of τ^E .

Also, the triplet (E, W, τ^E) is said to be a soft topological space.

Definition 1.3 [18] (Soft Homotopy of maps)

At this point, assume two softs continuous maps $f_\chi, g_\lambda: (E, W, \tau^E) \rightarrow (H, B, \tau^H)$ are called soft homotopic, written as $f_\chi \sim g_\lambda$, such that, there exists a soft continuous map $h_\pi: (E \times L, W \times W, \tau^E \times \tau^L) \rightarrow (H, B, \tau^H)$ where $\pi: W \times W \rightarrow B$ and $h: E \times L \rightarrow H$, if $(E, W, \tau^E), (H, B, \tau^H)$ be two soft topological spaces, such that:

$$h_\pi[(F, W) \times (\bar{0}, W)] = f_\chi[(F, W)],$$

and

$$h_\pi[(F, W) \times (\bar{1}, W)] = g_\lambda[(F, W)], \forall (F, W) \in S(E, W)$$

i.e. $h_\pi[(\tilde{n}, \bar{0})] = f_\chi(\tilde{n}), h_\pi[(\tilde{n}, \bar{1})] = g_\lambda(\tilde{n}), \forall \tilde{n} \in (L, W)$. Also, h_π between the maps is known as a soft homotopy f_χ and g_λ .

Definition 1.4 (Soft Fundamental Group) [17, 18]

Let (E, A, τ^E) be soft topological space and $\tilde{z}_0 \in (E, A, \tau^E)$, the set of soft paths homotopy classes of soft loops based at $\tilde{z}_0$ is called the soft fundamental group of (E, A, τ^E) relative to the base soft point $\tilde{z}_0$ if satisfies the following conditions:

- We want to have $[f_\chi] \cdot ([g_\lambda] \cdot [h_\pi]) = ([f_\chi] \cdot [g_\lambda]) \cdot [h_\pi]$ (Associative)
- $[f_\chi] \cdot [h_\pi] = [f_\chi]$ and $[f_\chi] \cdot [f_\chi^{-1}] = [h_\pi]$ so $[f_\chi^{-1}] = [f_\chi]^{-1}$

It is denoted by $\pi_1((E, A, \tau^E), \tilde{z}_0)$.

Definition 1.5 (Soft Isomorphism) [19, 20]

Let (E, A, τ^E) and (H, B, τ^H) be two soft topological spaces. A mapping $f_\chi: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ is called a soft homomorphism if it preserves the soft structure. If, moreover, f_χ is bijective, then f_χ is called a soft isomorphism. In this case we say that (E, A, τ^E) and (H, B, τ^H) are soft isomorphic, and we write:

$$(E, A, \tau^E) \cong (H, B, \tau^H)$$

Definition 1.6 (Soft Hausdorff Space) [19]

Let (E, A, τ^E) be a soft topological space. We say that (E, A, τ^E) is soft Hausdorff if for any two distinct soft points $\tilde{x}_a$ and $\tilde{y}_b$ in X (i.e., $x \neq y$ or $a \neq b$), there exist soft open sets $(F, A), (G, A) \in \tau^A$ such that:

$$\tilde{x}_a \in (F, A), \tilde{y}_b \in (G, A), (F, A) \cap (G, A) = \phi_A$$

Definition 1.7 (Soft Path Concatenation) [18]

Let $\tilde{z}_0, \tilde{z}_1, \tilde{z}_2$ be soft elements of $(\tilde{Z}, A)$ and $f_\chi \cdot g_\lambda: (I, A, \tau^I) \rightarrow (E, A, \tau^E)$ be two soft paths from $\tilde{z}_0$ to $\tilde{z}_1$ and from $\tilde{z}_1$ to $\tilde{z}_2$, respectively, such that $f_\chi(\bar{0}) = \tilde{z}_0$, $f_\chi(\bar{1}) = \tilde{z}_1 = g_\lambda(\bar{0})$, $g_\lambda(\bar{1}) = \tilde{z}_2$. Now, define a mapping $P_\Delta = f_\chi * g_\lambda: (I, A, \tau^I) \rightarrow (E, A, \tau^E)$ by:

$$P_\Delta(\tilde{z}) = \begin{cases} f_\chi(\bar{2}\tilde{z}), & \tilde{z} \in \left([0, \frac{1}{2}], A\right) \\ g_\lambda(\bar{2}\tilde{z} - \bar{1}), & \tilde{z} \in \left([\frac{1}{2}, 1], A\right) \\ \tilde{z}_1, & elsewhere \end{cases}$$

Definition 1.8 (Soft Fiber Space)

A soft fiber structure is a triple $((E, A, \tau^E), P_\sigma, (H, B, \tau^H))$ consisting of two soft topological spaces $(E, A, \tau^E), (H, B, \tau^H)$ and a soft continuous projection $P_\sigma: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$.

