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Some properties of a subclass involving close - to - convex of univalent and multivalent functions

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ABSTRACT: Our work included studying some subclasses of functions (univalent - multivalent) that are close to convex. which is divide into two parts. First part, we discussed the properties of parameter estimates, the implication relation, distortion theorem, and the radius of convexity of functions a subclass of univalent. Second part of this article, we discuss the same properties for subclasses of multivalent functions defined using the Salagen operator in the unit disc.

Keywords: Univalent; multivalent; close-to-convex univalent function; close-to-convex multivalent function

1. INTRODUCTION

Let $\mathbb{A}$ be family of analytic functions in $\boldsymbol{E} = \{ž \in \mathbb{C}: |ž| < \mathbf{1}\}$. of the form:

$$f(ž) = ž + \sum_{k=2}^{\infty} a_k ž^k \qquad (1.1)$$

Which have a series expansion and $\boldsymbol{S} \in \mathbb{A}$ are normalized specifically if $\boldsymbol{f(0) = f(0)' - 1 = 0}$ and the Schwarzian function consists are analytic in $\boldsymbol{E}$ the form:

$$W(ž) = \sum_{k=1}^{\infty} C_k ž^k \qquad (1.2)$$

Are univalent Such that $\boldsymbol{w(0) = 0}$ ***and*** $|\boldsymbol{w}(ž)| < \mathbf{1}$. Generally, denotes by$\boldsymbol{U}$, [1]. Introduce $\boldsymbol{f(E)} \in \mathbb{A}$ is said to be starlike of order $\boldsymbol{\alpha}$ concerning the origin giving the following necessary and sufficient conditions

$$\boldsymbol{Re}\left[\frac{ž\boldsymbol{f}(ž)'}{\boldsymbol{f}(ž)}\right] > \boldsymbol{\alpha}, ž \in \boldsymbol{E} \quad whre\ \alpha \in [0,1)$$

Denoted by $\boldsymbol{S}^*$.[2]. introduce the class of convex functions is denoted $\boldsymbol{K}$, such that the following for a function $\boldsymbol{f} \in \mathbb{A}$ be in $\boldsymbol{K}$

$$\boldsymbol{Re}\left[1 + \frac{žf(ž)''}{f(ž)'}\right] > \mathbf{0}, ž \in \boldsymbol{E}.$$

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The intimate relationship between starlike and convex functions given by S^*, [3].

$$f \in K \Leftrightarrow žf' \in S^*, ž \in E$$

Define the convex function of order α by following the condition

$$Re\left[1 + \frac{žf(ž)''}{f(ž)'}\right] > \alpha, ž \in E \quad where\ \alpha \in [0,1)$$

Analogously [4].

$$f \in K(\alpha) \Leftrightarrow žf' \in S^*(\alpha), ž \in E$$

suppose $g \in S^*$ is closed to convex if

$$Re\left[\frac{f(ž)'}{g(ž)}\right] > 0\,, ž \in E.$$

Denoted by C [5]. consider the pair function f is subordinate to g (written simply as $f \prec g$) in E if there exists a Schwarzian function $w \in U$ for which

$$f(ž) = g(w(ž)), ž \in E.$$

But when f, g univalent we get $f(0) = g(0)$ and $f(E) \subseteq g(E)$. [6]. introduce the multivalent functions and has exactly p roots in E

We denote by $\mathbb{A}_p$ the functions of the form:

$$f(ž) = ž^p + \sum_{k=1}^{\infty} a_{p+k} ž^{p+k}\,, p \in \mathbb{N}$$

It is multivalent in E. Obviously$\mathbb{A}_1 \equiv \mathbb{A}$, $f \in \mathbb{A}_p$ and is said to be multivalent starlike in E if

$$Re\left[\frac{žf(ž)'}{pf(ž)}\right] > 0, ž \in E$$

The class of all functions S_p^*. That is multivalent starlike functions of order α denoted by $S_p^*(\alpha)$ and $S_p^* \equiv S_p^*(0)$

can be defined analogously as follows:

A function $f \in \mathbb{A}_p$ is called multivalent convex in E if

$$Re\frac{1}{p}\left[1 + \frac{žf(ž)''}{f(ž)'}\right] > 0, ž \in E$$

The class of multivalent function of order $\alpha(0 \leq \alpha < p)$ if

$$Re\frac{1}{p}\left[1 + \frac{žf(ž)''}{f(ž)'}\right] > \alpha, ž \in E$$

is denoted by K_p such that all functions are denoted $k_p(\alpha)$ and $k_p(0) \equiv k_p$.

