Weakly Completely Prime Graph

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Abstract

In this paper, we presented a new properties of a weakly completely prime graph, where we added the algebraic properties in this graph and extracted from them some important theories and results, with the study of homomorphism in this graph when homomorphism is available in near rings, as well as the definition of homomorphism preserve vertex cover, with the proving of a number of theorems and supported relationships with examples.

Keywords: weakly c-prime ideal, subtractive ideal, nilpotent, vertex cover, preserve vertex cover, homomorphism of graphs.

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1.Introduction

Since 1974, the trend began towards writing papers with the specialization of Algebra graph theory [1-2], pilz [3] is introduced the basic concepts and relationships in the near-ring theory and is considered one of the founders of this discipline.
Introduce, Al-swidi and Omran [4] graph is called weakly completely prime graph $(W_{\mathcal{A}}(N))$ of a near ring N as the ideal $\{0\} \neq \mathcal{A}$ of near ring with the vertices of $(W_{\mathcal{A}}(N))$ are elements of N and the distinct pair a and b of vertices are adjacent iff $0 \neq a.b \in \mathcal{A}$ or $0 \neq b.a \in \mathcal{A}, \forall a, b \in N$,which depend on define (Anderson and Smith [5]) of weakly c-prime (w-comp) ideal of the ring.
In this part, we'll go over some important concepts.
(Groenewald [6]) Let the ideal $\mathcal{A}$ of a ring R. Is called c-prime ideal (completely prime ideal), if for $a.b\in\mathcal{A}$ then $a\in\mathcal{A}$ or $b\in\mathcal{A}$,for $a, b\in R$, and is c-semi prime ideal whenever $a = b$.
(pilz[3]) A mapping $\psi : (N_1,+, .) \rightarrow (N_2,'+','.')$ is called a near ring homomorphism if $\psi(a+b)=\psi(a)'+'\psi(b)$ and $\psi(a.b)=\psi(a)'.'\psi(b)$, for $a,b\in N_1$.
(pilz[3]) A near ring homomorphism is a mapping $f : (N_1,+, .) \rightarrow (N_2, '+', '.')$ is called endomorphism if $N_1 = N_2$.
(pilz[3]) A near ring homomorphism is called epimorphism if is a surjective(onto) homomorphism.
(Loomis [7]) Let $\emptyset\neq K\subset G$ a mapping $\gamma:K\rightarrow G$ is called inclusion mapping if and only if $\gamma(a)=a$ for all $a\in K$.
(Vasudev[8],Deo[9]) A graph G with sets of vertices V and the number of edges E, and two vertices if they have same ends of edge then are adjacent vertices. A walk in the graph is an alternative sequences of vertices and edges with the repetition of

vertices and edges are allowed and if initial and end vertices are different then is called open walk otherwise is called closed and the number of edges in walk is define a length. A Path in the graph G is an open walk without repetition of vertices and is denoted by P_k of length k-1 is a sequence of distinct edges $v_1v_2, v_2v_3, \dots\dots, v_{k-1}v_k$. A Cycle (Circuit) is a closed walk without repetition of vertices. Thus, the degree of each vertices of a cycle graph is two, and denoted by C_k. A subset W of V is said vertex cover of the graph G if the edges of the graph G has one end vertex in W. The degree(deg) of vertex is represent the number of incident edges in this vertex . the graph is called connected whenever every vertices in the graph there is a path, otherwise the graph is disconnected. the length of the shortest path between the vertices a_1, a_2 in G is called the distance between $a_1$and a_2 and denoted by $d(a_1, a_2)$. If two adjacent vertices have no same color then is called a proper coloring of G such that, the $\chi(G)$ is chromatic number of the minimum number of a proper colors which is needed to coloring the graph .A Homomorphism of graphs K to G, written as
φ : K → H is a mapping φ : V (K) → V (G) such that $\overline{\varphi(v1)\varphi(v2)}$ ∈ E(G), whenever $\overline{v_1v_22}$ ∈ E(K).

2. Main results

In this part we will present some new results with proving propositions and theorem, as well as finding relationships with homomorphism in the third part of the paper.