The relationship between the total space (E, A, τ^E) , the base space (H, B, τ^H) , and the projection map projection $P_\sigma: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ is illustrated in Figure (1). The diagram shows that for each element $y \in (H, B, \tau^H)$, the fiber $F = p^{-1}(y)$ represents the corresponding soft subset in the total space.

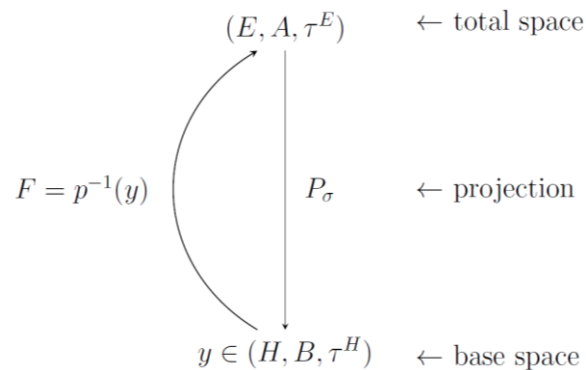


FIGURE 1. - Soft fiber structure

The space (E, A, τ^E) is called soft total [or soft fibered] space, P_σ is termed the projection, and (H, B, τ^H) is a soft base space for each $y \in (H, B, \tau^H)$. Then $F = p^{-1}(y)$ and F is called a soft fiber over y .

Example 1.1

Example of soft homotopy

Let $(E, A, \tau^E) = [\tilde{a}, \tilde{b}]$ and $(H, B, \tau^H)[0,1]$ and $f_\chi, g_\lambda: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ such that $f_\chi(\tilde{a}) = 0.2$, $f_\chi(\tilde{b}) = 0.8$ and $g_\lambda(\tilde{a}) = 0.4$, $g_\lambda(\tilde{b}) = 0.6$. Define $K_\theta: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (E, A, \tau^E)$ as follows:

$$K_\theta(\tilde{z}, t) = (1 - t)f_\chi(\tilde{z}) + tg_\lambda(\tilde{z})$$

$$K_{\theta}(\tilde{z}, 0) = f_{\chi}(\tilde{z}); t = 0$$

$$K_{\theta}(\tilde{z}, 1) = g_{\lambda}(\tilde{z}); t = 1$$

K_{θ} is soft homotopy.

2. SOFT RELATIVE HOMOTOPY

Definition 2.1 (Soft Relative Homotopic)

Let $(E, A, \tau^E), (H, B, \tau^H)$ be two soft topological space and let $f_{\chi}: (E, A, \tau^E) \rightarrow (H, B, \tau^H), g_{\lambda}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ be soft continuous map. Let (N, A, τ^N) be soft subset of (E, A, τ^E) such that the condition for every $\tilde{a} \in (N, A, \tau^N), f_{\chi}(\tilde{a}) = g_{\lambda}(\tilde{a})$ is satisfied. We say that f_{χ} and g_{λ} are soft relative homotopic of (N, A, τ^N) denoted by $f_{\chi} \simeq g_{\lambda}[(real(N, A, \tau^N))]$ if and only if there exist a (soft continuous) homotopy $K_{\theta}: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (H, B, \tau^H)$ between f_{χ} and g_{λ} such that:

- $K_{\theta}(\tilde{z}, 0) = f_{\chi}(\tilde{z})$ and $K_{\theta}(\tilde{z}, 1) = g_{\lambda}(\tilde{z})$ for all $\tilde{z} \in (E, A, \tau^E), t \in [0, 1]$
- $K_{\theta}(\tilde{a}, t) = f_{\chi}(\tilde{a}) = g_{\lambda}(\tilde{a})$ for all $\tilde{z} \in (N, A, \tau^N), t \in [0, 1]$

If $(N, A, \tau^N) = \emptyset$ we will simply say that f_{χ} soft homotopic to g_{λ} , and we will denote it by $f_{\chi} \simeq g_{\lambda}$.

Definition 2.2 (Null soft Homotopic)

Let (E, A, τ^E) and (H, B, τ^H) be two soft topological spaces and let $f_{\chi}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ be a soft continuous map. We say that f_{χ} is null soft homotopic if there exists $y \in (H, B, \tau^H)$ such that f_{χ} is soft homotopic to the soft constant map $C_y: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ that sends every $x \in (E, A, \tau^E)$ to y .

Definition 2.3 (Soft Contractible)

Let (E, A, τ^E) be soft topological space, we say that (E, A, τ^E) is soft contractible if (E, A, τ^E) be soft homotopic to some constant soft map (i.e. the identity soft map on (E, A, τ^E) is soft null homotopic).