A function $f \in \mathbb{A}_p$ is called a multivalent close-to convex in E if

We have a multivalent convex function h

$$\boldsymbol{Re}\frac{\mathbf{1}}{\boldsymbol{p}}\left[\frac{\check{z}\boldsymbol{f}(\check{z})'}{\boldsymbol{h}(\check{z})'}\right] > \boldsymbol{\alpha}, \check{z} \in \boldsymbol{E}\,.$$

The class of multivalent close-to-convex functions of order α and represented by $\boldsymbol{C_p(\alpha)}$

2. SUBCLASS OF CLOSE-TO-CONVEX UNIVALENT FUNCTIONS

Introduced the class k_s of close-to-convex and $f \in \mathbb{A}$ if [7].

$$Re\left[\frac{\check{z}^2 f(\check{z})}{g(-\check{z})g(\check{z})}\right] > 0 \;\; where\; g \in S^*(2)^{-1}, \check{z} \in E$$

extended the class k_s introducing the following class [8].

$$k_s(y) = \left\{f \in \mathbb{A}: Re\left[\frac{\check{z}^2 f(z)}{g(-\check{z})g(\check{z})}\right] > y, g \in S^*(2)^{-1}, y \in [0,1), \check{z} \in E\right\}.$$

introduced the class $X_t(y), t \neq 0, t \in [1,1], y \in [0,1)$. Is analytic and the function $g \in S^*(2)^{-1}$ then [9].

$$Re\left[\frac{tz^2 f(\check{z})'}{g(\check{z})g(t\check{z})}\right] > y, \check{z} \in E.$$

Definition 1.2 let $X_t(A, B), t \neq 0, t \in [1,1]$ denote the class $f \in \mathbb{A}$ satisfying the condition

$$\left[\frac{t\check{z}^2 f(\check{z})'}{g(\check{z})g(t\check{z})}\right] \prec \frac{1 + A\check{z}}{1 + B\check{z}}, -1 \leq \mathcal{B} < \mathcal{A} \leq 1, \check{z} \in E, g \in S^*(2)^{-1}$$

In particular

1) $X_t(1 - 2y, -1) \equiv X_t(y)$,
2) $X_{-1}(1 - 2y, -1) \equiv k_s(y)$,
3) $X_{-1}(1, -1) \equiv k_s$.

We assume that

$$-1 \leq D \leq \mathcal{B} < \mathcal{A} \leq C \leq 1, t \neq 0, t \in [1,1], \check{z} \in E.$$

Lemma1.3 [10]. Let

$$P(\check{z}) = \frac{1 + \mathcal{A}W(\check{z})}{1 + \mathcal{B}W(\check{z})} = 1 + \sum_{n=1}^{\infty} P_n \check{z}^n, W \in U \tag{1.3}$$

The limits are precise and are achieved for the functions.

$$P_n(\check{z}) = \frac{1 + \mathcal{A}\delta\check{z}^n}{1 + \mathcal{B}\delta\check{z}^n}, |\delta| = 1.$$

Lemma1.4 [11]. Let

$$g \in S^*(2)^{-1}, G(\check{z}) = \frac{g(\check{z})g(t\check{z})}{t\check{z}} = z\check{z} + \sum_{n=2}^{\infty} d_n \check{z}^n \in S^*, then |d_n| \leq n \tag{1.4}$$

Lemma 1.5 [12, 13]. If

$p(\check{z}) = \sum_{n=0}^{\infty} p_n \check{z}^n$ Have positive real part, for μ,

$$|p_2 - \mu p_1^2| \leq 2max\{1, |2\mu - 1|\}$$

The lemma is precise for the functions provided. $p(\check{z}) = \frac{1-i^2\check{z}^2}{1+i^2\check{z}^2}$, $p(\check{z}) = \frac{1+\check{z}}{1-z\check{z}}$.

Lemma 1.6 [14]. if

$$G(\check{z}) = \check{z} + \sum_{n=2}^{\infty} b_n \check{z}^n \in S^* \; for\; any\; complex\; number\; \lambda$$

$$|d_3 - \lambda d_2^2| \leq max\{1, |3 - 4\lambda|\}.$$

The lemma is exact for the functions provided k, if $\left|\lambda - \frac{3}{4}\right| \geq \frac{1}{4}$

Where $(k(\check{z}^2))^{\frac{1}{2}} = \frac{\check{z}}{1+i^2z^2}$ if $\left|\lambda - \frac{3}{4}\right| \geq \frac{1}{4}$.

