Generalize of some concepts on Linear Random Dynamical Systems

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Abstract. This work aims to study the concepts of proximal, distal, minimal, distil point, regionally proximal, equicontinuous and uniformly equicontinuous in the linear random dynamical systems with the study of some their properties.

Keywords. Liner Random Dynamical Systems, distal point of liner random dynamical systems, minimal of liner random dynamical systems, proximal of liner random dynamical systems.

1.Introduction.

One of the important concepts in dynamical systems is the study of concepts such as proximal, distal, minimal, regionally proximal, and uniformly equicontinuous. In 1970, W. A. Veech, [9] introduce definition of a point-distal flows. In 1976, E. Glasner [5] provided a definition of proximal flows. [1] J. Auslander discusses the concept of "minimal flows and their extensions in 1988.In 1991, A. Ludwig,[7] introduce “Random Dynamical Systems" and in 1998, A. Ludwig and others [8] discusses the Order-Preserving Random Dynamical Systems: Equilibria, attractor, applications, Dynamics and Stability of System.[6] In 2002, C. Igor, present a new study on "Monotone Random Systems Theory and Applications". [3] In 2009, J. Auslander and N. Markley submitted the almost periodic minimal flows". And many researchers see, for example, [2] and [4].

In our work, we will generalize these concepts from deterministic dynamical systems to randomness. In particular, we will discuss the concepts of minimality, distal, point-distal, and proximal in linear random dynamical systems where the phase space is considered a Banach space.

The classification of sets is one of the goals of random dynamics. We will be concerned with three types of minimal sets: equicontinuous, distal, and point-distal. Point-distal is necessarily minimal [1]. That is, "if is point-distal, then it contains no proper" a closed and invariant subset.

2. Some concepts on RDSs:

Here, we collected definitions and notions from the theory of RDSs that we needing our work, (see L. Arnold [7,8] and Igore [6]).

Definition 2.1 [6]:

The $(\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is said to be **metric dynamical system** (MDS) if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and the function $\theta: \mathbb{T} \times \Omega \to \Omega$ satisfy

(i) θ is measurable,

(ii) $\theta(0, \omega) = \omega$, for every $\omega \in \Omega$,

(iii) $\theta_{t+s}(\omega) = (\theta_t \circ \theta_s)(\omega)$ for every $\omega \in \Omega$, $t, s \in \mathbb{T}$ and

(iv) $\mathbb{P}(\theta_t F) = \mathbb{P}(F)$, $\forall\, F \in \mathcal{F}$ and $\forall\, t \in \mathbb{T}$.

Definition 2.2 [7]:

Let X be a topological space and θ be a MDS. A topological **random dynamical system** on X over θ is a function $\varphi: \mathbb{T} \times \Omega \times X \to X$, admit the following properties:

(i) $\varphi(\cdot, \omega, \cdot): \mathbb{T} \times X \to X$ is continuous for every $\omega \in \Omega$.

(ii) The mapping $\varphi(t, \omega) \coloneqq \varphi(t, \omega, \cdot): X \to X$ form a cocycle on $\theta(\cdot)$, that is satisfy

$$\varphi(0, \omega)x = x,\ \forall\, \omega \in \Omega \text{ and for evrey } x \in X,$$

$$\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)\ ,\forall\ s, t \in \mathbb{T}, \omega \in \Omega.$$

Definition 2.3 [7] (Linear RDSs):

A linear random dynamical system LRDS is an RDS (θ, φ) on the Banach space X such that $\varphi(t, \omega)$ is linear operators of X, $\forall\, \omega \in \Omega$, $t \in \mathbb{T}$.

Definition 2.4 [6]:

Suppose that (X, d) be a metric space which is a measurable space with Borel $\sigma-$field $\mathcal{B}(X)$ and $(\Omega, \mathcal{F})$ be a measurable space. The set-valued function $A: \Omega \to \mathcal{B}(X)$, $\omega \longmapsto A(\omega)$, is a **random set** if the mapping $\omega \longmapsto d(x, A(\omega))$ is measurable for each $x \in$ X. The random set $A(\omega)$ is called a ***random closed(compact) set***, if it is closed (compact) for all $\omega \in \Omega$.

Definition 2.5 (Equivalence of RDS) [7, 8]: Let(θ, φ_1) and (θ, φ_2) be two RDS over the same MDS θ with phase spaces X_1 and X_2 resp. These RDSs are said to be (topologically) equivalent (or conjugate) if there exists a mapping $T: \Omega \times X_1 \longrightarrow X_2$ with the properties:

(i) The mapping $x \longmapsto T(\omega, x)$ is a homeomorphism from X_1 onto X_2, $\forall\, \omega \in \Omega$;

(ii) The mappings $\omega \longmapsto T(\omega, x_1)$ and $\omega \longmapsto T^{-1}(\omega, x_2)$ are measurable, $\forall\, x_1 \in X_1$ and $x_2 \in X_2$;

(iii) The cocycles φ_1 and φ_2 are cohomologous, i.e.

$$\varphi_2\big(t, \omega, T(\omega, x)\big) = T(\theta_t\, \omega, \varphi_1(t, \omega, x)) \text{ for any } x \in X_1 .$$

(iv) $T_\omega: X_1 \longrightarrow X_2$ is linear map, $\forall\, \omega \in \Omega$.