Theorem 2.1

Every Soft relative homotopy in a soft topological space is a soft weak equivalence relation.

Proof

Let $(E, A, \tau^E), (H, B, \tau^H)$ and $(Z \times I, A \times A \times \tau^Z, \tau^I)$ be soft topological spaces, and $(N, A, \tau^N) \subseteq (E, A, \tau^E)$ be a soft subset. Consider the relation on the soft set $(R, A, \tau^R)[(E, A, \tau^E), (H, B, \tau^H)]$ if and only if g_{λ}, f_{χ} is soft homotopic relative to (N, A, τ^N) . To prove:

- **The relation is reflexive:**

Since for any soft function $f_{\chi}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$, define $F_{\Delta}: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (H, B, \tau^H)$ by $F_{\Delta}(\tilde{z}, t) = f_{\chi}(\tilde{z})$.

This yields a soft homotopy between f_{χ} and itself. Also, trivially, if $\tilde{z} \in (N, A, \tau^N)$ and $t \in [0, 1]$, then: $f_{\chi}(\tilde{z}) = K_{\theta}(\tilde{z}, t)$. So that K_{θ} is a soft relative homotopy to (N, A, τ^N) .

- **The relation is symmetric:**

For all $f_{\chi} \in R[(E, A, \tau^E), (H, B, \tau^H)]$, using the soft homotopy $K_{\theta}(\tilde{z}, t) = f_{\chi}(\tilde{z})$, for all $\tilde{z} \in (E \times I, A \times A \times \tau^E, \tau^I)$. On the other hand, if $f_{\chi} \simeq g_{\lambda}[(rel(N, A, \tau^N))]$ by means of the soft homotopy $K_{\theta}: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (H, B, \tau^H)$, then define $G_{\gamma}: (E, A, \tau^E) \times [0, 1] \rightarrow (H, B, \tau^H)$ by $G_{\gamma}(\tilde{z}, t) = K_{\theta}(\tilde{z}, 1 - t)$, for all $\tilde{z} \in (E, A, \tau^E)^t \in [0, 1]$, is a soft homotopy from g_{λ} to f_{χ} relative to (N, A, τ^N) , this is $g_{\lambda} \simeq f_{\chi}[(rel(N, A, \tau^N))]$.

- **The relation is transitive:**

Let $f_{\chi}, g_{\lambda}, h_{\pi} \in R[(E, A, \tau^E), (H, B, \tau^H)]$ satisfy $f_{\chi} \simeq g_{\lambda}[(rel(N, A, \tau^N))]$ then there exist a soft homotopy $F_{\Delta}: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (H, B, \tau^H)$ and since $g_{\lambda} = h_{\pi}[(rel(N, A, \tau^N))]$ then there exist a soft homotopy $G_{\gamma}: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (H, B, \tau^H)$, then define the soft map $K_{\theta}: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (H, B, \tau^H)$ by:

$$K_{\theta}(\tilde{z}, t) = \begin{cases} F_{\Delta}(2\tilde{z}, t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G_{\gamma}(\tilde{z}, 2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is $f_{\chi} \simeq h_{\pi}[rel(N, A, \tau^N)]$.

Therefore, by reflexivity, symmetry, and transitivity, the relation $\simeq$ defines a fuzzy equivalence relation on $R[(E, A, \tau^E), (H, B, \tau^H)]$.

Proposition 2.1

Let soft continuous maps $f_{\chi}, g_{\lambda}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ and $w_k, v_{\eta}: (H, B, \tau^H) \rightarrow (Z, A, \tau^Z)$ such that $f_{\chi} \simeq g_{\lambda}$ and $w_k \simeq v_{\eta}$ (relative $(f_{\chi}((N, A, \tau^N)))$), then $w_k \circ f_{\chi} \simeq v_{\eta} \circ g_{\lambda}$ (relative $((N, A, \tau^N))$).

Proof

Let $(E, A, \tau^E), (H, B, \tau^H)$ and (Z, A, τ^Z) be soft topological spaces, and $(N, A, \tau^N) \subseteq (E, A, \tau^E)$ is a soft subset. Let $K_{\theta}: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (H, B, \tau^H)$ be a soft homotopy from f_{χ} to g_{λ} relative to (N, A, τ^N) and $G_{\gamma}: (H \times I, B \times B \times \tau^H, \tau^I) \times [0, 1] \rightarrow (Z, A, \tau^Z)$ be a soft homotopy from w_k to v_{η} relative to $f_{\chi}[(N, A, \tau^N)]$. Define the soft map $K_{\theta}: \tau^H \rightarrow (Z \times I, A \times A \times \tau^Z, \tau^I)$ by $K_{\theta}(\tilde{z}, s) = G_{\gamma}[K_{\theta}(\tilde{z}, s)]$ for all $\tilde{z} \in (E, A, \tau^E)$ and $s \in [0, 1]$ this gives a soft homotopy from $w_k \circ f_{\chi} \simeq v_{\eta} \circ g_{\lambda}$ (relative $((N, A, \tau^N))$) as was to be shown.