3. MAIN RESULT

Theorem 2.1 if $f \in X_2(\mathcal{A}, \mathcal{B}), then$

$$|a_n| \leq 1 + \frac{(n-1)(\mathcal{A} - \mathcal{B})}{2}. \tag{2.1}$$

Proof. By definition,

$$f \in X_t(\mathcal{A},\mathcal{B}) \Leftrightarrow \frac{tž^2 f(ž)'}{g(ž)g(tž)} = \frac{1+\mathcal{A}W(ž)}{1+\mathcal{B}W(ž)}, W \in U, -1 \le \mathcal{B} < \mathcal{A} \le 1, ž \in E \tag{2.2}$$

As $f \in X_t(\mathcal{A},\mathcal{B})$,(2.2) can be, express as

$$\frac{žf(ž)'}{G(ž)} = P(ž), \qquad where\ G(z) = \frac{g(ž)g(tž)}{tž} \tag{2.3}$$

Using (1.1), (1.4) and (2.3) in (2.3) it yields

$$1+\sum_{n=2}^{\infty} na_n ž^{n-1} = \left(1+\sum_{n=2}^{\infty} nd_n\, ž^{n-1}\right)\left(1+\sum_{n=1}^{\infty} p_n ž^n\right). \tag{2.4}$$

Equating the coefficients of $ž^{n-1}$in (2.4), we get

$$na_n = d_n + d_{n-1}p_1 + d_{n-2}p_2 + \cdots + d_2 p_{n-2} + p_{n-1} \tag{2.5}$$

Applying Lemma 1.3 and Lemma 1.4, (2.6) gives

$$n|a_n| \le n + (\mathcal{A}-\mathcal{B})[(n-1)+(n-2)+\cdots+2+1]. \tag{2.6}$$

The result (2.3) can be, easily obtained from (2.7)

The following result due to [9]. Is obtaine when $\mathcal{A} = 1-2y$ and $\mathcal{B} = -1$ in theorem 2.1.

Corollary 2.2 if $f \in X_t(y), then$

$$|a_n| \le 1 + (n-1)(1-y)$$

For $\mathcal{A} = 1,\ \mathcal{B} = -1$ and $t = -1$, the theorem 2.1 leads to the following result

Corollary 2.2 if $f \in K_s\ then\ \ |a_n| \le n$

Theorem 2.3: let $-1 \le \mathcal{B}_2 \le \mathcal{B}_1 < \mathcal{A}_1 \le \mathcal{A}_2 \le 1, then$

$X_t(\mathcal{A}_1,\mathcal{B}_1) \subset X_t(\mathcal{A}_2,\mathcal{B}_2)$

Proof.

$$if\ f \in X_t(\mathcal{A}_1,\mathcal{B}_1)\ then\ \frac{tž^2 f(ž)'}{g(ž)g(tž)} \prec \frac{1+\mathcal{A}_1 ž}{1+,\mathcal{B}_1 ž} \prec \frac{1+\mathcal{A}_2 ž}{1+,\mathcal{B}_2 ž}$$

Which implies that $f \in X_t(A_2,B_2)$

Theorem 2.4 if $f \in X_t(\mathcal{A},\mathcal{B})$, then for $|z| = r, r \in (0,1)$we have

$$\frac{(1-\mathcal{A}r)}{(1-\mathcal{B}r)(1+r)^2} \le |f(ž)'| \le \frac{(1+\mathcal{A}r)}{(1+\mathcal{B}r)(1-r)^2} \tag{2.7}$$

And

$$\int_0^r \frac{(1-\mathcal{A}t)}{(1-\mathcal{B}t)(1+t)^2}dt \le |f(ž)'| \le \int_0^r \frac{(1+\mathcal{A}t)}{(1+\mathcal{B}t)(1-t)^2}dt \tag{2.8}$$

Proof. From (2.6) and (2.7) we get

$$|f(ž)'| = \frac{|G(ž)|}{|ž|}\left|\frac{1+\mathcal{A}w(ž)}{1+\mathcal{B}w(ž)}\right| \tag{2.9}$$

$$\frac{zf(ž)'}{G(ž)} = \frac{1+\mathcal{A}w(ž)}{1+\mathcal{B}w(ž)}$$

Clearly maps $|w(ž)| \le r$ on to the circle

$$\left|\frac{žf(ž)'}{G(ž)} - \frac{1-\mathcal{A}\mathcal{B}r^2}{1-\mathcal{B}^2r^2}\right| \le \frac{(\mathcal{A}-\mathcal{B})r}{(1-\mathcal{B}^2r^2)}, \quad |z| = r,$$

This means that

$$\frac{1-\mathcal{A}r}{1-\mathcal{B}r} \le \left|\frac{1+\mathcal{A}w(ž)}{1+\mathcal{B}w(ž)}\right| \le \frac{1+\mathcal{A}r}{1+\mathcal{B}r} \tag{2.10}$$

When G starlike

$$\frac{r}{(1+r)^2} \le |G(ž)| \le \frac{r}{(1-r)^2} \tag{2.11}$$

On integrating (2.7) from zero to r, we get (2.8)

On substituting $\mathcal{A} = 1-2y$ and $\mathcal{B} = -1$ in theorem 1.11 and [9]. leads the result

