Generalize of some concepts on Linear Random Dynamical Systems

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Abstract. This work aims to study the concepts of proximal, distal, minimal, distil point, regionally proximal, equicontinuous and uniformly equicontinuous in the linear random dynamical systems with the study of some their properties.

Keywords. Liner Random Dynamical Systems, distal point of liner random dynamical systems, minimal of liner random dynamical systems, proximal of liner random dynamical systems.

1. Introduction.


In our work, we will generalize these concepts from deterministic dynamical systems to randomness. In particular, we will discuss the concepts of minimality, distal, point-distal, and proximal in linear random dynamical systems where the phase space is considered a Banach space.

The classification of sets is one of the goals of random dynamics. We will be concerned with three types of minimal sets: equicontinuous, distal, and point-distal. Point-distal is necessarily minimal [1]. That is, "if is point-distal, then it contains no proper" a closed and invariant subset.
2. Some concepts on RDSs:

Here, we collected definitions and notions from the theory of RDSs that we need for our work. (see L. Arnold [7,8] and Igore [6]).

**Definition 2.1 [6]:**

The \((\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)\) is said to be **metric dynamical system** (MDS) if \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space and the function \(\theta: \mathbb{T} \times \Omega \to \Omega\) satisfy

(i) \(\theta\) is measurable,

(ii) \(\theta(0, \omega) = \omega\), for every \(\omega \in \Omega\),

(iii) \(\theta_{t+s}(\omega) = (\theta_t \circ \theta_s)(\omega)\) for every \(\omega \in \Omega, t, s \in \mathbb{T}\) and

(iv) \(\mathbb{P}(\theta_tF) = \mathbb{P}(F), \forall F \in \mathcal{F}\) and \(\forall t \in \mathbb{T}\).

**Definition 2.2 [7]:**

Let \(X\) be a topological space and \(\theta\) be a MDS. A topological **random dynamical system** on \(X\) over \(\theta\) is a function \(\varphi: \mathbb{T} \times \Omega \times X \to X\), admit the following properties:

(i) \(\varphi(\cdot, \omega, \cdot): \mathbb{T} \times X \to X\) is continuous for every \(\omega \in \Omega\).

(ii) The mapping \(\varphi(t, \omega) := \varphi(t, \omega, \cdot): X \to X\) form a cocycle on \(\theta(\cdot)\), that is satisfy

\[
\varphi(0, \omega)x = x, \quad \forall \omega \in \Omega \quad \text{and for every} \quad x \in X,
\]

\[
\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \quad \forall \ s, t \in \mathbb{T}, \omega \in \Omega.
\]

**Definition 2.3 [7] (Linear RDSs):**

A linear random dynamical system LRDS is an RDS \((\theta, \varphi)\) on the Banach space \(X\) such that \(\varphi(t, \omega)\) is linear operators of \(X, \forall \omega \in \Omega, t \in \mathbb{T}\).

**Definition 2.4 [6]:**

Suppose that \((X, d)\) be a metric space which is a measurable space with Borel \(\sigma\)-field \(\mathcal{B}(X)\) and \((\Omega, \mathcal{F})\) be a measurable space. The set-valued function \(A: \Omega \to \mathcal{B}(X), \omega \mapsto A(\omega)\), is a **random set** if the mapping \(\omega \mapsto d(x, A(\omega))\) is measurable for each \(x \in X\). The random set \(A(\omega)\) is called a **random closed(compact) set**, if it is closed (compact) for all \(\omega \in \Omega\).
Definition 2.5 (Equivalence of RDS) [7, 8]: Let \((\theta, \varphi_1)\) and \((\theta, \varphi_2)\) be two RDS over the same MDS \(\theta\) with phase spaces \(X_1\) and \(X_2\) resp. These RDSs are said to be (topologically) equivalent (or conjugate) if there exists a mapping \(T: \Omega \times X_1 \rightarrow X_2\) with the properties:

(i) The mapping \(x \mapsto T(\omega, x)\) is a homeomorphism from \(X_1\) onto \(X_2\), \(\forall \omega \in \Omega\);
(ii) The mappings \(\omega \mapsto T(\omega, x_1)\) and \(\omega \mapsto T^{-1}(\omega, x_2)\) are measurable, \(\forall x_1 \in X_1\) and \(x_2 \in X_2\);
(iii) The cocycles \(\varphi_1\) and \(\varphi_2\) are cohomologous, i.e.
\[
\varphi_2(t, \omega, T(\omega, x)) = T(\theta, \omega, \varphi_1(t, \omega, x)) \quad \text{for any } x \in X_1.
\]
(iv) \(T_\omega: X_1 \rightarrow X_2\) is linear map, \(\forall \omega \in \Omega\).

Definition 2.6 [7, 8]:
Let \((\mathcal{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)\) be a MDS. A random variable \(\delta: \Omega \rightarrow \mathbb{R}^+\) is said to be tempered random variable (t.r.v.), if there exists a full measurable subset \(\tilde{\Omega}\) of \(\Omega\) such that
\[
\lim_{n \rightarrow \infty} \frac{1}{\log \delta(\theta_n \omega)} = 0, \quad \forall \omega \in \tilde{\Omega},
\]
where \(\log^+ := \max(0, \log)\).