3. SOFT WEAK EQUIVALENCE

Definition 3.1 (Soft Homotopy Equivalence)

Let (E, A, τ^E) and (H, B, τ^H) be two soft topological space. A soft continuous map $f_{\chi}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ is said a soft homotopy equivalence if and only if there exists a soft continuous map $g_{\lambda}: (H, B, \tau^H) \rightarrow (E, A, \tau^E)$ s.t.:

$$f_{\chi} \circ g_{\lambda} = id_{(H, B, \tau^H)}$$

and:

$$g_{\lambda} \circ f_{\chi} = id_{(E, A, \tau^E)}$$

The soft continuous map g_{λ} is called be a soft homotopy inverse to f_{χ} .

Example 3.1

Let (E, A, τ^E) and (H, B, τ^H) be two soft topological spaces where $(E, A, \tau^E): A \rightarrow P(E)$ and $(H, B, \tau^H): B \rightarrow P(H)$ are soft sets. Define $f_{\chi}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$, $f_{\chi}(\tilde{z}) = 0$, $g_{\lambda}: (H, B, \tau^H) \rightarrow (E, A, \tau^E)$, $g_{\lambda}(\tilde{u}) = 0$.

Then both f_{χ} and g_{λ} are soft continuous maps. We can define a soft homotopy between $g_{\lambda} \circ f_{\chi}$

and $id_{(H, B, \tau^H)}$ by $M(\tilde{z}, t) = (1-t)g_{\lambda}(\tilde{z})$, $K(\tilde{u}, t) = (1-t)f_{\chi}(\tilde{u})$. Hence

$$f_{\chi} \circ g_{\lambda} \simeq id_{(H, B, \tau^H)}$$

and:

$$g_{\lambda} \circ f_{\chi} \simeq id_{(E, A, \tau^E)}$$

Definition 3.2 (Soft Weak Homotopy Equivalence)

Let $f_{\chi}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ be soft continuous map of soft spaces is said a soft weak homotopy equivalence if it satisfies by soft isomorphism:

$$f_{\chi*}: \pi_k((E, A, \tau^E), \tilde{z}_0) \rightarrow \pi_k((H, B, \tau^H), f_{\chi}(\tilde{z}))$$

for all $k \geq 0$ and all soft base points $\tilde{z} \in (E, A, \tau^E)$.

Example 3.2

Let (E, A, τ^E) be soft topological space. Consider the soft product $(Z \times I, A \times A \times \tau^Z, \tau^I)$, Where $I = [0, 1]$.

Define the soft continuous maps $f_{\chi}: (Z \times I, A \times A \times \tau^Z, \tau^I) \rightarrow (E, A, \tau^E)$, $f_{\chi}(\tilde{z}, t) = \tilde{z}$,

$g_{\lambda}: (E, A, \tau^E) \rightarrow (Z \times I, A \times A \times \tau^Z, \tau^I)$, $g_{\lambda}(\tilde{z}) = (\tilde{z}, 0)$. We have $f_{\chi} \circ g_{\lambda} = id_{(E, A, \tau^E)}$.

A soft homotopy between $g_{\lambda} \circ f_{\chi}$ and $id_{(Z \times I, A \times A \times \tau^Z, \tau^I)}$ is given by $F((\tilde{z}, t), \tilde{u}) = (\tilde{z}, (1 - \tilde{u})t)$.

Therefore $g_{\lambda} \circ f_{\chi} \simeq id_{(Z \times I, A \times A \times \tau^Z, \tau^I)}$, Consequently on each soft homotopy group $\pi_k((E, A, \tau^E), \tilde{z}_0)$,

$$f_{\chi*}: \pi_k((Z \times I, A \times A \times \tau^Z, \tau^I), (\tilde{z}_0, 0)) \rightarrow \pi_k((E, A, \tau^E), \tilde{z}_0)$$

Proposition 3.1

Every soft homotopy equivalence in a soft topological space is a soft weak homotopy equivalence.