Definition 2.6 [7, 8]:

Let $(\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a MDS. A random variable $\delta: \Omega \longrightarrow \mathbb{R}^+$ is said to be tempered random variable (t.r.v.), if there exists a full measurable subset $\widetilde{\Omega}$ of Ω such that $\lim\limits_{n \to \pm\infty} \frac{1}{|n|} \log^+ \delta(\theta_n \omega) = 0, \forall\, \omega \in \widetilde{\Omega}$, where $\log^+ \coloneqq \max\{0, \log\}$.

Definition 2.7:

Let (φ, θ) be an LRDS, and $\gamma_x^t: \omega \to D(\omega)$ be a multifunction defined by $\gamma_x^t(\omega) = \cup \{\varphi(t, \theta_{-t}\omega)x: t \in T\}$ the set γ_x^t is called a ***trajectory*** .

3. Proximal of RDSs.

In this section, the concepts of proximal, distal, minimal, regionally proximal, and uniformly equicontinuous in the LRDSs are studied and mentioning some of their characteristics.

Definition 3.1 : Let (θ, φ) be an LRDS. The pair of random variables $x, y \in X^\Omega$ is called ***proximal*** if there exists a divergent net $\{t_\lambda\}$ in G and a full measure invariant subset $\widetilde{\Omega}$ of Ω such that, $\forall\, \omega \in \widetilde{\Omega}$ we have

$$\lim_{\lambda} \left\| \varphi(t_\lambda, \theta_{-t_\lambda}\omega) x(\theta_{-t_\lambda}\omega) - \varphi(t_\lambda, \theta_{-t_\lambda}\omega) y(\theta_{-t_\lambda}\omega) \right\| = 0,$$

distal otherwise. A random variable x is called **distal** if $x, y \in X^\Omega$ is proximal only for $x = y$.

Definition 3.2: An LRDS (θ, φ) is said to be ***proximal*** if every pairs of random variables are proximal, ***distal*** if all pairs of random variables are distal.

The set of all pairs of random variable proximal denoted by

$$\mathcal{P}(X) \coloneqq \left\{(x, y) \in X^\Omega \times X^\Omega : \exists net\ \{t_\lambda\}\ in\ \mathbb{R}: \lim_{\lambda} \left\| \varphi(t_\lambda, \theta_{-t_\lambda}\omega) x(\theta_{-t_\lambda}\omega) - \varphi(t_\lambda, \theta_{-t_\lambda}\omega) y(\theta_{-t_\lambda}\omega) \right\| = 0\right\}$$

In particular, if the LRDS is proximal and distal , then $X^{\Omega} \times X^{\Omega} = \Delta_x$, $\Delta = \{(x, x): x \in X^{\Omega}\}$, hence (φ, θ) is the trivial on a one-point space.

Example 3.3 Let (θ, φ) be an LRDS defined as follows: $X = T = \mathbb{R}$ and $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space with $\Omega = [0,1]$, $\mathcal{B}$ be a Borel σ −algebra and $\mathbb{P}$ is a Lebsegue measure. Define $\varphi: \mathbb{R} \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, by $\varphi(t, \omega, x) = tx$ and $\theta: \mathbb{R} \times \Omega \longrightarrow \Omega$ by $\theta(t, \omega) = \omega$ .Then (θ, φ) is proximal.

Lemma 3.4: The LRDS (θ, φ) is distal if and only if $P(X) = \Delta$.

Proof. Let(θ, φ) be distal then all $x, y \in X^{\Omega}$ are distal, thus

$\lim_{\lambda} \|\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x(\theta_{-t_\lambda}\omega) - \varphi(t_\lambda, \theta_{-t_\lambda}\omega)y(\theta_{-t_\lambda}\omega)\| > 0$, (i.e. the is no $x \neq y \in X^{\Omega}$

Such that

$\lim_{\lambda} \|\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x(\theta_{-t_\lambda}\omega) - \varphi(t_\lambda, \theta_{-t_\lambda}\omega)y(\theta_{-t_\lambda}\omega)\| = 0$, then $x = y$, thus $P(X)$contain only Δ

Now

Let $P(X) = \Delta$,therefor for all $x \neq y$ then $\lim_{\lambda} \|\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x(\theta_{-t_\lambda}\omega) - \varphi(t_\lambda, \theta_{-t_\lambda}\omega)y(\theta_{-t_\lambda}\omega)\| \neq 0$

Thus, $\lim_{\lambda} \|\varphi(t_\lambda, \theta_{-t_\lambda}\omega)x(\theta_{-t_\lambda}\omega) - \varphi(t_\lambda, \theta_{-t_\lambda}\omega)y(\theta_{-t_\lambda}\omega)\| > 0$ for all $x, y \in X^{\Omega}$ then all pairs of random variables are distal.

Definition3.5 : Let (θ, φ) be a LRDS . Then

(a) $x \in X^{\Omega}$ is called a ***distal point*** of (θ, φ) if there exists no point other than itself in $\overline{\gamma_x^t(\omega)}$ to be proximal to it under (θ, φ).

(b) (θ, φ) is called **point-distal** if there exists a random variable $x \in X^{\Omega}$ such that

(1) x is distal, and

(2) $\overline{\gamma_x^t(\omega)} = X$.