Corollary 2.5: if $f \in X_t(y)$, then

$$\frac{1-(1-2y)r}{(1+r)^3} \le |f(z)'| \le \frac{1-(1-2y)r}{(1-r)^3}$$

And

$$\int_0^r \frac{1-(1-2y)t}{(1+t)^3}dt \le |f(z)| \le \int_0^r \frac{1+(1-2y)t}{(1-t)^3}dt$$

Theorem 2.6: let $f \in X_t(\mathcal{A},\mathcal{B}), then\ f$ is convex in $|z| = r_1$ and r_1 is smallest positive root in (0, 1) of

$$\mathcal{A}\mathcal{B}r^3 - \mathcal{A}(\mathcal{B}-2)r^2 - (2\mathcal{B}-1)r - 1 = 0 \qquad (2.12)$$

Proof. as $f \in X_t(\mathcal{A},\mathcal{B})$, from (2.13) we have

$$\check{z}f(\check{z})' = G(\check{z})P(\check{z}) \qquad (2.13)$$

On differentiating after taking logarithm in (2.11), hence

$$1 + \frac{\check{z}f(\check{z})''}{f(\check{z})'} = \frac{zG(\check{z})'}{G(\check{z})} + \frac{\check{z}p(\check{z})'}{P(\check{z})} \qquad (2.14)$$

Since $G \in S^*$ by definition $Re\left[\frac{zG(\check{z})'}{G(\check{z})}\right] \ge \frac{1-r}{1+r}$

Therefore (2.14) yields

$$Re\left[1 + \frac{\check{z}f(\check{z})''}{f(\check{z})'}\right] \ge \frac{1-r}{1+r} - \left|\frac{\check{z}P(\check{z})'}{P(\check{z})}\right|$$

$$\ge \frac{1-r}{1+r} - \frac{r(\mathcal{A}-\mathcal{B})}{(1+\mathcal{A}r)(1+\mathcal{B}r)}$$

$$\ge \frac{-\mathcal{A}\mathcal{B}r^3 + \mathcal{A}(\mathcal{B}-2)r^2 + (2\mathcal{B}-1) + 1}{(1+r)(1+\mathcal{A}r)(1+\mathcal{B}r)}$$

Hence, $f(z)$ in $|z| < r_1$ and r_1 is the smallest positive root in (0, 1) of

$$\mathcal{A}\mathcal{B}r^3 - \mathcal{A}(\mathcal{B}-2)r^2 - (2\mathcal{B}-1)r - 1 = 0$$

Corollary 2.7: if $f \in X_t(y)$, then f is convex in $|z| < r_0 = 2 - \sqrt{3}$

4. SUBCLASS OF CLOSE-TO-CONVEX MULTIVALENT FUNCTIONS

Introduced the differential operator $D_\delta^n f(\check{z})$ as follows: [15].

$$D_\delta^n f(\check{z}) = z^p + \sum_{k=p+n}^{\infty}\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n a_k z^k, p \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$$

With $D_\delta^0 f(0) = 0$

$D_\delta^n f(\check{z})$ is popularly known as generalized Sãlãgean operator.

We first present a glimpse of some well-known fundamental subclasses as follows:

1. $C' = \left\{Re\left(\frac{\check{z}f(\check{z})'}{h(z)}\right) > 0, f \in \mathbb{A}, h \in K\ and\ \check{z} \in E\right\}$, the subclass introduced and studied in [16].

2. $C'(\mathcal{A},\mathcal{B}) = \left\{\frac{\check{z}f(\check{z})'}{h(\check{z})} \prec \frac{1+A\check{z}}{1+B\check{z}}, f \in \mathbb{A}, h \in K, -1 \le \mathcal{B} < \mathcal{A} \le 1\ and\ \check{z} \in E\right\}$, the subclass introduced and studied in [17].

3. $CS^*(\mathcal{A},\mathcal{B}) = \left\{\frac{f(\check{z})}{g(\check{z})} \prec \frac{1+\mathcal{A}\check{z}}{1+\mathcal{B}\check{z}}, f \in \mathbb{A}, g \in S^*, -1 \le \mathcal{B} < \mathcal{A} \le 1\ and\ \check{z} \in E\right\}$, the subclass introduced and studied in [18].

4. $CS^*{}_1(\mathcal{A},\mathcal{B}) = \left\{\frac{f(\check{z})}{h(\check{z})} \prec \frac{1+\mathcal{A}\check{z}}{1+\mathcal{B}\check{z}}, f \in \mathbb{A}, h \in K, -1 \le \mathcal{B} < \mathcal{A} \le 1\ and\ \check{z} \in E\right\}$, the subclass introduced and studied in [19].

5. $C(\mathcal{A},\mathcal{B}) = \left\{\frac{zf(\check{z})'}{g(\check{z})} \prec \frac{1+\mathcal{A}\check{z}}{1+\mathcal{B}\check{z}}, f \in \mathbb{A}, g \in S^*, -1 \le \mathcal{B} < \mathcal{A} \le 1\ and\ \check{z} \in E\right\}$, the subclass introduced and studied in [20].