Proof

Let (E, A, τ^E) and (H, B, τ^H) be two soft topological space, suppose that $f_{\chi}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ is soft homotopy equivalence i.e. there exist a soft continuous map $g_{\lambda}: (H, B, \tau^H) \rightarrow (E, A, \tau^E)$ s.t. $f_{\chi} \circ g_{\lambda} = id_{(H, B, \tau^H)}$ and $g_{\lambda} \circ f_{\chi} = id_{(E, A, \tau^E)}$. In the presence of a soft homotopy with the fundament group of π_k , then for all $x \in (E, A, \tau^E)$, $y \in (H, B, \tau^H)$ and all $k \geq 0$, $f_{\chi*}: \pi_k((E, A, \tau^E), \tilde{z}_0) \rightarrow \pi_k((H, B, \tau^H), f_{\chi}(\tilde{z}))$ and $g_{\lambda*}: \pi_k((H, B, \tau^H), y_0) \rightarrow \pi_k((E, A, \tau^E), f(y))$, $f_{\chi*} \circ g_{\lambda*} = id$ and $g_{\lambda*} \circ f_{\chi*} = id$ so that $f_{\chi*}$ and $g_{\lambda*}$ are inverse soft isomorphisms, Thus a soft homotopy equivalence is a soft weak homotopy equivalence.

Conversely

The soft weak homotopy equivalence is not necessarily to be a soft homotopy equivalence as in the following example.

Example 3.3

Let $(E, A, \tau^E) = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ and $(H, B, \tau^H) = \mu_{S^1}$ is soft Hausdorff space then the soft map $f_{\chi}: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ be soft weak homotopy equivalence, but we can prove that it cannot be a soft homotopy equivalence, because $g_{\lambda}: \frac{1}{n} \rightarrow 0$.

Proposition 3.2

Let $F_b = P_{\sigma}^{-1}(\tilde{z})$ be soft fibers with $P_{\sigma}: (N, A, \tau^N) \rightarrow (M, B, \tau^M)$ be a soft fibration. Each soft path component of (M, B, τ^M) are all soft homotopy equivalent.

Proof

Let $f_{\chi}, g_{\lambda}: (I, A, \tau^I) \rightarrow (M, B, \tau^M)$ be two soft paths from $\tilde{z}_0$ to $\tilde{z}_1$ and from $\tilde{z}_1$ to $\tilde{z}_2$, respectively, such that $f_{\chi}(\bar{0}) = \tilde{z}_0$, $f_{\chi}(\bar{1}) = \tilde{z}_1 = g_{\lambda}(\bar{0})$, $g_{\lambda}(\bar{1}) = \tilde{z}_2$, to soft homotopy:

$$H(t): F[f_{\chi}(0)] \rightarrow (M, B, \tau^M), \quad s.t. H(t)(F[f_{\chi}(0)]) = f_{\chi}(0)$$

$$H(t): F[g_{\lambda}(0)] \rightarrow (M, B, \tau^M), \quad s.t. H(t)(F[g_{\lambda}(0)]) = f_{\chi}(0)$$

The inclusion $i_{\delta}: F[f_{\chi}(0)] \rightarrow (N, A, \tau^N)$, $i_{\delta}: F[g_{\lambda}(0)] \rightarrow (N, A, \tau^N)$ provides a soft lift $H_{\pi}^*(0): F[f_{\chi}(0)] \rightarrow (N, A, \tau^N)$, $F[g_{\lambda}(0)] \rightarrow (N, A, \tau^N)$. Since P_{σ} is soft fibration then P_{σ} has (So.H.L.P) with respect to all any soft space (Z, A, τ^Z) for pairs $[(Z \times I, A \times A \times \tau^Z \times \tau^I), (Z \times \{0, 1\}, A \times A \times \tau^Z, \tau^{\{0, 1\}})]$ since $(I \times I \times I \cup 0) \simeq (I \times I, I \times 0)$, then after taking products with (Z, A, τ^Z) .

We have a soft homotopy $H_\pi^*(t): F[f_\chi(0)] \rightarrow (N, A, \tau^N)$, $H_\pi^*(t): F[g_\lambda(0)] \rightarrow (N, A, \tau^N)$ such that $H_\pi^*(t): F[f_\chi(0)] \subset F[f_\chi(t)]$, $H_\pi^*(t): F[g_\lambda(0)] \subset F[g_\lambda(t)]$, for all $t \in \{0,1\}$, particularly, $H_\pi^*(t)$ gives a soft map. In this work, instead of using the notation L_α for the induced fuzzy (resp. soft) map associated with a path α , we shall use the symbols f' and g' to denote the induced maps corresponding to the soft paths f and g , respectively.

That is, $f'_\chi: F[f_\chi(0)] \rightarrow F[f_\chi(1)]$, $g'_\lambda: F[g_\lambda(0)] \rightarrow F[g_\lambda(1)]$. The association $f_\chi \mapsto f'_\chi \cdot g_\lambda$ must possess the following basic properties:

- 1) If $f_\chi \simeq g_\lambda \text{ rel } \partial I$, then $f'_\chi \simeq g'_\lambda$.
- 2) A composition of soft paths $(f_\chi * g_\lambda)' \simeq f'_\chi \circ g'_\lambda$.