Lemma3.6: If (θ, φ_1) and (θ, φ_2) be two LRDS's then so is $(\theta, \varphi_1 \times \varphi_2)$, where

$$\varphi_1 \times \varphi_2(t, \omega, (x, y)) = (\varphi_1(t, \omega, x), \varphi_2(t, \omega, y))$$

Proof

Put $\varphi := \varphi_1 \times \varphi_2: \mathbb{R} \times \Omega \times (X_1 \times X_2) \rightarrow ((X_1 \times X_2)$

Defined by

$\varphi(t, \omega)(x, y)) := \varphi_1 \times \varphi_2(t, \omega, (x, y)) = (\varphi_1 (t, \omega)x, \varphi_2(t, \omega)y)$, $\forall\, x \in X_1$ and $y \in X_2$

i) $\varphi(0, \omega)(x, y)) = (\varphi_1 (0, \omega)x, \varphi_2(0, \omega)y)$

$$= (x, y)$$

ii) $\varphi\big(t,\omega)(x,y)\big)$ is a measurable

iii) $\varphi\big(t+s,\omega)(x,y)\big) = (\varphi_1\,(t+s,\omega)x, \varphi_2(t+s,\omega)y)$

$$= (\varphi_1\,(t,\theta_s\omega) \circ (\varphi_1\,(s,\omega)x, \varphi_2(t,\theta_s\omega) \circ (\varphi_2(s,\omega)y)$$

$$= \varphi(t,\theta_s\omega)(\varphi_1\,(s,\omega)x, \varphi_2(s,\omega)y)$$

$$= \varphi(t,\theta_s\omega)(\varphi(s,\omega)(x,y))$$

Theorem 3.7: Let $(\theta,\varphi_1) \cong_T (\theta,\varphi_2)$ with T is linear. If (θ,φ_1) is proximal LRDS, then so is (θ,φ_2).

Proof: Assume that (θ,φ_1) is proximal LRDS. Let $x_1^2, x_2^2 \in X_2^\Omega$. Then there exists $x_1^1, x_2^1 \in X_1^\Omega$ such that $x_1^1 \coloneqq T^{-1}(x_1^2)$ and $x_2^1 \coloneqq \mathrm{T}^{-1}(x_2^2)$. By hypothesis there exists a divergent net $\{t_\lambda\}$ in G and $\mathbb{P}\big\{\omega: x_j^i(\omega) \neq y_j^i(\omega), i,j = 1{,}2\big\} = 1$ and a full measure invariant subset $\widehat{\Omega}$ of Ω such that, $\forall\ \omega \in \widehat{\Omega}$ we have

$$\lim_\lambda \big\|\varphi_1\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_1^1\big(\theta_{-t_\lambda}\omega\big) - \varphi_1\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_2^1(\theta_{-t_\lambda}\omega)\big\| = 0.$$

Now,

$$\lim_\lambda \big\|\varphi_2\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_1^2\big(\theta_{-t_\lambda}\omega\big) - \varphi_1\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_2^2(\theta_{-t_\lambda}\omega)\big\|.$$

$$= \lim_\lambda \big\|\varphi_2\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_1^2\big(\theta_{-t_\lambda}\omega\big) - \varphi_1\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_2^2(\theta_{-t_\lambda}\omega)\big\|$$

$$= \lim_\lambda \Big\|\varphi_2\big(t_\lambda,\theta_{-t_\lambda}\omega\big)T_{\theta_{-t_\lambda}\omega}\Big(x_1^1\big(\theta_{-t_\lambda}\omega\big)\Big) - \varphi_1\big(t_\lambda,\theta_{-t_\lambda}\omega\big)T_{\theta_{-t_\lambda}\omega}(x_2^1\big(\theta_{-t_\lambda}\omega\big))\Big\|$$

$$= \lim_\lambda \big\|T_\omega\varphi_2\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_1^1\big(\theta_{-t_\lambda}\omega\big) - T_\omega\varphi_1\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_2^1\big(\theta_{-t_\lambda}\omega\big)\big\|$$

$$= \lim_\lambda \Big\|T_\omega\Big(\varphi_2\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_1^1\big(\theta_{-t_\lambda}\omega\big) - \varphi_1\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_2^1\big(\theta_{-t_\lambda}\omega\big)\Big)\Big\|$$

$$\leq c \lim_{t_\lambda \to +\infty} \big\|\varphi_2\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_1^1\big(\theta_{-t_\lambda}\omega\big) - \varphi_1\big(t_\lambda,\theta_{-t_\lambda}\omega\big)x_2^1\big(\theta_{-t_\lambda}\omega\big)\big\| = 0$$

where c be constant then (θ,φ_2) is proximal LRDS. ■

Theorem 3.8: Let (θ,φ_1) and (θ,φ_2) be two RDS's. Then$(\theta,\varphi_1)$and (θ,φ_2) are proximal if and only if $(\theta,\varphi_1 \times \varphi_2)$ is proximal.

Proof:

Assume that (θ,φ_1) and (θ,φ_2) are proximal,

Let $z_1, z_2 \in (X^\Omega \times Y^\Omega)$. Then $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.
we will use the fact $(X^\Omega \times Y^\Omega) \times (X^\Omega \times Y^\Omega) \cong (X^\Omega \times X^\Omega) \times (Y^\Omega \times Y^\Omega)$,and

$\varphi \coloneqq \varphi_1 \times \varphi_2 \left(t, \omega, \left(x\left(\theta_{-t_\lambda}\omega\right), y\left(\theta_{-t_\lambda}\omega\right)\right)\right) = (\varphi_1 \left(t, \omega, x\left(\theta_{-t_\lambda}\omega\right)\right), \varphi_2 \left(t, \omega, y\left(\theta_{-t_\lambda}\omega\right)\right))$