6. $K_c'(\alpha) = \left\{\left(\frac{f(\check{z})'}{g(\check{z})'}\right) > \alpha, f \in \mathbb{A}, 0 \le \alpha < \mathcal{A} < 1, h \in C'\ and\ \check{z} \in E\right\}$, the subclass introduced and studied in [21].

7. $K_c'(\mathcal{A},\mathcal{B}) = \left\{\left(\frac{f(\check{z})'}{h(\check{z})'}\right) \prec \frac{1+\mathcal{A}\check{z}}{1+\mathcal{B}\check{z}}\alpha, f \in \mathbb{A}, h(z)' \in C', -1 \le \mathcal{B} < \mathcal{A} \le 1\ and\ \check{z} \in E\right\}$, the subclass introduced and studied in[22], in particular

$$K_c'(1-2\alpha,-1) = K_c'(\alpha).$$

8. $K_c'(\mathcal{A},\mathcal{B};C,D) = \left\{\left(\frac{f(\check{z})'}{h(\check{z})'}\right) \prec \frac{1+C\check{z}}{1+D\check{z}}, f \in \mathbb{A}, h \in C'(A,B), -1 \le D \le \mathcal{B} < \mathcal{A} \le C < 1\ and\ \check{z} \in E\right\}$, the subclass introduced and studied in [23].

In particular

$$K_c'(1,-1;C,D) = K_c'(C,D).$$

Based on these subclasses, we introduce the following subclasses of analytic functions using the generalized Sãlãgean operator and establish various results regarding coefficient bounds, distortion and argument theorems for these subclasses. Some well-known results follow as corollaries of our results.

Definition 3.1 let $K_c(\delta;p;\mathcal{A},\mathcal{B};C,D)$ denote the function $f\in\mathbb{A}_p$ and satisfying

$$\frac{D_\delta^p f(ž)}{žg(ž)'}\prec\frac{1+Cž}{1+Dž}\ \ where\ \ g(ž)=ž+\sum_{k=2}^{\infty}b_k\,ž^k\in C(\mathcal{A},\mathcal{B}).$$

Definition 3.2 let $K_c'(\delta;p;\mathcal{A},\mathcal{B};C,D)$ denote the function $f\in\mathbb{A}_p$ and satisfying

$$\frac{D_\delta^p f(ž)}{žh(z)'}\prec\frac{1+Cž}{1+Dž}\ \ where\ \ h(ž)=ž+\sum_{k=2}^{\infty}d_k\,ž^k\in C'(\mathcal{A},\mathcal{B}).$$

Definition 3.3 let $KCS^*(\delta;p;\mathcal{A},\mathcal{B};C,D)$ denote the function $f\in\mathbb{A}_p$ and satisfying

$$\frac{D_\delta^p f(ž)}{g(ž)}\prec\frac{1+Cž}{1+Dž}\ \ where\ \ g(ž)=ž+\sum_{k=2}^{\infty}b_k\,ž^k\in CS^*(\mathcal{A},\mathcal{B}).$$

Definition 3.4 let $KCS_1^*(\delta;p;\mathcal{A},\mathcal{B};C,D)$ denote the function $f\in\mathbb{A}_p$ and satisfying

$$\frac{D_\delta^p f(ž)}{h(ž)}\prec\frac{1+Cž}{1+Dž}\ \ where\ \ h(ž)=ž+\sum_{k=2}^{\infty}d_k\,ž^k\in CS_1^*(\mathcal{A},\mathcal{B}).$$

Now, we will use some lemmas

Lemma 3.5 [20]. let $g\in C(\mathcal{A},\mathcal{B})$, for $n\geq 2$, then $|b_n|\leq\frac{\mathcal{A}n+\mathcal{A}\mathcal{B}-\mathcal{A}-\mathcal{B}n}{2}$.

Lemma 3.6 [20]. Let $g\in C(\mathcal{A},\mathcal{B})$ for $|ž|=r<1$ then

$$\frac{1-\mathcal{A}r}{(1-\mathcal{B}r)(1+r)^2}\leq|g(ž)'|\leq\frac{1+\mathcal{A}r}{(1+\mathcal{B}r)(1-r)^2}$$

Lemma 3.7 [20]. Let $g\in C(\mathcal{A},\mathcal{B})$ for $|ž|=r<1$ then

$$|arg\,g(ž)'|\leq 2sin^{-1}r+sin^{-1}\frac{(\mathcal{A}-\mathcal{B})r}{1-\mathcal{A}\mathcal{B}r^2}.$$