Definition 2.7:
Let \((\varphi, \theta)\) be an LRDS, and \(\gamma^f_\mathcal{T}: \omega \rightarrow D(\omega)\) be a multifunction defined by
\[
\gamma^f_\mathcal{T}(\omega) = \cup \{\varphi(t, \theta^{-t} \omega)x: t \in \mathcal{T}\}
\]
the set \(\gamma^f_\mathcal{T}\) is called a trajectory.

3. Proximal of RDSs.

In this section, the concepts of proximal, distal, minimal, regionally proximal, and uniformly equicontinuous in the LRDSs are studied and mentioning some of their characteristics.

Definition 3.1: Let \((\theta, \varphi)\) be an LRDS. The pair of random variables \(x, y \in X^\Omega\) is called proximal if there exists a divergent net \(\{\lambda_k\}\) in \(\mathcal{G}\) and a full measure invariant subset \(\tilde{\Omega}\) of \(\Omega\) such that, \(\forall \omega \in \tilde{\Omega}\) we have
\[
\lim_{\lambda_k} \|\varphi(t_{\lambda_k} \theta_{-t_{\lambda_k}} \omega)x(\theta_{-t_{\lambda_k}} \omega) - \varphi(t_{\lambda_k} \theta_{-t_{\lambda_k}} \omega)y(\theta_{-t_{\lambda_k}} \omega)\| = 0,
\]
distal otherwise. A random variable \(x\) is called distal if \(x, y \in X^\Omega\) is proximal only for \(x = y\).

Definition 3.2: An LRDS \((\theta, \varphi)\) is said to be proximal if every pairs of random variables are proximal, distal if all pairs of random variables are distal.

The set of all pairs of random variable proximal denoted by
\[
\mathcal{P}(X) := \{(x, y) \in X^\Omega \times X^\Omega: \exists \text{net } \{\lambda_k\} \text{ in } \mathbb{R}: \lim_{\lambda_k} \|\varphi(t_{\lambda_k} \theta_{-t_{\lambda_k}} \omega)x(\theta_{-t_{\lambda_k}} \omega) - \varphi(t_{\lambda_k} \theta_{-t_{\lambda_k}} \omega)y(\theta_{-t_{\lambda_k}} \omega)\| = 0\}
\]
In particular, if the LRDS is proximal and distal, then \( X^\Omega \times X^\Omega = \Delta_x \), \( \Delta = \{(x, x) : x \in X^\Omega\} \), hence \((\varphi, \theta)\) is the trivial on a one-point space.

**Example 3.3** Let \((\theta, \varphi)\) be an LRDS defined as follows: \( X = T = \mathbb{R} \) and \((\Omega, \mathcal{B}, \mathbb{P})\) be a probability space with \( \Omega = [0,1] \), \( \mathcal{B}\) be a Borel \( \sigma\)–algebra and \( \mathbb{P}\) is a Lebesgue measure. Define \( \varphi: \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R} \), by \( \varphi(t, \omega, x) = tx \) and \( \theta: \mathbb{R} \times \Omega \to \Omega \) by \( \theta(t, \omega) = \omega \). Then \((\theta, \varphi)\) is proximal.

**Lemma 3.4:** The LRDS \((\theta, \varphi)\) is distal if and only if \( P(X) = \Delta \).

**Proof.** Let \((\theta, \varphi)\) be distal then all \( x, y \in X^\Omega \) are distal, thus
\[
\lim_{\lambda} \| \varphi(t_\lambda, \theta_{-t_\lambda} \omega) x(\theta_{-t_\lambda} \omega) - \varphi(t_\lambda, \theta_{-t_\lambda} \omega) y(\theta_{-t_\lambda} \omega) \| > 0 , \text{ i.e. the is no } x \neq y \in X^\Omega
\]
Such that
\[
\lim_{\lambda} \| \varphi(t_\lambda, \theta_{-t_\lambda} \omega) x(\theta_{-t_\lambda} \omega) - \varphi(t_\lambda, \theta_{-t_\lambda} \omega) y(\theta_{-t_\lambda} \omega) \| = 0 , \text{ then } x = y , \text{ thus } P(X) \text{ contain only } \Delta
\]
Now Let \( P(X) = \Delta \), therefore for all \( x \neq y \) then
\[
\lim_{\lambda} \| \varphi(t_\lambda, \theta_{-t_\lambda} \omega) x(\theta_{-t_\lambda} \omega) - \varphi(t_\lambda, \theta_{-t_\lambda} \omega) y(\theta_{-t_\lambda} \omega) \| \neq 0
\]
Thus, \( \lim_{\lambda} \| \varphi(t_\lambda, \theta_{-t_\lambda} \omega) x(\theta_{-t_\lambda} \omega) - \varphi(t_\lambda, \theta_{-t_\lambda} \omega) y(\theta_{-t_\lambda} \omega) \| > 0 \) for all \( x, y \in X^\Omega \) then all pairs of random variables are distal.

