Homotopy Covers of Graphs and Lifting Property

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Abstract— The aim of this paper is creating requirements for a graph cover to have a Homotopy lifting property of topological space covers, or A-homotopy lifting property. Also, it defines the homotopy cover for a graph and gives a definition of the covering graph, and develops the path-lifting property.

Keywords—Graph, Homotopy theory, Cover, Lifting Properties.

1- Introductions

Covers of graphs were initially researched by looking at graphs as one-dimensional topological spaces. This viewpoint has led to the extension of numerous topological space cover features to graphs, such as covers and deck transfer motions [1],[2].

Covers of topological spaces provide the lifting of Homotopy between them when seen as one-dimensional spaces in addition to the unique lifting of pathways. In graphs, the Homotopy of pathways are not particularly fascinating. Recent developments in graph Homotopy, however, have been made. Two ideas stand out in particular: A-Homotopy [3],[4],[5] and X-Homotopy [6],[7],[8],[9],[10],[11], [12]. In [2], Hardeman demonstrates how certain graphs with no three or four cycles cover lift a Homotopy. In this article, we describe the Homotopy covers of the graphs and demonstrate that these covers adhere to the Homotopy lift property.

2- Preliminaries

A covering space in topology is a continuous map $P: \tilde{U} \rightarrow U$ it keeps the space's local structure. .when considering a graph

as a space, these covering spaces a gain fail to recognize the structure of the graph, namely, that is vertices and edges. So there are covering graph, that is, graph homomorphism p: $\tilde{Y} \rightarrow Y$ that maintain the local structure of the graphs. Specially, In particular, graph $\tilde{Y}$ should look like graph Y locally with the map p formalizing the structure. In topology, given a covering space $P: \tilde{U} \rightarrow U$ and continuous map $\alpha: L \rightarrow Y$ also, there are also lifts $\tilde{\alpha}: L \rightarrow \tilde{Y}$, which factor α through the space $\tilde{Y}$ there are lifting properties in topology that determine when a lift does or does not exist. Although the literature on A-Homotopy theory currently does not have an analogous name or set of attributes, the following three definitions provide a clearer understanding of what covering graphs are.

Definition 2.1[3]

A graph homomorphism $\propto: Y_1 \rightarrow Y_2$ is map of sets $V(Y_1) \rightarrow V(Y_2)$ such that if $\{a, b\} \in E(Y_1)$ .Then either $\propto (a) = \propto (b)$ or $\{\propto (a), \propto (b)\} \in E(Y_2)$ that is adjacent vertices in Y_1 are mapped to the same vertex of Y_2 or adjacent vertices of G_2.

Definition 2.2
Let Y be a graph x $\in$ V (Y). The closed neighborhood of x mean N[x] is the collection of vertices surrounding x as well as x itself exactly N[x] $= \{m \in V(Y)|\{m, x\} \in E(Y) \text{or } m = x\}$.

Definition 2.3[1]

The graph homomorphism $P: Y_1 \rightarrow Y_2$ is a local isomorphism if P is onto and for each vertex x $\in$ V (Y_2), also the vertex y $\in p^{-1}(x)$ iterative mapping P $|_{N[y]}$: N [y] $\rightarrow$ N[x] is bijective.

Remark 2.4
As opposed to the limitation $P|_{N[y]}$: N[y] $\rightarrow$ N[x] as previously defined, which is a bijection between the sets of vertices N[y] , and N[x] it is not always a bijection between the induced subgraphs' edges. $Y_1(N[y])$ and $Y_2(N[x])$.

Example 2.5

Let C_n be an n-cycle on n $\geq$ 3 and vertices labeled [0],[1],[2]…[n-1] fig.1 depicts a local isomorphism $p: C_8 \rightarrow C_4$ defined by
$$p([r]) = [r \bmod 4] \text{ for } r \in \{0,1,\dots,8\}$$