Lemma 3.8 [18]. Let $g\in CS^*(\mathcal{A},\mathcal{B})$ for $|ž|=r<1$ then

$$\frac{r(1-\mathcal{A}r)}{(1-\mathcal{B}r)(1+r)^2}\leq|g(z)|\leq\frac{r(1+\mathcal{A}r)}{(1+\mathcal{B}r)(1-r)^2}$$

Lemma 3.9 [18]. Let $g\in CS^*(\mathcal{A},\mathcal{B})$ for $|ž|=r<1$ then

$$\left|arg\frac{g(ž)}{ž}\right|\leq 2sin^{-1}r+sin^{-1}\frac{(\mathcal{A}-\mathcal{B})r}{1-\mathcal{A}\mathcal{B}r^2}.$$

Theorem 3.10 if $f\in K_c(\delta;p;\mathcal{A},\mathcal{B};C,D)$ then $n\geq 2$,

$$|a_n|\leq\frac{1}{[1+(n-1)\delta]^p}\left[n+(C-D)+\frac{n^2(\mathcal{A}-\mathcal{B})-n(\mathcal{A}-\mathcal{B})}{2}+(C-D)\sum_{k=2}^{n-1}k\left(1+\frac{k(\mathcal{A}-\mathcal{B})-(\mathcal{A}-\mathcal{B})}{2}\right)\right]\quad(3.1)$$

The boundaries are clearly defined

Proof. Using definition 3.1. We get

$$D_\delta^p f(ž)=\left(\frac{1+Cw(ž)}{1+Dw(ž)}\right)žg(ž)',w\in U\quad(3.2)$$

Broadening our scope (3.2)

$$1+\sum_{k=2}^{\infty}[1+(k-1)\delta]^p\,a_kž^{k-1}=\left[1+\sum_{k=1}^{\infty}p_kž^k\right]\left[1+\sum_{k=2}^{\infty}kb_kž^{ž-1}\right].\quad(3.3)$$

in equation (3.3), and applying the triangle inequality in conjunction with Lemma 3.3, we get

$$[1+(n-1)\delta]^p|a_n|\leq n|b_n|+(C-D)\left[1+\sum_{k=2}^{n-1}k|b_k|\right].\quad(3.4)$$

Using lemma 3.5 the result (3.1), we get (3.4), but when $n=2$ equality holds in (1.8) for all $f_n$by the following expression

$$D_\delta^p f_n(ž)=\frac{ž}{(1-\delta_1ž)^2}\left(\frac{1+\mathcal{A}\delta_2ž^{n-1}}{1+\mathcal{B}\delta_2ž^{n-1}}\right)\left(\frac{1+C\delta_3ž^{n-1}}{1+D\delta_3ž^{n-1}}\right),|\delta_1|=|\delta_2|=|\delta_3|=1\quad(3.5)$$

substituting $\delta=1$ and $p=1$ in theorem above we get some corollaries [23].

Theorem 3.11 $f\in K_c'(\delta;p;\mathcal{A},\mathcal{B};C,D)$, then for $n\geq 2$

$$|a_n|\leq\frac{1}{[1+(n-1)\delta]^p}+\left[1+(n-1)(\mathcal{A}-\mathcal{B})+(C-D)+(C-D)\sum_{k=2}^{n-1}\big(1+(\mathcal{A}-\mathcal{B})(7k-1)\big)\right]\quad(3.6)$$

The bounds are sharp

Proof. with some proof of Theorem 2.9, the result of Theorem 3.10 follows

$$D_\delta^p f_n(ž)=\frac{ž}{(1-\delta_1ž)}\left(\frac{1+\mathcal{A}\delta_2ž^{n-1}}{1+\mathcal{B}\delta_2ž^{n-1}}\right)\left(\frac{1+C\delta_3ž^{n-1}}{1+D\delta_3ž^{n-1}}\right),|\delta_1|=|\delta_2|=|\delta_3|=1\quad(3.7)$$

corollary 311 if $f\in K_c'(\mathcal{A},\mathcal{B};C,D)$, then

$$|a_n|\leq\frac{1}{n}+\frac{(n-1)}{n}\left[(C-D)+(\mathcal{A}-\mathcal{B})+\frac{(\mathcal{A}-\mathcal{B})(C-D)(n-2)}{2}\right]$$

for $\mathcal{A} = \delta = p = 1$ and $\mathcal{B} = -1$, theorem 2.8we get

Corollary 3.12 if $f \in K_c'(C, D)$, then

$$|a_n| \le \frac{(2n-1) + (n-1)^2(C-D)}{n}$$

putting $\mathcal{A} = \delta = p = 1, \mathcal{B} = -1, C = 1 - 2\alpha$ and $D = -1$ leads the corollary [21].

Corollary 3.13 if $f \in K_c'(\alpha)$, then $|a_n| \le \frac{2}{n}(1-\alpha)(n-1)^2 + \frac{(2n-1)}{n}$.