To prove (1), let $f_\chi \simeq g_\lambda$ proof required $f'_\chi \simeq g'_\lambda$. Let $(t, s) \in I \times I$ such that $(f_\chi \cdot g_\lambda)(t, s)$ be a soft homotopy from $f_\chi(t)$ to $g_\lambda(s)$. This determines a soft homotopy $H(t, s): F[f_\chi(0)] \rightarrow (M, B, \tau^M)$, $H(t, s): F[g_\lambda(0)] \rightarrow (M, B, \tau^M)$ such that $H(r, s)[F[g_\lambda(0)]] = f_\chi(r, s)$, $H(r, s)[F[f_\chi(0)]] = g_\lambda(r, s)$, let $H_\pi^*(t, 0)$ is the inclusion $i_\delta: F[f_\chi(0)] \hookrightarrow (N, A, \tau^N)$, $i_\delta: F[g_\lambda(0)] \hookrightarrow (N, A, \tau^N)$, for all $t \in I$ and let $H_\pi^*(0, s)$ and $H_\pi^*(1, s)$ be soft lifts define by f'_χ and g'_λ . By using the soft homotopy lifting property for $(F[f_\chi(0)] \times I, F[f_\chi(0)] \times \partial I)$, $(F[g_\lambda(0)] \times I, F[g_\lambda(0)] \times \partial I)$, we can lengthen this soft lifts to soft lifts $H_\pi^*(t, s)$, for all $(t, s) \in I \times I$. When we take $s = 1$ we get a soft homotopy, let $f_\chi \simeq g_\lambda$ proof required $f'_\chi \simeq g'_\lambda$. Property (2) verified since $H_\pi^*(t)$ is soft lift for g'_λ and $H_\pi^*(t, 1)$ is soft lift for f'_χ then we get soft lift defined $f'_\chi \circ g'_\lambda$ by:

$$P_\Delta(\tilde{z}) = (f_\chi * g_\lambda)(\tilde{z}) = \begin{cases} f_\chi(\tilde{z}), & \tilde{z} \in \left([0, \frac{1}{2}], A\right) \\ g_\lambda(\tilde{z} - 1), & \tilde{z} \in \left([\frac{1}{2}, 1], A\right) \\ \tilde{z}_1, & \text{elsewhere} \end{cases}$$

From 1 and 2 we get f'_χ is a soft homotopy equivalence with soft homotopy inverse f_χ^{-1} .

Definition 3.3 (Soft Fiber-preserving)

Let $P_1\sigma: (N_1, A_1, \tau^N) \rightarrow (M, B, \tau^M)$ and $P_2\sigma: (N_2, A_2, \tau^N) \rightarrow (M, B, \tau^M)$ be two soft fibration. A soft map $g_\lambda: (N_1, A_1, \tau^N) \rightarrow (N_2, A_2, \tau^N)$ is said soft fiber-preserving if:

$$P_1\sigma = P_2\sigma \circ g_\lambda$$

Figure (2) illustrates the relationship between the two soft fibrations $P_1\sigma$ and $P_2\sigma$ and the soft map g_λ . The diagram shows that the composition $P_2\sigma \circ g_\lambda$ coincides with $P_1\sigma$, thereby characterizing the soft fiber-preserving property.

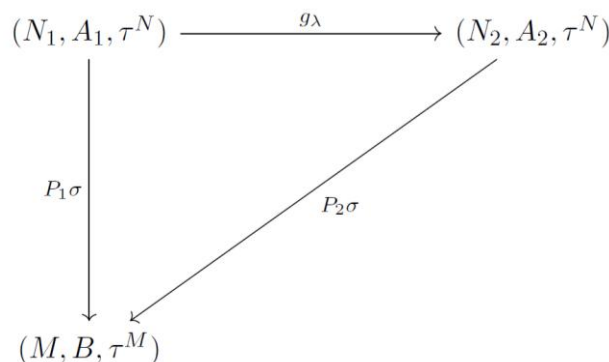


FIGURE 2. - The soft fiber-preserving

Proposition 3.3

The soft pullback fibrations is soft fiber homotopy equivalent.