**Definition 3.5:** Let \((\theta, \varphi)\) be a LRDS. Then
(a) \( x \in X^\Omega \) is called a **distal point** of \((\theta, \varphi)\) if there exists no point other than itself in \( \gamma^\theta_x(\omega) \) to be proximal to it under \((\theta, \varphi)\).
(b) \((\theta, \varphi)\) is called **point-distal** if there exists a random variable \( x \in X^\Omega \) such that
1. \( x \) is distal, and
2. \( \gamma^\theta_x(\omega) = X \).

**Lemma 3.6:** If \((\theta, \varphi_1)\) and \((\theta, \varphi_2)\) be two LRDS’s then so is \((\theta, \varphi_1 \times \varphi_2)\), where
\[
\varphi_1 \times \varphi_2 (t, \omega, (x, y)) = (\varphi_1(t, \omega, x), \varphi_2(t, \omega, y))
\]

**Proof**
Put \( \varphi := \varphi_1 \times \varphi_2: \mathbb{R} \times \Omega \times (X_1 \times X_2) \to ((X_1 \times X_2) \\
Defined by
\[
\varphi(t, \omega)(x, y) := \varphi_1 \times \varphi_2(t, \omega, (x, y)) = (\varphi_1(t, \omega)x, \varphi_2(t, \omega)y) , \forall x \in X_1 \text{ and } y \in X_2
\]
i) \( \varphi(0, \omega)(x, y) = (\varphi_1(0, \omega)x, \varphi_2(0, \omega)y) \)
\[ (x, y) \]

ii) \( \varphi(t, \omega)(x, y) \) is a measurable

iii) \( \varphi(t + s, \omega)(x, y) = (\varphi_1(t + s, \omega)x, \varphi_2(t + s, \omega)y) = (\varphi_1(t, \theta_s \omega) \circ (\varphi_1(s, \omega)x, \varphi_2(t, \theta_s \omega) \circ (\varphi_2(s, \omega)y) = \varphi(t, \theta_s \omega)(\varphi_1(s, \omega)x, \varphi_2(s, \omega)y) = \varphi(t, \theta_s \omega)(\varphi(s, \omega)(x, y)) \]

**Theorem 3.7:** Let \( (\theta, \varphi_1) \equiv_T (\theta, \varphi_2) \) with \( T \) is linear. If \( (\theta, \varphi_1) \) is proximal LRDS, then so is \( (\theta, \varphi_2) \).

**Proof:** Assume that \( (\theta, \varphi_1) \) is proximal LRDS. Let \( x_1^1, x_2^1 \in X_1^\Omega \). Then there exists \( x_1^1, x_2^1 \in X_1^\Omega \) such that \( x_1^1 := T^{-1}(x_2^1) \) and \( x_2^1 := T^{-1}(x_2^1) \). By hypothesis there exists a divergent net \( \{t_k\} \) in \( G \) and \( \mathbb{P}\{\omega: x_j^i(\omega) \neq y_j^i(\omega), i, j = 1, 2\} = 1 \) and a full measure invariant subset \( \tilde{\Omega} \) of \( \Omega \) such that, \( \forall \omega \in \tilde{\Omega} \) we have

\[
\lim_{\lambda} \| \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega)x_1^1(\theta_{-t_\lambda} \omega) - \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega)x_2^1(\theta_{-t_\lambda} \omega) \| = 0.
\]

Now,

\[
\lim_{\lambda} \| \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega)x_1^2(\theta_{-t_\lambda} \omega) - \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega)x_2^2(\theta_{-t_\lambda} \omega) \|.
\]

\[
= \lim_{\lambda} \| \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega)x_1^2(\theta_{-t_\lambda} \omega) - \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega)x_2^2(\theta_{-t_\lambda} \omega) \|
\]

\[
= \lim_{\lambda} \| \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega)x_1^2(\theta_{-t_\lambda} \omega) - \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega)x_1^2(\theta_{-t_\lambda} \omega) \|
\]

\[
= \lim_{\lambda} \| \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega)T_{\theta_{-t_\lambda} \omega} - \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega)x_1^2(\theta_{-t_\lambda} \omega) \|
\]

\[
\leq c \lim_{\lambda \to +\infty} \| \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega)x_1^2(\theta_{-t_\lambda} \omega) - \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega)x_1^2(\theta_{-t_\lambda} \omega) \| = 0
\]

where \( c \) be constant then \( (\theta, \varphi_2) \) is proximal LRDS.

**Theorem 3.8:** Let \( (\theta, \varphi_1) \) and \( (\theta, \varphi_2) \) be two RDS's. Then \( (\theta, \varphi_1) \) and \( (\theta, \varphi_2) \) are proximal if and only if \( (\theta, \varphi_1 \times \varphi_2) \) is proximal.