Then $(x_1, x_2) \in X^\Omega \times X^\Omega$, $(y_1, y_2) \in Y^\Omega \times Y^\Omega$. By hypothesis there exists a divergent net $\{t_\lambda\}$ in G and $\mathbb{P}\{\omega: z_1(\omega) \neq z_2(\omega)\} = 1$, and a full measure invariant subset $\widetilde{\Omega}$ of Ω such that, $\forall\ \omega \in \widetilde{\Omega}$ we get,

$$\lim_\lambda \left\|\varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_1\left(\theta_{-t_\lambda}\omega\right) - \varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_1\left(\theta_{-t_\lambda}\omega\right)\right\| = 0. \quad \text{And}$$

$$\lim_\lambda \left\|\varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_2\left(\theta_{-t_\lambda}\omega\right) - \varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_2\left(\theta_{-t_\lambda}\omega\right)\right\| = 0$$

Then

$$\lim_\lambda \left\|\varphi\left(t_\lambda, \theta_{-t_\lambda}\omega\right)z_1\left(\theta_{-t_\lambda}\omega\right) - \varphi\left(t_\lambda, \theta_{-t_\lambda}\omega\right)z_2\left(\theta_{-t_\lambda}\omega\right)\right\|$$

$$= \lim_\lambda \left\|\varphi_1 \times \varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)(x_1\left(\theta_{-t_\lambda}\omega\right), y_1\left(\theta_{-t_\lambda}\omega\right)) - \varphi_1 \times \varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)(x_2\left(\theta_{-t_\lambda}\omega\right), y_2\left(\theta_{-t_\lambda}\omega\right))\right\|$$

$$= \lim_\lambda \left\|\varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_1\left(\theta_{-t_\lambda}\omega\right) - \varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_1\left(\theta_{-t_\lambda}\omega\right)\right\|_1$$

$$+ \lim_\lambda \left\|\varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_2\left(\theta_{-t_\lambda}\omega\right) - \varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_2\left(\theta_{-t_\lambda}\omega\right)\right\|_2 = 0\ .$$

Thus $(\theta, \varphi_1 \times \varphi_2)$ is proximal.

On the other hand, suppose that $(\theta, \varphi_1 \times \varphi_2)$ is proximal. Let $z_1, z_2 \in X^\Omega \times Y^\Omega$. Such that $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. By hypothesis there exist a divergent net $\{t_\lambda\}$ in G and a full measure invariant subset $\widetilde{\Omega}$ of Ω such that, $\forall\ \omega \in \widetilde{\Omega}$ we get,

$$\lim_\lambda \left\|\varphi\left(t_\lambda, \theta_{-t_\lambda}\omega\right)z_1\left(\theta_{-t_\lambda}\omega\right) - \varphi\left(t_\lambda, \theta_{-t_\lambda}\omega\right)z_2\left(\theta_{-t_\lambda}\omega\right)\right\| = 0.$$

Since

$$\lim_\lambda \left\|\varphi\left(t_\lambda, \theta_{-t_\lambda}\omega\right)z_1\left(\theta_{-t_\lambda}\omega\right) - \varphi\left(t_\lambda, \theta_{-t_\lambda}\omega\right)z_2\left(\theta_{-t_\lambda}\omega\right)\right\| = 0$$

$$= \lim_\lambda \left\|\varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_1\left(\theta_{-t_\lambda}\omega\right), \varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_1\left(\theta_{-t_\lambda}\omega\right)\right] - [\varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_2\left(\theta_{-t_\lambda}\omega\right), \varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_2\left(\theta_{-t_\lambda}\omega\right)$$

$$= \lim_\lambda \left\|\varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_1\left(\theta_{-t_\lambda}\omega\right) - \varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_2\left(\theta_{-t_\lambda}\omega\right)\right\|_1$$

$$+ \lim_\lambda \| \varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_1\left(\theta_{-t_\lambda}\omega\right) - \varphi_2\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_2\left(\theta_{-t_\lambda}\omega\right) \|_2 = 0$$

Since $\left\|\varphi_i\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_i, \varphi_i\left(t_\lambda, \theta_{-t_\lambda}\omega\right)y_i\right\|_i \geq 0$, for $i = 1,2$, then

$\lim_\lambda \left\|\varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_1 - \varphi_1\left(t_\lambda, \theta_{-t_\lambda}\omega\right)x_2\right\|_1 = 0$, and

$$\lim_{\lambda}\left\|\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)y_1-\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)y_2\right\|_2=0$$

This means that (θ,φ_1) and (θ,φ_2) are proximal. ■

Theorem 3.9:

Let (θ,φ_1) and (θ,φ_2)be two RDS's. Then $(\theta,\varphi_1\times\varphi_2)$ is distal if and only if either (θ,φ_1) or (θ,φ_2) is distal.

Proof:

Suppose that either (θ,φ_1) or (θ,φ_2) is distal. Let $z_1,z_2\in(X^\Omega\times Y^\Omega)$. Then $z_1=(x_1,x_2)$ and $z_2=(y_1,y_2)$.