Proof

Let $P_\sigma: (N, A, \tau^N) \rightarrow (M, B, \tau^M)$ be a soft fibration and let $g_\lambda(t): (Z, A, \tau^Z) \rightarrow (M, B, \tau^M)$ a soft homotopy. Suppose that $G_\phi: (Z \times I, A \times A, \tau^Z \times \tau^I) \rightarrow (M, B, \tau^M)$ be the soft homotopy $g_\lambda(t)$ and $Q_\phi: (S, B, \tau^S) \rightarrow (Z \times I, A \times A, \tau^Z \times \tau^I)$ be soft fibration such that Q_ϕ contains $q_{1\delta}: (S_1, B_1, \tau^S) \rightarrow (Z, A, \tau^Z)$ and $q_{2\delta}: (S_2, B_2, \tau^S) \rightarrow (Z, A, \tau^Z)$ over $(Z \times \{0\}, A \times A, \tau^Z \times \tau^{\{0\}})$ and $(Z \times \{1\}, A \times A, \tau^Z \times \tau^{\{1\}})$. So, it is enough to prove the restricted soft fibrations $F(t) = P_\sigma^{-1}[(M, B, \tau^M) \times \{t\}] \rightarrow (M, B, \tau^M)$ are all soft fiber homotopy equivalent, for all $t \in \{0,1\}$ for soft fibration $P_\sigma: (N, A, \tau^N) \rightarrow (M \times I, B \times B, \tau^M \times \tau^I)$. To prove this, building tradition of soft homotopy equivalences.

Let $f_\chi, g_\lambda: (I, A, \tau^I) \rightarrow (M, B, \tau^M)$ be soft path gives to a soft fiber preserving map $f'_\chi: F[f_\chi(0)] \rightarrow F[f_\chi(1)]$, $g'_\lambda: F[g_\lambda(0)] \rightarrow F[g_\lambda(1)]$, by soft lifting the soft homotopy $H_\pi(t): F[f_\chi(0)] \rightarrow (M \times I, B \times B, \tau^M \times \tau^I)$, $H_\pi(t): F[g_\lambda(0)] \rightarrow (M \times I, B \times B, \tau^M \times \tau^I)$.

$$H_\pi(\tilde{z}, t) = \begin{cases} (P_\sigma(\tilde{z}), f_\chi(2t)), & 0 \leq t \leq \frac{1}{2} \\ (P_\sigma(\tilde{z}), g_\lambda(2t - 1)), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$\forall \tilde{z} \in Z$ we start from the inclusion $i_\delta: [f_\chi(0)] \hookrightarrow (N, A, \tau^N)$, $i_\delta: [g_\lambda(0)] \hookrightarrow (N, A, \tau^N)$. As before, we explain two properties (1) and (2), with a note in (1) the soft homotopy $f'_\chi \simeq g'_\lambda$ is soft fiber-preserving since it is obtained by lifting a soft homotopy $g_\lambda(t): E[f_\chi(0)] \times \{0,1\} \rightarrow (M \times I, B \times B, \tau^M \times \tau^I)$, $g_\lambda(t): E[g_\lambda(0)] \times \{0,1\} \rightarrow (M \times I, B \times B, \tau^M \times \tau^I)$ of the form $g_t(\tilde{z}, u) = [P_\sigma(\tilde{z}), -] \forall \tilde{z} \in Z$.

From (1) and (2) we have f_χ is a soft fiber homotopy equivalence where f'_χ is inverse.

Corollary 3.1

If $P_\sigma: (N, A, \tau^N) \rightarrow (M, B, \tau^M)$ be soft fibration such that (M, B, τ^M) be a soft contractible base then P_σ is soft homotopy equivalent to a product soft fibration $(M, B, \tau^M) \times (E, A, \tau^E) \rightarrow (M, B, \tau^M)$.

Proof

Let $P_\sigma: (N, A, \tau^N) \rightarrow (M, B, \tau^M)$ be a soft fibration and let the identity soft map $id: (M, B, \tau^M) \rightarrow (M, B, \tau^M)$ is (N, A, τ^N) itself. The soft pullback of (N, A, τ^N) and since (M, B, τ^M) is soft contractible then a soft constant map $(M, B, \tau^M) \rightarrow (M, B, \tau^M)$ is a product $(M, B, \tau^M) \times (F, A, \tau^F)$.

Proposition 3.4

If $P_\sigma: (N, A, \tau^N) \rightarrow (M, B, \tau^M)$ be a soft fibration, then inclusion $(N, A, \tau^N) \hookrightarrow (N_p, A_p, \tau^N)$ is a soft homotopy equivalence.