**Proof:**

Assume that \( (\theta, \varphi_1) \) and \( (\theta, \varphi_2) \) are proximal.
Let $z_1, z_2 \in (X^\Omega \times Y^\Omega)$. Then $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.
we will use the fact $(X^\Omega \times Y^\Omega) \times (X^\Omega \times Y^\Omega) \equiv (X^\Omega \times X^\Omega) \times (Y^\Omega \times Y^\Omega)$, and
\[
\varphi := \varphi_1 \times \varphi_2 \left( t, \omega, \left( x(\theta_{-t_\lambda} \omega), y(\theta_{-t_\lambda} \omega) \right) \right) = \left( \varphi_1 \left( t, \omega, x(\theta_{-t_\lambda} \omega) \right), \varphi_2 \left( t, \omega, y(\theta_{-t_\lambda} \omega) \right) \right)
\]
Then $(x_1, x_2) \in X^\Omega \times X^\Omega$, $(y_1, y_2) \in Y^\Omega \times Y^\Omega$. By hypothesis there exists a divergent net $\{t_\lambda\}$ in $G$ and $\mathbb{P}\{\omega: z_1(\omega) \neq z_2(\omega)\} = 1$, and a full measure invariant subset $\tilde{\Omega}$ of $\Omega$ such that, $\forall \omega \in \tilde{\Omega}$ we get,
\[
\lim_\lambda \| \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega_1(\theta_{-t_\lambda} \omega)) - \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega) y_1(\theta_{-t_\lambda} \omega) \| = 0. \text{ And }
\lim_\lambda \| \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega_2(\theta_{-t_\lambda} \omega)) - \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega) y_2(\theta_{-t_\lambda} \omega) \| = 0
\]
Then
\[
\lim_\lambda \| \varphi(t_\lambda, \theta_{-t_\lambda} \omega z_1(\theta_{-t_\lambda} \omega) - \varphi(t_\lambda, \theta_{-t_\lambda} \omega) z_2(\theta_{-t_\lambda} \omega) \| = 0
\]
\[
= \lim_\lambda \| \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega x_1(\theta_{-t_\lambda} \omega) - \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega) y_1(\theta_{-t_\lambda} \omega) \|_1
\]
\[
+ \lim_\lambda \| \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega y_1(\theta_{-t_\lambda} \omega) - \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega) y_2(\theta_{-t_\lambda} \omega) \|_2 = 0.
\]
Thus $(\theta, \varphi_1 \times \varphi_2)$ is proximal.

On the other hand, suppose that $(\theta, \varphi_1 \times \varphi_2)$ is proximal. Let $z_1, z_2 \in X^\Omega \times Y^\Omega$. Such that $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. By hypothesis there exist a divergent net $\{t_\lambda\}$ in $G$ and a full measure invariant subset $\tilde{\Omega}$ of $\Omega$ such that, $\forall \omega \in \tilde{\Omega}$ we get,
\[
\lim_\lambda \| \varphi(t_\lambda, \theta_{-t_\lambda} \omega z_1(\theta_{-t_\lambda} \omega) - \varphi(t_\lambda, \theta_{-t_\lambda} \omega) z_2(\theta_{-t_\lambda} \omega) \| = 0.
\]
Since
\[
\lim_\lambda \| \varphi(t_\lambda, \theta_{-t_\lambda} \omega z_1(\theta_{-t_\lambda} \omega) - \varphi(t_\lambda, \theta_{-t_\lambda} \omega) z_2(\theta_{-t_\lambda} \omega) \| = 0
\]
\[
= \lim_\lambda \| \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega x_1(\theta_{-t_\lambda} \omega), \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega) y_1(\theta_{-t_\lambda} \omega) \| - \| \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega x_2(\theta_{-t_\lambda} \omega), \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega) y_2(\theta_{-t_\lambda} \omega) \|
\]
\[
= \| \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega x_1(\theta_{-t_\lambda} \omega) - \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega) x_2(\theta_{-t_\lambda} \omega) \|_1
\]
\[
+ \| \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega y_1(\theta_{-t_\lambda} \omega) - \varphi_2(t_\lambda, \theta_{-t_\lambda} \omega) y_2(\theta_{-t_\lambda} \omega) \|_2 = 0
\]
Since $\| \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega x_i, \varphi_i(t_\lambda, \theta_{-t_\lambda} \omega) y_i \| \geq 0$, for $i = 1, 2$, then
\[
\lim_\lambda \| \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega x_1 - \varphi_1(t_\lambda, \theta_{-t_\lambda} \omega) x_2 \|_1 = 0, \text{ and}
\]
\[ \lim_{\lambda} \| \varphi_1(t_{\lambda}, \theta_{-\tau_{\lambda}}) y_1 - \varphi_1(t_{\lambda}, \theta_{-\tau_{\lambda}}) y_2 \|_2 = 0 \]

This means that \((\theta, \varphi_1)\) and \((\theta, \varphi_2)\) are proximal. ■

**Theorem 3.9:**
Let \((\theta, \varphi_1)\) and \((\theta, \varphi_2)\) be two RDS's. Then \((\theta, \varphi_1 \times \varphi_2)\) is distal if and only if either \((\theta, \varphi_1)\) or \((\theta, \varphi_2)\) is distal.

**Proof:**
Suppose that either \((\theta, \varphi_1)\) or \((\theta, \varphi_2)\) is distal. Let \(z_1, z_2 \in (X^\Omega \times Y^\Omega)\). Then \(z_1 = (x_1, x_2)\) and \(z_2 = (y_1, y_2)\).