Theorem 3.15 if $f \in KCS^*(\delta; p, \mathcal{A}, \mathcal{B}; C, D)$ for $n \ge 2$,

$$|a_n| \le \frac{1}{[1 + (n-1)\delta]^p}\left[n + \frac{(n-1)}{4}\{2n(\mathcal{A} - \mathcal{B}) + 4(C-D) + (\mathcal{A} - \mathcal{B})(C-D)(n-2)\}\right] \quad (3.8)$$

The bounds are sharp.

Proof. From theorem 3.10, for $n = 2$, holds (2.8) for function f in (2.7)

Theorem 3.14 if $f \in KCS_1^*(\delta; p, \mathcal{A}, \mathcal{B}; C, D)$ for $n \ge 2$,

$$|a_n| \le \frac{1}{[1 + (n-1)\delta]^p}\left[1 + (n-1)\left\{(\mathcal{A} - \mathcal{B}) + (C-D) + \frac{(\mathcal{A} - \mathcal{B})(C-D)(n-2)}{2}\right\}\right] \quad (3.9)$$

The bounds are sharp.

Proof. From theorem 3.10, the holds (3.9) we obtained for $n = 2$ for the functions f defined in (3.7).

Theorem 3.15 if $f \in K_c(\delta; p, \mathcal{A}, \mathcal{B}; C, D)$ for $|ž| = r \in (0,1)$, we get

$$\frac{r(1 - \mathcal{A}r)(1 - Cr)}{(1+r)^2(1 - \mathcal{B}r)(1 - Dr)} \le \left|D_\delta^p f(z)\right| \le \frac{r(1 + \mathcal{A}r)(1 + Cr)}{(1-r)^2(1 + \mathcal{B}r)(1 + Dr)} \quad (3.10)$$

The result is holds

Proof. From (3.2), we get

$$\left|D_\delta^p(f)\right| = \left|\frac{1 + Cw(ž)}{1 + Dw(ž)}\right| |z||g(z)'|, w \in U \quad (3.11)$$

Following the procedure of theorem (3.7) we have

$$\frac{1 - Cr}{1 - Dr} \le \left|\frac{1 + Cw(ž)}{1 + Dw(ž)}\right| \le \frac{1 + Cr}{1 + Dr} \quad (3.12)$$

Using (3.12) and lemma (3.11) in (3.10), the result (3.10) is clear sharpness functions f defined in (3.5)

Theorem 3.16 if $f \in K_c'(\delta; p, \mathcal{A}, \mathcal{B}; C, D)$ for $|ž| = r \in (0,1)$, we have

$$\frac{r(1 - Ar)(1 - Cr)}{(1+r)(1 - \mathcal{B}r)(1 - Dr)} \le \left|D_\delta^p f(z)\right| \le \frac{r(1 + \mathcal{A}r)(1 + Cr)}{(1-r)(1 + \mathcal{B}r)(1 + Dr)} \quad (3.13)$$

The result is sharp.

Proof. With the some proof of Theorem 3.14, the result (3.13) we get the functions f defined in (3.7).

putting $\delta = 1 = p$

Corollary 3.17 let $f \in K_c'(\mathcal{A}, \mathcal{B}; C, D)$ then

$$\frac{(1 - \mathcal{A}r)(1 - Cr)}{(1+r)(1 - \mathcal{B}r)(1 - Dr)} \le |f(z)'| \le \frac{(1 + \mathcal{A}r)(1 + Cr)}{(1-r)(1 + \mathcal{B}r)(1 + Dr)}$$

For $\delta = \mathcal{A} = p = 1$ and $\mathcal{B} = -1$, theorem 3.16 we get the corollary. [22].

Corollary 3.18 if $f \in K_c'(C, D)$ then

$$\frac{(1-r)(1 - Cr)}{(1+r)^2(1 - Dr)} \le |f(ž)'| \le \frac{(1+r)(1 + Cr)}{(1+r)^2(1 + Dr)}$$

putting For $\delta = \mathcal{A} = p = 1$ and $\mathcal{B} = -1 = D$ and $C = 1 - 2\alpha$ in theorem 2.16 follows: [21].

Corollary 3.19 if $f \in K_c'(\alpha)$ then

$$\frac{(1-r)(1 - (1 - 2\alpha)r)}{(1+r)^3} \le |f(z)'| \le \frac{(1+r)(1 + (1 - 2\alpha)r)}{(1-r)^3}$$

Theorem 3.20 if $f \in KCS^*(\delta; p; \mathcal{A}, \mathcal{B}; C, D)$ then for $|ž| = r \in (0,1)$ we have

$$\frac{r(1 - \mathcal{A}r)(1 - Cr)}{(1+r)^2(1 - \mathcal{B}r)(1 - Dr)} \le \left|D_\delta^p f(ž)\right| \le \frac{r(1 + \mathcal{A}r)(1 + Cr)}{(1-r)^2(1 + \mathcal{B}r)(1 + Dr)} \quad (3.14)$$

The result is sharp.