We will use the fact $(X^\Omega\times Y^\Omega)\times(X^\Omega\times Y^\Omega)\cong(X^\Omega\times X^\Omega)\times(Y^\Omega\times Y^\Omega)$, and

$$\varphi\coloneqq\varphi_1\times\varphi_2\left(t,\omega,\left(x(\theta_{-t_\lambda}\omega),y(\theta_{-t_\lambda}\omega)\right)\right)=(\varphi_1\left(t,\omega,x(\theta_{-t_\lambda}\omega)\right),\varphi_2\left(t,\omega,y(\theta_{-t_\lambda}\omega)\right))$$

Then $(x_1,x_2)\in X^\Omega\times X^\Omega$, $(y_1,y_2)\in Y^\Omega\times Y^\Omega$. By hypothesis there exists a divergent net $\{t_\lambda\}$ in G and a full measure invariant subset $\widetilde{\Omega}$ of Ω such that, $\forall\,\omega\in\widetilde{\Omega}$ and $\mathbb{P}\{\omega: z_1(\omega)\neq z_2(\omega)\}=1$ we have

$$\lim_{\lambda}\left\|\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)x_1(\theta_{-t_\lambda}\omega)-\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)x_2(\theta_{-t_\lambda}\omega)\right\|>0\ ,$$

or

$$\lim_{\lambda}\left\|\varphi_2(t_\lambda,\theta_{-t_\lambda}\omega)y_1(\theta_{-t_\lambda}\omega)-\varphi_2(t_\lambda,\theta_{-t_\lambda}\omega)y_2(\theta_{-t_\lambda}\omega)\right\|>0.$$

Then

$$\lim_{\lambda}\left\|\varphi(t_\lambda,\theta_{-t_\lambda}\omega)z_1-\varphi(t_\lambda,\theta_{-t_\lambda}\omega)z_2\right\|$$
$$=\lim_{\lambda}\left\|\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)x_1(\theta_{-t_\lambda}\omega)-\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)x_2(\theta_{-t_\lambda}\omega)\right\|_1$$
$$+\lim_{\lambda}\left\|\varphi_2(t_\lambda,\theta_{-t_\lambda}\omega)y_1(\theta_{-t_\lambda}\omega)-\varphi_2(t_\lambda,\theta_{-t_\lambda}\omega)y_2(\theta_{-t_\lambda}\omega)\right\|_2>0\ .$$

Thus $(\theta,\varphi_1\times\varphi_2)$ is distal.

Conversely, suppose that $(\theta,\varphi_1\times\varphi_2)$ is distal. Let $z_1,z_2\in X^\Omega\times Y^\Omega$, $z_1=(x_1,y_1)$ $z_2=(x_2,y_2)$. By hypothesis there exist a divergent net $\{t_\lambda\}$ in G and a full measure subset $\widetilde{\Omega}$ of Ω such that

$$\lim_{\lambda}\left\|\varphi(t_\lambda,\theta_{-t_\lambda}\omega)z_1-\varphi(t_\lambda,\theta_{-t_\lambda}\omega)z_2\right\|>0.$$

Since

$$\lim_{\lambda}\left\|\varphi(t_\lambda,\theta_{-t_\lambda}\omega)z_1-\varphi(t_\lambda,\theta_{-t_\lambda}\omega)z_2\right\|$$
$$=\lim_{\lambda}\left\|\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)x_1(\theta_{-t_\lambda}\omega)-\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)x_2(\theta_{-t_\lambda}\omega)\right\|_1$$
$$+\lim_{\lambda}\left\|\varphi_2(t_\lambda,\theta_{-t_\lambda}\omega)y_1(\theta_{-t_\lambda}\omega)-\varphi_2(t_\lambda,\theta_{-t_\lambda}\omega)y_2(\theta_{-t_\lambda}\omega)\right\|_2>0$$

Then either

$$\lim_{\lambda}\left\|\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)x_1(\theta_{-t_\lambda}\omega)-\varphi_1(t_\lambda,\theta_{-t_\lambda}\omega)x_2(\theta_{-t_\lambda}\omega)\right\|_1>0,$$

or

$$\lim_{\lambda}\left\|\varphi_2(t_\lambda,\theta_{-t_\lambda}\omega)y_1(\theta_{-t_\lambda}\omega)-\varphi_2(t_\lambda,\theta_{-t_\lambda}\omega)y_2(\theta_{-t_\lambda}\omega)\right\|_2>0$$

This means that either (θ,φ_1) or (θ,φ_2) is distal. ■

<u>Proposition 3.10:</u> If (θ,φ) is a point − distal with $x\in X^{\Omega}$ a distal point, then each of $\gamma_x^t(\omega)$ is a distal point.

<u>Proof:</u> since $\gamma_y^t(\omega)\subset\overline{\gamma_x^t(\omega)}$, then for all $y\in\gamma_x^t(\omega)$ imply that $y\in\overline{\gamma_x^t(\omega)}$, since y is a distal point (i.e. there exists no point other than itself in $\overline{\gamma_x^t(\omega)}$ to be proximal to it under (θ,φ)) there for each x of $\gamma_x^t(\omega)$ is a distal point. ■

<u>Definition 3.11:</u>

If (θ,φ) an RDS. The pair $(x,y)\in X^{\Omega}\times X^{\Omega}$ is called ***regionally proximal*** if there exist $x_n\to x$, $y_n\to y$, and a full measure invariant subset $\tilde{\Omega}$ of Ω such that, $\forall\ \omega\in\tilde{\Omega}$ and $\mathbb{P}\{\omega: x(\omega)\neq y(\omega)\}=1$ we obtain

$$\lim_{n\to+\infty}\left\|\left(\varphi(t_n,\theta_{-t_n}\omega)x_n(\theta_{-t_n}\omega)-\varphi(t_n,\theta_{-n}\omega)y_n(\theta_{-t_n}\omega)\right)\right\|=0,$$

Otherwise, is called ***regionally distal***.