We will use the fact \((X^\Omega \times Y^\Omega) \times (X^\Omega \times Y^\Omega) \cong (X^\Omega \times X^\Omega) \times (Y^\Omega \times Y^\Omega)\), and

\[ \varphi := \varphi_1 \times \varphi_2 \left( t, \omega, \left( x(t_{\lambda} \omega), y(t_{\lambda} \omega) \right) \right) = (\varphi_1(t, \omega, x(t_{\lambda} \omega)), \varphi_2(t, \omega, y(t_{\lambda} \omega))) \]

Then \((x_1, x_2) \in X^\Omega \times X^\Omega\), \((y_1, y_2) \in Y^\Omega \times Y^\Omega\). By hypothesis there exists a divergent \(\{t_{\lambda}\}\) in \(G\) and a full measure invariant subset \(\tilde{\Omega}\) of \(\Omega\) such that, \(\forall \omega \in \tilde{\Omega}\) and \(\mathbb{P}\{\omega: z_1(\omega) \neq z_2(\omega)\} = 1\) we have

\[ \lim_{\lambda} \| \varphi_1(t_{\lambda}, \theta_{-\tau_{\lambda}}) x_1(\theta_{-\tau_{\lambda}}) - \varphi_1(t_{\lambda}, \theta_{-\tau_{\lambda}}) x_2(\theta_{-\tau_{\lambda}}) \| > 0, \]

or

\[ \lim_{\lambda} \| \varphi_2(t_{\lambda}, \theta_{-\tau_{\lambda}}) y_1(\theta_{-\tau_{\lambda}}) - \varphi_2(t_{\lambda}, \theta_{-\tau_{\lambda}}) y_2(\theta_{-\tau_{\lambda}}) \| > 0. \]

Then

\[ \lim_{\lambda} \| \varphi(t_{\lambda}, \theta_{-\tau_{\lambda}}) z_1 - \varphi(t_{\lambda}, \theta_{-\tau_{\lambda}}) z_2 \| = \lim_{\lambda} \| \varphi_1(t_{\lambda}, \theta_{-\tau_{\lambda}}) x_1(\theta_{-\tau_{\lambda}}) - \varphi_1(t_{\lambda}, \theta_{-\tau_{\lambda}}) x_2(\theta_{-\tau_{\lambda}}) \|_1 + \lim_{\lambda} \| \varphi_2(t_{\lambda}, \theta_{-\tau_{\lambda}}) y_1(\theta_{-\tau_{\lambda}}) - \varphi_2(t_{\lambda}, \theta_{-\tau_{\lambda}}) y_2(\theta_{-\tau_{\lambda}}) \|_2 > 0. \]

Thus \((\theta, \varphi_1 \times \varphi_2)\) is distal.

Conversely, suppose that \((\theta, \varphi_1 \times \varphi_2)\) is distal. Let \(z_1, z_2 \in X^\Omega \times Y^\Omega\), \(z_1 = (x_1, y_1)\)

\(z_2 = (x_2, y_2)\). By hypothesis there exist a divergent \(\{t_{\lambda}\}\) in \(G\) and a full measure subset \(\tilde{\Omega}\) of \(\Omega\) such that

\[ \lim_{\lambda} \| \varphi(t_{\lambda}, \theta_{-\tau_{\lambda}}) z_1 - \varphi(t_{\lambda}, \theta_{-\tau_{\lambda}}) z_2 \| > 0. \]

Since
\[
\begin{align*}
\lim_{\lambda} \| \varphi(t_\lambda, \theta_{-t_\lambda})z_1 - \varphi(t_\lambda, \theta_{-t_\lambda})z_2 \| \\
= \lim_{\lambda} \| \varphi_1(t_\lambda, \theta_{-t_\lambda}) x_1(\theta_{-t_\lambda}) - \varphi_1(t_\lambda, \theta_{-t_\lambda}) x_2(\theta_{-t_\lambda}) \|_1 \\
+ \lim_{\lambda} \| \varphi_2(t_\lambda, \theta_{-t_\lambda}) y_1(\theta_{-t_\lambda}) - \varphi_2(t_\lambda, \theta_{-t_\lambda}) y_2(\theta_{-t_\lambda}) \|_2 > 0
\end{align*}
\]

Then either
\[
\lim_{\lambda} \| \varphi_1(t_\lambda, \theta_{-t_\lambda}) x_1(\theta_{-t_\lambda}) - \varphi_1(t_\lambda, \theta_{-t_\lambda}) x_2(\theta_{-t_\lambda}) \|_1 > 0,
\]
or
\[
\lim_{\lambda} \| \varphi_2(t_\lambda, \theta_{-t_\lambda}) y_1(\theta_{-t_\lambda}) - \varphi_2(t_\lambda, \theta_{-t_\lambda}) y_2(\theta_{-t_\lambda}) \|_2 > 0
\]

This means that either \((\theta, \varphi_1)\) or \((\theta, \varphi_2)\) is distal.

**Proposition 3.10:** If \((\theta, \varphi)\) is a point-distal with \(x \in X^\Omega\) a distal point, then each of \(\gamma^x_\lambda(\omega)\) is a distal point.

**Proof:** since \(\gamma^x_\lambda(\omega) \subseteq \gamma^x_\lambda(\omega)\), then for all \(y \in \gamma^x_\lambda(\omega)\) imply that \(y \in \gamma^x_\lambda(\omega)\), since \(y\) is a distal point (i.e. there exists no point other than itself in \(\gamma^x_\lambda(\omega)\) to be proximal to it under \((\theta, \varphi)\)) there for each \(x\) of \(\gamma^x_\lambda(\omega)\) is a distal point.