Proof

Let $i_\delta: (N, A, \tau^N) \hookrightarrow (N_p, A_p, \tau^N)$ be inclusion. As $P_\sigma: (N, A, \tau^N) \rightarrow (M, B, \tau^M)$ be soft fibration then P_σ has soft homotopy lifting property with respect to all any soft space (E, A, τ^E) the soft homotopy $h_\pi(t): (N_p, A_p, \tau^N) \rightarrow (M, B, \tau^M)$ define by $h_t(e, \alpha) = \alpha(t)$, the soft lifting $h_\pi(t): (N_p, A_p, \tau^N) \rightarrow (N, A, \tau^N)$ define by $h_t^*(e, \alpha) = e$ such that the first coordinate of a soft homotopy $g_\lambda(t): (N_p, A_p, \tau^N) \rightarrow (N_p, A_p, \tau^N)$ while it is the second coordinate is the restricted by the soft paths α to the interval $[t, 1]$. $g_\lambda(t)$ is soft fiber-preserving since in the soft paths the end point has not changed. Saves us $g_\lambda(0) = id$, $g_\lambda(1)[(N_p, A_p, \tau^N)] \subset (N_p, A_p, \tau^N)$ and $g_\lambda(t)[(N_p, A_p, \tau^N)] \subset (N_p, A_p, \tau^N)$, for all $t \in [0,1]$. Then from $i_\delta \circ g_\lambda(1) \simeq id$ via $g_\lambda(t)$ and $g_\lambda(1) \circ i_\delta \simeq id$ via $g_\lambda(t)[(N_p, A_p, \tau^N)]$ we get i_δ is a soft homotopy equivalence.

5. RELATIONSHIP BETWEEN SOFT RELATIVE HOMOTOPY AND SOFT WEAK EQUIVALENCE

In this section, we discuss the relationship between the concepts of Soft Relative Homotopy and Soft Weak Equivalence.

The concept of Soft Relative Homotopy deals with the relation between two soft continuous maps $f_\chi: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$, $g_\lambda: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ within the same soft topological spaces.

We say that f_χ and g_λ are soft homotopic relative to a soft subset $(N, A, \tau^N) \subset (E, A, \tau^E)$, written as

$$f_\chi \simeq g_\lambda[(real(N, A, \tau^N))]$$

$K_\theta: (E \times I, A \times A \times \tau^E, \tau^I) \rightarrow (H, B, \tau^H)$ between f_χ and g_λ such that:

- $K_\theta(\tilde{z}, 0) = f_\chi(\tilde{z})$ and $K_\theta(\tilde{z}, 1) = g_\lambda(\tilde{z})$ for all $\tilde{z} \in (E, A, \tau^E)$, $t \in [0, 1]$
- $K_\theta(\tilde{a}, t) = f_\chi(\tilde{a}) = g_\lambda(\tilde{a})$ for all $\tilde{z} \in (N, A, \tau^N)$, $t \in [0, 1]$.

On the other hand, Soft Weak Equivalence extends this idea to a higher, more global level.

A soft continuous map $f_\chi: (E, A, \tau^E) \rightarrow (H, B, \tau^H)$ is called a soft weak homotopy equivalence if it induces isomorphisms on all soft homotopy groups:

$$f_{\chi*}: \pi_k((E, A, \tau^E), \tilde{z}_0) \rightarrow \pi_k((H, B, \tau^H), f_\chi(\tilde{z}))$$

for all $k \geq 0$ and all soft base points $\tilde{z} \in (E, A, \tau^E)$.

Hence, while Soft Relative Homotopy focuses on local deformations between maps within the same space, Soft Weak Equivalence concerns the global structural similarity between two soft spaces from a homotopical viewpoint. Every soft homotopy equivalence (defined via the relative homotopy relation) is necessarily a soft weak homotopy equivalence, but the converse does not always hold.

In summary,

Soft Relative Homotopy $\Rightarrow$ Soft Homotopy Equivalence $\Rightarrow$ Soft Weak Equivalence.

This connection shows that the theory of soft weak equivalence is built upon the framework of soft (relative) homotopy, providing a bridge between local and global homotopical properties of soft topological spaces."

6. CONCLUSION

In this paper, we developed and analyzed the concept of soft relative homotopy and introduced new formulations for soft fiber spaces, soft fiber-preserving maps, and soft weak homotopy equivalence. These constructions refine the classical homotopy framework within the context of soft topological spaces, providing a more rigorous foundation for studying deformations and equivalences among soft mappings. The results demonstrate that soft weak equivalence preserves soft homotopy classes, thus offering a flexible and generalized tool for identifying structural similarities between soft spaces. From a theoretical perspective, the introduced notions serve as a fundamental basis for advancing the theory of soft fibrations, soft lifting properties, and the categorical structures underlying soft topology. Future research may focus on developing soft homology and related algebraic invariants, extending soft weak equivalence to categorical and functorial frameworks, or exploring its potential applications in fuzzy systems, uncertain topological models, and data analysis. Such extensions could strengthen the connections between soft topology and other mathematical and applied disciplines, opening new avenues for both theoretical and practical exploration.

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