<u>Definition 3.12:</u>

An RDS (θ,φ) is called ***regionally proximal*** if all pairs of points are regionally proximal, and It is called ***regionally distal*** if all pairs of points are regionally distal.

<u>Proposition 3.13:</u>

Every proximal LRDS is regionally proximal.

Proof:

Assume (θ,φ) is proximal LRDS and Let $(x,y)\in X\times X$.

Define a sequences $\{x_n\}$ and $\{y_n\}$ in X by $x_n=x$ and $y_n=y$, then $x_n\to x$ and $y_n\to y$. Now

$$\lim_{n\to+\infty}\left\|(\varphi(n,\theta_{-n}\omega)x_n-\varphi(n,\theta_{-n}\omega)y_n)\right\|.$$
$$=\lim_{n\to+\infty}\left\|(\varphi(n,\theta_{-n}\omega)x-\varphi(n,\theta_{-n}\omega)y)\right\|=0$$

That is (θ,φ) is regionally proximal. ■

Not: By proposition (3.12) thus example (3.3) satisfies the definition (3.11).

<u>Proposition 3.14 :</u> Put $(\theta,\varphi_1)\cong_T(\theta,\varphi_2)$ with T is linear. If (θ,φ_1) is regionally proximal RDS, then (θ,φ_2) also.

Proof: Assume that (θ,φ_1) is regionally proximal LRDS and let $y_1,y_2\in X_2$. Then there exists $x_1,x_2\in X_1$ such that $x_1\coloneqq T_\omega^{-1}(y_1)$ and $x_2\coloneqq T_\omega^{-1}(y_2)$. By hypothesis there exists $x_n^1\longrightarrow x_1$, $x_n^2\longrightarrow x_2$ and a full measure invariant subset $\widehat{\Omega}$ of Ω, $\forall\ \omega\in\widehat{\Omega}$ and $\mathbb{P}\{\omega: x_i(\omega)\neq y_i(\omega), i=1,2\}=1$ such that

$$\lim_{n\to+\infty}\|\varphi_1(n,\theta_{-n}\omega)x_n^1-\varphi_1(n,\theta_{-n}\omega)x_n^2\|=0.$$

Set $y_n^1=T_\omega(x_n^1)$ and $y_n^2=T_\omega(x_n^2)$. Since $x_n^1\longrightarrow x_1$ and $x_n^2\longrightarrow x_2$ and T_ω is continuous foe every $\omega\in\Omega$, then $y_n^1\longrightarrow y_1$, $y_n^2\longrightarrow y_2$

Now,

$$\lim_{n\to+\infty}\|\varphi_2(n,\theta_{-n}\omega)y_n^1-\varphi_2(n,\theta_{-n}\omega)y_n^2\|$$
$$=\lim_{n\to+\infty}\|\varphi_2(n,\theta_{-n}\omega)T_\omega(x_n^1)-\varphi_2(n,\theta_{-n}\omega)T_\omega(x_n^2)\|$$
$$=\lim_{n\to+\infty}\|T(\theta_n\,\theta_{-n}\omega,\varphi_1(n,\theta_{-n}\omega,x_n^1))\ -T(\theta_n\,\theta_{-n}\omega,\varphi_1(n,\theta_{-n}\omega,x_n^2))\ \|$$
$$=\lim_{n\to+\infty}\|T(\omega,\varphi_1(n,\theta_{-n}\omega,x_n^1))\ -T(\omega,\varphi_1(n,\theta_{-n}\omega,x_n^2))\ \|$$
$$=\lim_{n\to+\infty}\|T_\omega(\varphi_1(n,\theta_{-n}\omega,x_n^1))\ -T_\omega(\varphi_1(n,\theta_{-n}\omega,x_n^2))\ \|$$
$$=\lim_{n\to+\infty}\|T_\omega(\varphi_1(n,\theta_{-n}\omega,x_n^1)\ -\varphi_1(n,\theta_{-n}\omega,x_n^2))\ \|$$
$$\leq\lim_{n\to+\infty}c\|\varphi_1(n,\theta_{-n}\omega,x_n^1)\ -\varphi_1(n,\theta_{-n}\omega,x_n^2)\ \|=0,$$

where c be constant. Thus

$$\lim_{n\to+\infty}\|\varphi_2(n,\theta_{-n}\omega)y_n^1-\varphi_2(n,\theta_{-n}\omega)y_n^2\|=0$$

hence (θ,φ_2) is regionally proximal RDS . ■

<u>Theorem 3.15 :</u>

Let (θ, φ_1) and (θ, φ_2) be two LRDSs. Then (θ, φ_1) and (θ, φ_2) are regionally proximal if and only if $(\theta, \varphi_1 \times \varphi_2)$ is regionall proximal .

Proof: Assume that (θ, φ_1) and (θ, φ_2) are regionally proximal.

Let $(z_1, z_2) \in (X \times Y) \times (X \times Y)$.

Then $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

In fact, $(X \times Y) \times (X \times Y) \cong (X \times X) \times (Y \times Y)$.