**Definition 3.11:**
If \((\theta, \varphi)\) an RDS. The pair \((x, y) \in X^\Omega \times X^\Omega\) is called **regionally proximal** if there exist \(x_n \to x, y_n \to y\), and a full measure invariant subset \(\Omega\) of \(\Omega\) such that, \(\forall \omega \in \Omega\) and \(\mathbb{P}\{\omega: x(\omega) \neq y(\omega)\} = 1\) we obtain
\[
\lim_{n \to +\infty} \| (\varphi(t_n, \theta_{-t_n}) x_n(\theta_{-t_n}) - \varphi(t_n, \theta_{-t_n}) y_n(\theta_{-t_n})) \| = 0.
\]
Otherwise, is called **regionally distal**.

**Definition 3.12:**
An RDS \((\theta, \varphi)\) is called **regionally proximal** if all pairs of points are regionally proximal, and It is called **regionally distal** if all pairs of points are regionally distal.

**Proposition 3.13:**
Every proximal LRDS is regionally proximal.

**Proof:**
Assume \((\theta, \varphi)\) is proximal LRDS and Let \((x, y) \in X \times X\).
Define a sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) by \(x_n = x\) and \(y_n = y\), then \(x_n \to x\) and \(y_n \to y\). Now
\[
\lim_{n \to +\infty} \left\| (\phi(n, \theta_n \omega)x_n - \phi(n, \theta_n \omega)y_n) \right\| .
\]
\[
= \lim_{n \to +\infty} \left\| (\phi(n, \theta_n \omega)x - \phi(n, \theta_n \omega)y) \right\| = 0
\]
That is \((\theta, \phi)\) is regionally proximal. ■

**Not:** By proposition (3.12) thus example (3.3) satisfies the definition (3.11).

**Proposition 3.14:** Put \((\theta, \phi_1) \equiv_T (\theta, \phi_2)\) with \(T\) is linear. If \((\theta, \phi_1)\) is regionally proximal RDS, then \((\theta, \phi_2)\) also.

**Proof:** Assume that \((\theta, \phi_1)\) is regionally proximal LRDS and let \(y_1, y_2 \in X_2\). Then there exists \(x_1, x_2 \in X_1\) such that \(x_1 := T_{\omega}^{-1}(y_1)\) and \(x_2 := T_{\omega}^{-1}(y_2)\). By hypothesis there exists \(x_n^1 \to x_1, x_n^2 \to x_2\) and a full measure invariant subset \(\Omega\) of \(\Omega, \forall \omega \in \Omega\) and \(P\{\omega; x_i(\omega) \neq y_i(\omega), i = 1, 2\} = 1\) such that
\[
\lim_{n \to +\infty} \left\| \phi_1(n, \theta_n \omega)x_n^1 - \phi_1(n, \theta_n \omega)x_n^2 \right\| = 0.
\]
Set \(y_n^1 = T_\omega(x_n^1)\) and \(y_n^2 = T_\omega(x_n^2)\). Since \(x_n^1 \to x_1\) and \(x_n^2 \to x_2\) and \(T_\omega\) is continuous foe every \(\omega \in \Omega\), then \(y_n^1 \to y_1, y_n^2 \to y_2\)

Now,
\[
\lim_{n \to +\infty} \left\| \phi_2(n, \theta_n \omega)y_n^1 - \phi_2(n, \theta_n \omega)y_n^2 \right\|
\]
\[
= \lim_{n \to +\infty} \left\| \phi_2(n, \theta_n \omega)T_\omega(x_n^1) - \phi_2(n, \theta_n \omega)T_\omega(x_n^2) \right\|
\]
\[
= \lim_{n \to +\infty} \left\| T_\omega(\phi_2(n, \theta_n \omega,x_n^1)) - T_\omega(\phi_2(n, \theta_n \omega,x_n^2)) \right\|
\]
\[
= \lim_{n \to +\infty} \left\| \phi_1(n, \theta_n \omega,x_n^1) - \phi_1(n, \theta_n \omega,x_n^2) \right\|
\]
\[
\leq \lim_{n \to +\infty} c \left\| \phi_1(n, \theta_n \omega,x_n^1) - \phi_1(n, \theta_n \omega,x_n^2) \right\| = 0,
\]
where \(c\) be constant. Thus
\[
\lim_{n \to +\infty} \left\| \phi_2(n, \theta_n \omega)y_n^1 - \phi_2(n, \theta_n \omega)y_n^2 \right\| = 0
\]
hence \((\theta, \phi_2)\) is regionally proximal RDS. ■

**Theorem 3.15:**
Let \((\theta, \varphi_1)\) and \((\theta, \varphi_2)\) be two LRDSs. Then \((\theta, \varphi_1)\) and \((\theta, \varphi_2)\) are regionally proximal if and only if \((\theta, \varphi_1 \times \varphi_2)\) is regionally proximal.