The edges of the induced subgraphs $C_8([[5]]), C_8(N[[6]])$ and $C_4(N[[1]]), C_4(N[[2]])$ are shown in red while there is an edge $\{[0], [3]\}$ in C_4 there is no edge in $\{[4], [7]\}$.
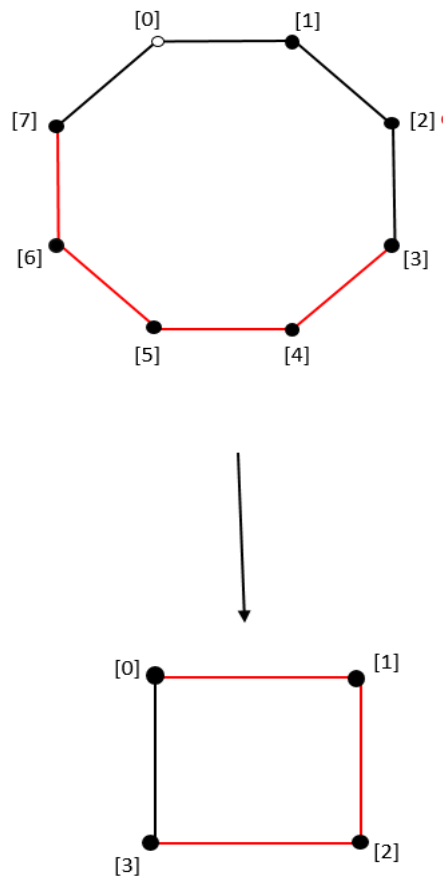

Fig. 1. The local isomorphism $p: C_8 \rightarrow C_4$

We create a new subgraph with the restriction that it must be bijective on both its vertices and edges. For $x \in V(Y_1)$ let N_z denoted the subgraph of Y_1 with a vertex on both vertices and edges. For $z \in V(Y_1)$ let N_z indicate the subgraph of Y_1 with vertex set $V(N_z) = N[z]$ and edge set $E(N_z) = \{\{z, x\} \mid x \in N_z, x \neq z\}$ .If $P: Y_1 \rightarrow Y_2$ is a local isomorphism then p induces a graph homomorphism from the subgraph N_z to the subgraph. $N_{P(z)}$ For $z \in V(Y_1)$, In other words, a graph homomorphism exists. $P|N_z: N_z \rightarrow N_{P(z)}$

This entails the following remark since it is bijective on the vertices and edges of the subgraphs.

Remark 2.6

Let $p: Y_1 \to Y_2$ is a local isomorphism, $z \in V(Y_1)$ then the graph homomorphism $P|_{N_z}$ is invertible and it is inverse.

$(P|_{N_z})^{-1}: N_{p(z)} \to N_z$. Is a graph homomorphism.

3- Covering Graph

Definition 3.1[1]

Let Y and $\tilde{Y}$ be graphed and let $P: \tilde{Y} \to Y$ be a graph homomorphism. If P Is a local isomorphism the pair $(\tilde{Y}, P)$ is a covering graph of Y.

Here, we illustrate some covering graph examples and explain how they vary from covering spaces.

Example 3.2

Let C_n be a cycle with $n \geq 3$ and vertices [0], [1]… [n-1].

If the graph homomorphism $P_n: I_\infty \to C_n$ is realized by $P_n[r] = [r \bmod n]$] then the pair (I_∞, p_n) forms a covering graph of the circle C_n.

Traditional Homotopy theory, as a topological space, all cycles are Homotopy equal to the circle. Using the real line to encircle the circle in example 3.2 is comparable $\mathbb{R}$ by mapping into the circle as a helix this is illustrated in fig.2.
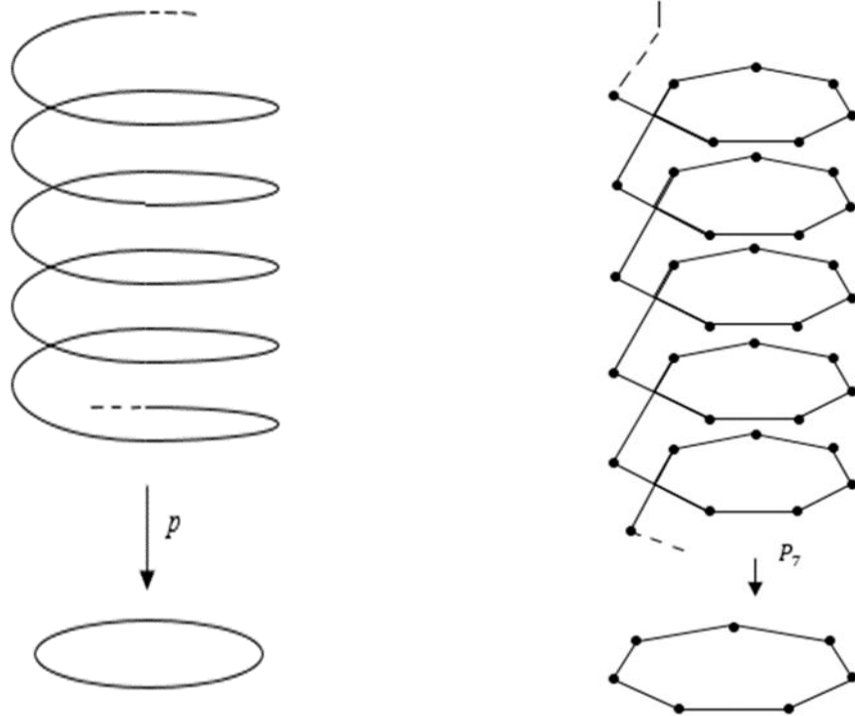
Example 3.3