Such that $(y_1, y_2) \in Y \times Y$, and $(x_1, x_2) \in X \times X$ and by hypothesis there exists $x_n^1 \longrightarrow x_1$, $x_n^2 \longrightarrow x_2$ and $y_n^1 \longrightarrow y_1$, $y_n^2 \longrightarrow y_2$ and $\mathbb{P}\{\omega: z_1(\omega) \neq z_2(\omega)\} = 1$, and a full measure invariant subset

$$\widehat{\Omega} \text{ of } \Omega \text{ such that, } \forall\, \omega \in \widehat{\Omega} \lim_{n\to+\infty} \|\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2\| = 0$$

$$\text{and} \qquad \lim_{n\to+\infty} \|\varphi_1(n, \theta_{-n}\omega)y_n^1 - \varphi_1(n, \theta_{-n}\omega)y_n^2\| = 0$$

Set $z_n^1 = (x_n^1, y_n^1)$, and $z_n^2 = (x_n^2, y_n^2)$, such that $z_n^1 \longrightarrow (x_1, y_1)$, $z_n^2 \longrightarrow (x_2, y_2)$

Now,

$$\lim_{n\to+\infty} \|(\varphi_1 \times \varphi_2)(n, \theta_{-n}\omega)z_n^1 - (\varphi_1 \times \varphi_2)(n, \theta_{-n}\omega)z_n^2\|$$

$$= \lim_{n\to+\infty} \|(\varphi_1(n, \theta_{-n}\omega)x_n^1, \varphi_2(n, \theta_{-n}\omega)y_n^1) - (\varphi_1(n, \theta_{-n}\omega)x_n^2, \varphi_2(n, \theta_{-n}\omega)y_n^2)\|$$

$$= \lim_{n\to+\infty} [\|(\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2\|_1 + \| \varphi_2(n, \theta_{-n}\omega)y_n^1 - \varphi_2(n, \theta_{-n}\omega)y_n^2)\|_2]$$

$$= \lim_{n\to+\infty} \left\|(\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2\right\|_1 + \lim_{n\to+\infty} \left\| \varphi_2(n, \theta_{-n}\omega)y_n^1 - \varphi_2(n, \theta_{-n}\omega)y_n^2)\right\|$$

$$= 0 + 0 = 0$$

This means that $(\theta, \varphi_1 \times \varphi_2)$ is regionally proximal RDS.

Conversely, since $(\theta, \varphi_1 \times \varphi_2)$ is regionally proximal RDS.

Then $\lim_{n\to+\infty} \|(\varphi_1 \times \varphi_2)(n, \theta_{-n}\omega)z_n^1 - (\varphi_1 \times \varphi_2)(n, \theta_{-n}\omega)z_n^2\|$

$$= \lim_{n\to+\infty} \|(\varphi_1(n, \theta_{-n}\omega)x_n^1, \varphi_2(n, \theta_{-n}\omega)y_n^1) - (\varphi_1(n, \theta_{-n}\omega)x_n^2, \varphi_2(n, \theta_{-n}\omega)y_n^2)\|$$

$$= \lim_{n\to+\infty} [\|(\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2\|_1 + \| \varphi_2(n, \theta_{-n}\omega)y_n^1 - \varphi_2(n, \theta_{-n}\omega)y_n^2)\|_2] = 0$$

thus

$$\lim_{n\to+\infty} \left\|(\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2\right\|_1 = 0, and \lim_{n\to+\infty} \left\| \varphi_2(n, \theta_{-n}\omega)y_n^1 - \varphi_2(n, \theta_{-n}\omega)y_n^2)\right\|$$

$$= 0$$

Then (θ, φ_1) and (θ, φ_2) are regionally proximal

4.Manimality of LRDSs:

Definition4.1: Let (θ, φ) be a LRDS, and A a nonempty random set in X .

(a) A is called minimal if satisfies the following:

1) A is an invariant.

2) A is a closed.

3) no proper subset of A has their properties.

(**b**) If X itself is a minimal set , then we call (θ, φ) a ***minimal RDS***.

(**c**) $x \in X^{\Omega}$ is called a ***minimal point*** if $\overline{\gamma_x(\omega)}$ is a minimal random set. If every random variable $x \in X^{\Omega}$ is minimal point, then (θ, φ) is called ***pointwise minimal***.

Theorem 4.2:

A is minimal random set if and only if $\overline{\gamma_x^t(\omega)} = A(\omega)$, $\forall x \in A(\omega)$.

Proof: Suppose that A is a minimal random set then

$$\varphi(t, \theta_{-t}\omega)x \in A(\omega), \forall\, x \in A, t \in T$$

Therefore $\cup \{\varphi(t, \theta_{-t}\omega)x : t \in T\} \subset A(\omega)$, i.e., $\gamma_x^t(\omega) \subset A(\omega)$.

So $\overline{\gamma_x^t(\omega)} \subset \overline{A(\omega)}$ $=A(\omega)$. Since $\gamma_x^t(\omega)$ is a non-empty invariant set, then $\overline{\gamma_x^t(\omega)} \neq \emptyset$ is an invariant. Hence $\overline{\gamma_x^t(\omega)} \neq \emptyset$ an invariant, closed set with $\overline{\gamma_x^t(\omega)} \subset A(\omega)$. Since A is minimal, then $\overline{\gamma_x^t(\omega)} = A$.

Conversely, suppose that

$$\overline{\gamma_x^t(\omega)} = A(\omega), \forall x \in A.$$

Then A is non-empty invariant, closed. Let B non-empty invariant, closed and $B \subset A$

Let $x \in B$ then $x \in A$ and $\gamma_x^t(\omega) \subset B(\omega)$ thus

$$\overline{\gamma_x^t(\omega)} \subset B(\omega) . \qquad \dots(1)$$

Since $x \in A$ then

$$\overline{\gamma_x^t(\omega)} = A(\omega) \qquad \dots(2)$$

from (1) and (2) we get $A(\omega) \subset B(\omega)$then $A = B$ then $A(\omega)$is minimal.