**Proof:** Assume that \((\theta, \varphi_1)\) and \((\theta, \varphi_2)\) are regionally proximal.

Let \((z_1, z_2) \in (X \times Y) \times (X \times Y)\).

Then \(z_1 = (x_1, y_1)\) and \(z_2 = (x_2, y_2)\).

In fact, \((X \times Y) \times (X \times Y) \cong (X \times X) \times (Y \times Y)\).

Such that \((y_1, y_2) \in Y \times Y\), and \((x_1, x_2) \in X \times X\) and by hypothesis there exists \(x_n^1 \to x_1\),

\[ x_n^2 \to x_2 \text{ and } y_n^1 \to y_1, \quad y_n^2 \to y_2 \]

and \(P\{\omega: z_1(\omega) \neq z_2(\omega)\} = 0\), and a full measure invariant subset \(\tilde{\Omega}\) of \(\Omega\) such that, \(\forall \omega \in \tilde{\Omega}\) \(\lim_{n \to +\infty} \|\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2\| = 0\)

and \(\lim_{n \to +\infty} \|\varphi_1(n, \theta_{-n}\omega)y_n^1 - \varphi_1(n, \theta_{-n}\omega)y_n^2\| = 0\)

Set \(z_n^1 = (x_n^1, y_n^1)\), and \(z_n^2 = (x_n^2, y_n^2)\), such that \(z_n^1 \to (x_1, y_1)\), \(z_n^2 \to (x_2, y_2)\)

Now,

\[
\lim_{n \to +\infty} \|(\varphi_1 \times \varphi_2)(n, \theta_{-n}\omega)z_n^1 - (\varphi_1 \times \varphi_2)(n, \theta_{-n}\omega)z_n^2\| = \lim_{n \to +\infty} \|(\varphi_1(n, \theta_{-n}\omega)x_n^1, \varphi_2(n, \theta_{-n}\omega)y_n^1) - (\varphi_1(n, \theta_{-n}\omega)x_n^2, \varphi_2(n, \theta_{-n}\omega)y_n^2)\| = \lim_{n \to +\infty} \|(\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2, \varphi_2(n, \theta_{-n}\omega)y_n^1 - \varphi_2(n, \theta_{-n}\omega)y_n^2)\| = \lim_{n \to +\infty} \|\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2\|_1 + \lim_{n \to +\infty} \|\varphi_2(n, \theta_{-n}\omega)y_n^1 - \varphi_2(n, \theta_{-n}\omega)y_n^2\| = 0 + 0 = 0
\]

This means that \((\theta, \varphi_1 \times \varphi_2)\) is regionally proximal RDS.

Conversely, since \((\theta, \varphi_1 \times \varphi_2)\) is regionally proximal RDS.

Then \(\lim_{n \to +\infty} \|(\varphi_1 \times \varphi_2)(n, \theta_{-n}\omega)z_n^1 - (\varphi_1 \times \varphi_2)(n, \theta_{-n}\omega)z_n^2\| = \lim_{n \to +\infty} \|(\varphi_1(n, \theta_{-n}\omega)x_n^1, \varphi_2(n, \theta_{-n}\omega)y_n^1) - (\varphi_1(n, \theta_{-n}\omega)x_n^2, \varphi_2(n, \theta_{-n}\omega)y_n^2)\| = \lim_{n \to +\infty} \|(\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2, \varphi_2(n, \theta_{-n}\omega)y_n^1 - \varphi_2(n, \theta_{-n}\omega)y_n^2)\| = 0\)

thus

\[
\lim_{n \to +\infty} \|(\varphi_1(n, \theta_{-n}\omega)x_n^1 - \varphi_1(n, \theta_{-n}\omega)x_n^2\|_1 = 0, \quad \text{and} \quad \lim_{n \to +\infty} \|\varphi_2(n, \theta_{-n}\omega)y_n^1 - \varphi_2(n, \theta_{-n}\omega)y_n^2\| = 0
\]

Then \((\theta, \varphi_1)\) and \((\theta, \varphi_2)\) are regionally proximal
4. Minimality of LRDSs:

**Definition 4.1:** Let $(\theta, \varphi)$ be a LRDS, and $A$ a nonempty random set in $X$.

(a) $A$ is called minimal if satisfies the following:

1) $A$ is an invariant.
2) $A$ is a closed.
3) no proper subset of $A$ has their properties.

(b) If $X$ itself is a minimal set, then we call $(\theta, \varphi)$ a **minimal RDS**.

(c) $x \in X^\Omega$ is called a **minimal point** if $\gamma_x(\omega)$ is a minimal random set. If every random variable $x \in X^\Omega$ is minimal point, then $(\theta, \varphi)$ is called **pointwise minimal**.

**Theorem 4.2:**

$A$ is minimal random set if and only if $\gamma_x^\perp(\omega) = A(\omega), \forall x \in A(\omega)$.

**Proof:** Suppose that $A$ is a minimal random set then

$$\varphi(t, \theta, \omega)x \in A(\omega), \forall x \in A, t \in T$$

Therefore $\cup \{\varphi(t, \theta, \omega)x : t \in T\} \subset A(\omega)$, i.e., $\gamma_x^\perp(\omega) \subset A(\omega)$.