The graph homomorphism: $P: C_{2n} \to C_n$ is realized by P ([r]) = [r mod n] for all $r \in [0, \dots, 2n-1]$ then the pair (C_{2n}, p), forms a covering graph of the circle C_n.

The local isomorphism p depicted in fig.1 is an example of covering graph of C_n by C_{2n} with k $= 4$ .Example 3.3 is like projecting the topological circle onto a different circle. Way that the first twice encircles the second. As we move further, we will define lift and discuss lifting qualities that are not covered in the literature that is currently available on A-Homotopy theory. The following definition is adapted from [9], except instead of a continuous map, it uses graph homomorphism.

4- Path Lifting Property

Definition 4.1[4]

The graph $I_n^m = I_n \times \dots \dots \times I_n$.Is the m-fold Cartesian product of I_n for some integers m, n ≥ 0 with the recognized vertex $0 = (0, \dots \dots, 0)$.

Remark 4.2

Let I_∞^m denote the m-fold Cartesian product of I_∞ we will only use non-based graph homomorphisms $\propto: I_\infty^m \to$ K with m $= 1, 2$.

This will give us the path in graph K and the graph Homotopy between the paths.

Definition 4.3[13]

Let $\alpha: I_\infty^m \to$ Y be a graph homomorphism and $c_1, c_2, \dots, c_m \in \mathbb{Z}$. We say that α stabilizes direction +r with $1 \leq r \leq m$ if there is an integer P_α^r such that for all n $\geq P_\alpha^r$

$$\propto (c_1, \dots c_{r-1}, \text{n}, c_{r+1}, \dots, c_m) = \propto (c_{1,\dots,} c_{r-1}, P_\propto^r, c_{r+1}, \dots, c_m)$$

We say that $\propto$ stabilizes in the direction – r with $1 \leq r \leq m$ if there is an integer $n_\propto^r$ such that for all n $\leq n_\propto^r$

$$\propto (c_1, \dots \dots, c_{r-1}, \text{n}, c_{r+1}, c_m) = \propto (c_1, \dots \dots, c_{r-1}, n_\alpha^r, c_{i+1}, c_m)$$

We always take $P_\propto^r$ to be the least integer and n_α^r being the largest number such that the previous claims are accurate. If $\propto$ is constant on the rth-axis then we take $P_\propto^r = n_\alpha^r = 0$

The integers P_α^r and n_α^r afford us the points at which the graph homomorphism $\propto$ stabilizes on the rth −axis in the positive and negative orientations

on the rth axis. Respectively, when a graph homomorphism $\propto: I_\infty^m \to Y$ stabilizes in every direction the region of I_∞^r induced by the vertex set $\prod_{r \in [m]} [n_\alpha^r, P_\alpha^r]$ is called the active region of α. For each path $\alpha: I_\infty \to Y$, we say that α stars at $\alpha(n_\propto^1)$ and α ends at $\alpha(p_\alpha^1)$ when these integers exist.

Definition 4.4[13]

If a graph homomorphism $\propto: I_\infty^m \to Y$ stabilizes in every direction $-r$ and $+r$ for $1 \leq r \leq m$ then we say that α is a stable graph homomorphism. Let $S_m(Y)$ be the set of stable graph homomorphism. From the infinite m-cube I_∞^m to the graph Y.

Definition 4.5

Let Y be a graph and let $(\tilde{Y}, P)$ be the covering graph of Y given a graph homomorphism $\alpha: G \to Y$ a lift of α is a graph homomorphism $\tilde{\alpha}: G \to \tilde{Y}$ such that $p \circ \tilde{\alpha} = a$.