Proof. Using definition 3.3, we get

$$\left|D_\delta^p f(ž)\right| = \left|\frac{1 + Cw(ž)}{1 + Dw(ž)}\right| |g(ž)|, w \in U \quad (3.15)$$

Using (3.8), and lemma (3.8), in (3.15), the result (3.14) is obvious.

Sharpness follows for the functions define in (3.5).

Theorem 3.21 if $f \in KCS_1^*(\delta; p; \mathcal{A}, \mathcal{B}; C, D)$ then for $|ž| = r \in (0,1)$ we have

$$\frac{r(1 - \mathcal{A}r)(1 - Cr)}{(1+r)(1 - \mathcal{B}r)(1 - Dr)} \le \left|D_\delta^p f(z)\right| \le \frac{r(1 + \mathcal{A}r)(1 + Cr)}{(1-r)(1 + Br)(1 + Dr)}$$

The result is sharp.
Proof. With the same proof (3.20), Sharpness follows for the functions define in (3.7).
Theorem 3.22 if $f \in K_c(\delta; p; \mathcal{A}, \mathcal{B}; C, D)$ then

$$\left|arg\frac{D_\delta^p f(\check{z})}{\check{z}}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right) + \sin^{-1}\left(\frac{(\mathcal{A}-\mathcal{B})r}{1-\mathcal{A}\mathcal{B}r^2}\right) + 2\sin^{-1}r \tag{3.16}$$

The result is sharp
Proof. From (3.2) we get

$$\left|arg\frac{D_\delta^p f(\check{z})}{\check{z}}\right| = \left|arg\left(\frac{1+Cw(\check{z})}{1+Dw(\check{z})}\right)\right| + |arg\, g(\check{z})'| \tag{3.17}$$

It has been proved in [69] that

$$\left|arg\left(\frac{1+Cw(\check{z})}{1+Dw(\check{z})}\right)\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right) \tag{3.18}$$

Using (3.18), and lemma (3.7), in (3.17), the result (3.16), follows.
Where f defined in (3.5) and the result is sharp.
Theorem 3.22 if $f \in K_c'(\delta; p; \mathcal{A}, \mathcal{B}; C, D)$, such that

$$\left|arg\frac{D_\delta^p f(\check{z})}{\check{z}}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right) + \sin^{-1}\left(\frac{(\mathcal{A}-\mathcal{B})r}{1-\mathcal{A}\mathcal{B}r^2}\right) + 2\sin^{-1}r$$

Then the result is sharp.
Proof. With the same proof (3.21).
Theorem 3.23 if $f \in KCS^*(\delta; p; \mathcal{A}, \mathcal{B}; C, D)$, such that

$$\left|arg\frac{D_\delta^p f(\check{z})}{\check{z}}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right) + \sin^{-1}\left(\frac{(\mathcal{A}-\mathcal{B})r}{1-\mathcal{A}\mathcal{B}r^2}\right) + 2\sin^{-1}r \tag{3.19}$$

Then the result is sharp.
Proof. Using definition (3.4) we get

$$\frac{D_\delta^p f(\check{z})}{\check{z}} = \left(\frac{1+Cw(\check{z})}{1+Dw(\check{z})}\right)\frac{g(\check{z})}{\check{z}} \tag{3.20}$$

Which implies

$$\left|arg\frac{D_\delta^p f(\check{z})}{\check{z}}\right| = \left|arg\left(\frac{1+Cw(\check{z})}{1+Dw(\check{z})}\right)\right| + \left|arg\frac{g(\check{z})}{\check{z}}\right| \tag{3.21}$$

On using (3.18), lemma 3.10 in (3.21) we obtain the result (3.19)
Sharpness follows for the functions defined in (3.5).
Theorem 3.24 if $f \in KCS_1^*(\delta; p; \mathcal{A}, \mathcal{B}; C, D)$, such that

$$\left|arg\frac{D_\delta^p f(\check{z})}{\check{z}}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right) + \sin^{-1}\left(\frac{(\mathcal{A}-\mathcal{B})r}{1-\mathcal{A}\mathcal{B}r^2}\right) + 2\sin^{-1}r \tag{3.22}$$

Then the result is sharp.
Proof. With the same proof (3.23), and the result is sharp function in f in (3.7)

5. CONCLUSION

This study shows that the investigation of partial classes of analytic functions close to single- and multi-valued functions allows the derivation of precise coefficient limits, distortion theorems, containment relations, and concavity radii. The introduction of the general operator also provides a framework for generating new classes with sharp results from which known results can be retrieved as special cases, thus strengthening the theoretical structure of complex analysis and expanding its applications.

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