So $\gamma_x^\perp(\omega) \subset A(\omega) \Rightarrow A(\omega)$. Since $\gamma_x^\perp(\omega)$ is a non-empty invariant set, then $\gamma_x^\perp(\omega) \neq \emptyset$ is an invariant. Hence $\gamma_x^\perp(\omega) \neq \emptyset$ an invariant, closed set with $\gamma_x^\perp(\omega) \subset A(\omega)$. Since $A$ is minimal, then $\gamma_x^\perp(\omega) = A$.

Conversely, suppose that

$$\gamma_x^\perp(\omega) = A(\omega), \forall x \in A.$$ 

Then $A$ is non-empty invariant, closed. Let $B$ non-empty invariant, closed and $B \subset A$

Let $x \in B$ then $x \in A$ and $\gamma_x^\perp(\omega) \subset B(\omega)$ thus

$$\gamma_x^\perp(\omega) \subset B(\omega). \quad \cdots(1)$$

Since $x \in A$ then

$$\gamma_x^\perp(\omega) = A(\omega) \quad \cdots(2)$$

from (1) and (2) we get $A(\omega) \subset B(\omega)$ then $A = B$ then $A(\omega)$ is minimal.

**Theorem 4.3:** Let $(\theta, \varphi)$ be a RDS then $\gamma_x^\perp(\omega)$ is minimal if and only if $y \in \gamma_x^\perp(\omega)$ imply to
\( x \in \gamma^L_x(\omega) \) for all \( x, y \in X \).

**Proof:**

Let \( \gamma^L_x(\omega) \) is minimal. If \( y \in \gamma^L_x(\omega) \), then by Theorem (2.3) \( \gamma^L_x(\omega) = \gamma^L_y(\omega) \), so \( x \in \gamma^L_y(\omega) \).

Conversely, suppose that \( \forall y \in X: y \in \gamma^L_x(\omega) \Rightarrow x \in \gamma^L_y(\omega) \). To prove, \( \gamma^L_x(\omega) \) is minimal. We have \( \gamma^L_x(\omega) \) is non-empty, closed, and invariant random set. Let \( M(\omega) \neq \emptyset \), closed and invariant subset of \( \gamma^L_x(\omega) \). If \( y \in M(\omega) \), then \( y \in \gamma^L_x(\omega) \) and by hypothesis, \( x \in \gamma^L_y(\omega) \). Since \( M(\omega) \) is closed and invariant, then \( \gamma^L_x(\omega) \subseteq M(\omega) \). Then \( x \in M(\omega) \).

In the same way, we prove that \( \gamma^L_x(\omega) = M(\omega) \). This means that \( \gamma^L_x(\omega) \) is minimal. \( \blacksquare \)

**Lemma 4.4:** If \( (\theta, \varphi) \) is an LRDS, point-distant then \( (\theta, \varphi) \) is minimal. In specific, a minimal and distal is a point–distal

**Proof:** Since \( (\theta, \varphi) \) is a point-distant, then \( \overline{\gamma^L_x(\omega)} = X, \forall x \in X \) and by Theorem (2.3) then \( X \) is minimal itself, therefor \( (\theta, \varphi) \) a minimal.

Now, let \( (\theta, \varphi) \) a minimal then \( X \) itself is a minimal, therefor by Theorem (3.2) \( \overline{\gamma^L_x(\omega)} = X, \forall x \in X \)

Since \( (\theta, \varphi) \) is a distal, therefore, every \( x \in X \) is a distal. \( \blacksquare \)

**Definition 4.5:** Let \( (\theta, \varphi) \) be a LRDS is called to be **equicontinuous** if for all t. r. v. \( \varepsilon > 0 \), \( \exists \) t. r. v. \( \delta > 0 \) such that:

\[
\| \varphi(t, \theta_{-t}\omega)x - \varphi(t, \theta_{-t}\omega)y \| < \varepsilon(\omega), \forall x \in X, \forall y \in B_\delta(x), \forall t \in T.
\]

If \( \delta \equiv \delta(\varepsilon) \) then is said to be uniformly equicontinuous.

**Remark** Every uniformly equicontinuous is equicontinuous.

**Definition 4.6:** Let \( (\theta, \varphi) \) be a LRDS. It is called **uniformly distal** if for each t. r. v. \( \varepsilon > 0 \), \( \exists \) t. r. v. \( \delta > 0 \) such that

\[
\| \varphi(t, \theta_{-t}\omega)x - \varphi(t, \theta_{-t}\omega)y \| > \varepsilon(\omega)
\]

implies that, \( \| x - y \| > \delta(\omega) \), \( \forall x, y \in X \).

**Theorem 4.7:**

If \( (\theta, \varphi) \) is uniformly equicontinuous then it is a distal.

**Proof.** Let \( (\theta, \varphi) \) be an uniformly equicontinuous and \( x_1, x_2 \in X \),
\( x_1 \neq x_2 \). There is a random variable \( \varepsilon > 0 \), \( \exists \) t. r. v. \( \delta > 0 \) such that
\[
\| \varphi(t, \theta_{-t}\omega)x - \varphi(t, \theta_{-t}\omega)y \| > \varepsilon(\omega), \forall x \in X, \forall y \in B_{\delta}(x), \forall t \in G. \]

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