Theorem 4.6 (path lifting property)

Let $(\tilde{Y}, P)$ be a covering graph of (Y, y_0). for each stable graph homomorphism $\alpha: I_\infty \to Y$ with $\alpha(n_\alpha^1) = y_0$ and each vertex $\tilde{y}_0 \in p^{-1}(y_0)$ there exists unique lift $\tilde{\alpha}$ of α starting at the vertex $\tilde{y}_0$.
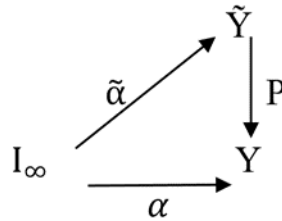
Proof

Let α and $\tilde{y}_0$ be as in the statement define $\tilde{\alpha}: I_\infty \to \tilde{Y}$ by $\tilde{\alpha}(r) = \tilde{y}_0$ for all $r \leq n_\alpha^1$ and recursively by

$$\tilde{\alpha}(r) = (P|_{N_{\alpha(r-1)}})^{-1}(a(r)) \qquad \text{for } r > n_\alpha^1$$

This means that $\tilde{\alpha}$ is defined using a different restriction for each $r \geq n_\alpha^1$. It is not immediately obvious that this produces a graph homomorphism since $\tilde{\alpha}$ is defined recursively. We will use induction for $r \geq n_\alpha^1$. By remark 2.6, the graph homomorphism $(P|_{N_{\alpha(r-1)}})^{-1}$ exists. Since $\alpha(r)$ is the domain of $(P|_{N_{\tilde{\alpha}(r-1)}})^{-1}: N_{p(\tilde{\alpha}(r-1))} \to N_{\tilde{\alpha}(r-1)}$ for each $r \geq n_\alpha^1$ $\tilde{\alpha}$ is well defined. For each base case, consider the edge $\alpha(n_\alpha^1) \, \alpha(n_\alpha^1 + 1) \in E(Y)$.

There is an edge $\tilde{\alpha}(n_\alpha^1)\tilde{\alpha}(n_\alpha^1 + 1) \in E(\check{Y})$ since $(P|_{N_{\check{y}_0}})^{-1}: N_{y_0} \to N_{\check{y}_0}$ is a graph homomorphism .For the inductive hypothesis, suppose that
$$\tilde{\alpha}(r-1)\,\tilde{\alpha}(r) \in E(\check{Y}) \text{ for some } r > n_\alpha^1$$
By definition
$$\tilde{\alpha}(r) = (P|N_{\alpha(r-1)})^{-1}(\alpha(r)) \text{ and } \tilde{\alpha}(r+1) = (P|N_{\alpha(r)})^{-1}(\alpha(r+1))$$
Here we have two different restrictions of p however
$\tilde{\alpha}(r) \in N[\tilde{\alpha}(r-1)] \cap N[\tilde{\alpha}(r)]$ by the inductive hypothesis.
Thus $\left(P|N_{\tilde{\alpha}(r-1)}\right)^{-1}\left(\alpha(r)\right) = (P|N_{\tilde{\alpha}(r)})^{-1}(\alpha(r))$
Which implies that $\tilde{\alpha}$ is a graph homomorphism.
The map $\tilde{\alpha}$ is a lift of α. Since by definition of $\tilde{\alpha}$ for all $r > n_\alpha^1$
$$P(\tilde{\alpha}(r)) = P((\left(P|N_{\alpha(r-1)}\right)^{-1})), (\alpha(r))) = \alpha(r)$$
To show that $\tilde{\alpha}$ is unique for each choice of $\check{y}_0 \in P^{-1}(y_0)$. Let $\tilde{\omega} : I_\infty \to \check{Y}$ be another graph homomorphism such that $\tilde{\omega}(n_{\tilde{\omega}}^1)) = \check{y}_0$ and $P \circ \tilde{\omega} = \alpha$.
Thus $P(\tilde{\omega}(r)) = \alpha(r) = y_0$ for all $r \leq n_\alpha^1$. By applying $(P|N_{\alpha(r)})^{-1}$ we have that $\tilde{\omega}(r) = \check{y}_0$ for all $r \leq n_\alpha^1$. For the inductive hypotheses, suppose $\tilde{\omega}(r) = \tilde{\alpha}(r)$ for some $r \geq n_\alpha^1$. Since $\tilde{\alpha}$ and $\tilde{\omega}$ are graph homomorphisms, this implies that $\tilde{\alpha}(r+1).\tilde{\omega}(r+1 \in N[\tilde{\alpha}(r)]$. Since $P \circ \tilde{\omega} = \alpha = P \circ \tilde{\alpha}$. It follows that
$$P|N_{\tilde{\alpha}(r)}\left(\tilde{\omega}(r+1)\right) = P|N_{\tilde{\alpha}(r)}(\tilde{\alpha}(r+1))$$
Thus by applying $(P|N_{\tilde{\alpha}(r)})^{-1}$, we have that $\tilde{\omega}(r+1) = \tilde{\alpha}(r+1)$.

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