Theorem 4.3: Let (θ, φ) be a RDS then $\overline{\gamma_x^t(\omega)}$ is minimal if and only if $y \in \overline{\gamma_x^t(\omega)}$ imply to

$x \in \gamma_y^t(\omega)$ for all $x, y \in X$.

Proof:

Let $\overline{\gamma_x^t(\omega)}$is minimal. If $y \in \overline{\gamma_x^t(\omega)}$, then by Theorem (2.3) $\overline{\gamma_x^t(\omega)} = \overline{\gamma_y^t(\omega)}$, so $x \in \overline{\gamma_y^t(\omega)}$.

Conversely, suppose that ∀ y ∈ X: y $\in \overline{\gamma_x^t(\omega)} \Rightarrow x \in \overline{\gamma_y^t(\omega)}$. To prove, $\overline{\gamma_x^t(\omega)}$ is minimal. We have $\overline{\gamma_x^t(\omega)}$ is non-empty, closed, and invariant random set. let $M(\omega) \neq \emptyset$, closed and invariant subset of $\overline{\gamma_x^t(\omega)}$. If $y \in M(\omega)$, then $y \in \overline{\gamma_x^t(\omega)}$ and by hypothesis, $x \in \overline{\gamma_y^t(\omega)}$. Since $M(\omega)$ is closed and invariant, then $\overline{\gamma_y^t(\omega)} \subset$ M(ω). Then $x \in M(\omega)$.

In the same way, we prove that $\overline{\gamma_x^t(\omega)} = M(\omega)$. This means that $\overline{\gamma_x^t(\omega)}$ is minimal. ■

Lemma 4.4: If (θ, φ) is an $LRDS_s$ point- distal then (θ, φ) is minimal .In specific, a minimal and distal is a point –distal

Proof: Since (θ, φ) is a point- distal, then $\overline{\gamma_x^t(\omega)} = X$, ∀ $x \in X$ and by Theorem (2.3) then X is minimal itself, therefor (θ, φ) a minimal.

Now, let (θ, φ) a minimal then X itself is a minimal, therefor by Theorem (3.2) $\overline{\gamma_x^t(\omega)} = X$, $\forall x \in X$

Since (θ, φ) is a distal, therefore, every $x \in X$ is a distal. ■

Definition 4.5: Let (θ, φ) be a LRDS is called to be ***equicontinuous*** if for all t. r. v. $\varepsilon > 0$, ∃ t. r. v. $\delta > 0$ such that:

$$\|\varphi(t, \theta_{-t}\omega)x - \varphi(t, \theta_{-t}\omega)y\| < \varepsilon(\omega) \ , \forall\, x \in X, \forall y \in B_\delta(x), \forall\, t \in T.$$

If $\delta \equiv \delta(\varepsilon)$ then is said to be uniformly equicontinuous.

Remark Every uniformly equicontinous is equicontinuous.

Definition 4.6: Let (θ, φ) be a LRDS, It is called **uniformly distal** if for each t. r. v. $\varepsilon > o$, ∃ t. r. v. $\delta > 0$ such that

$$\|\varphi(t, \theta_{-t}\omega)x - \varphi(t, \theta_{-t}\omega)y\| > \varepsilon(\omega)$$

implies that, $\|x - y\| > \delta(\omega)$, $\forall\, x, y \in X$.

Theorem 4.7:

If (θ, φ) is uniformly equicontinuous then it is a distal.

Proof. Let (θ, φ) be an uniformly equicontinuous and $x_1, x_2 \in X$,

$x_1 \neq x_2$. There is a random variable $\varepsilon > 0$, $\exists$ t. r. v. $\delta > 0$ such that
$\|\varphi(t,\theta_{-t}\omega)x - \varphi(t,\theta_{-t}\omega)y\| > \varepsilon(\omega)$, $\forall\, x \in X, \forall y \in B_\delta(x), \forall\, t \in G$. ■

References:

[1] J. Auslander, Minimal Flows and Their Extensions, North-Holland Math. Studies Vol. 153. North-Holland, Amsterdam, 1988.

[2] J. Auslander and X. Dai introduce, The minimality, distality and equicontinuity for semigroup actions on compact hausdorff spaces, Discrete and continuous dynamical systems, Vol. 39, N. 8, pp.4647- 4711, 2019.

[3] J. Auslander and N. Markley, locally almost periodic minimal flows, J. Difference Eq. Appl., 97-109, 2009.

[4] X. Dai and E. Glasner, On universal minimal proximal flows of topological groups, Proc. Amer. Math. Soc., 147, 1149-1164, 2019.

[5] E. Glasner, Proximal Flows, Lecture Notes in Math., 517, Springer-Verlag, 1976.

[6] C. Igor, Monotone Random Systems Theory and Applications, springer –verlag Berlin Heidelberg New York, 2002.

[7] A. Ludwig, "Random Dynamical Systems" springer-verlag Berlin Heidelberg New York, 1991.

[8] A. Ludwig; I., D., Chueshov, Order-Preserving Random Dynamical Systems: Equilibria, attractor, applications, Dynamics and Stability of System, 265-280, 1998.

[9] W. A. Veech, Point-distal flows, Amer. J. Math., 205-242, 1970.

Article submitted 1 March 2023. Accepted at 29 March.

Published at 30 Jun 2023.