Definition 2.1. A subtractive ideal $\mathcal{A}$ of N is an ideal such that if $x, x + y \in \mathcal{A}$ then $y \in \mathcal{A}$.

Theorem 2.2. Let the proper subtractive ideal $\mathcal{A} \neq \{0\}$ be a w-comp ideal of N. If $\mathcal{A}$ is not c-prime ideal with $u.v = 0$ for some $u, v \notin \mathcal{A}$.Then $u.\mathcal{A} = \mathcal{A}.u = \{0\}$.

proof.

Assume that $u.\mathcal{A} \neq \{0\}$, then for some $a \in \mathcal{A}$, $u.a \neq 0$ and $0 \neq u.(u + a) \in \mathcal{A}$. As $\mathcal{A}$ is w-comp ideal, so that $u \in \mathcal{A}$ or $u + a \in \mathcal{A}$, then $u \in \mathcal{A}$, a contradiction. Therefore $u.\mathcal{A} = 0$ and in the same way, show that $\mathcal{A}.u = 0$.

Now, assume that, u. $\mathcal{A} \neq \{0\}$, now to show that $\mathcal{A}$ is c-prime ideal, let u.a∈ $\mathcal{A}$, then u.a≠0 for all a∈ $\mathcal{A}$, as u.a∈ $\mathcal{A}$ then 0≠u.a∈ $\mathcal{A}$, since $\mathcal{A}$ is w-comp ideal, therefore u∈ $\mathcal{A}$ or a∈ $\mathcal{A}$. Then $\mathcal{A}$ is c-prime ideal and in the same way to show that $\mathcal{A}$.u≠{0}.

□

Proposition 2.3. Let the ideal $\mathcal{A} \neq \{0\}$ be a w-comp ideal of N then the ideal $\mathcal{A}$ is a vertex cover of $W_{\mathcal{A}}(N)$ whenever a vertices of N\\$\mathcal{A}$ is not isolated in $W_{\mathcal{A}}(N)$.

Proof.

Let $\overline{ab} \in E(W_{\mathcal{A}}(N))$, then 0≠a.b∈ $\mathcal{A}$ or 0≠b.a∈ $\mathcal{A}$, since $\mathcal{A}$ is w-comp ideal, then a ∈ $\mathcal{A}$ or b ∈ $\mathcal{A}$, therefore the ideal $\mathcal{A}$ is a vertex cover of $W_{\mathcal{A}}(N)$. And if a≠b∈N\ $\mathcal{A}$ such that $\overline{ab} \in E(W_{\mathcal{A}}(N))$, then a.b∈ $\mathcal{A}$ or b.a∈ $\mathcal{A}$, without loss of generality let

$a.b \in \mathcal{A}$, so that a$\in \mathcal{A}$ or $b \in \mathcal{A}$, a contradiction, therefore the ideal $\mathcal{A}$ is a vertex cover of $W_{\mathcal{A}}(N)$. □

Example 2.4. Let $N = \{0,1,2,3,4,5,6,7\}$ be a $n$ear ring defined in Table 1. And let $\mathcal{A} = \{0,2,5,7\}$ be an ideal of N.

***TABLE* 1.** *Multiplication and Addition table of* $N = \{0,1,2,3,4,5,6,7\}$

+	0	1	2	3	4	5	6	7		*	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7		0	0	0	0	0	0	0	0	0
1	1	2	3	0	5	6	7	4		1	0	1	2	3	4	5	6	7
2	2	3	0	1	6	7	4	5		2	0	2	0	2	0	0	0	0
3	3	0	1	2	7	4	5	6		3	0	3	2	1	4	5	6	7
4	4	7	6	5	0	3	2	1		4	0	4	2	6	4	0	6	2
5	5	4	7	6	1	0	3	2		5	0	5	0	5	0	5	0	5
6	6	5	4	7	2	1	0	3		6	0	6	2	4	4	0	6	2
7	7	6	5	4	3	2	1	0		7	0	7	0	7	0	5	0	5

and is w-comp ideal of N then is a vertex cover of $W_{\mathcal{A}}(N)$,which is illustrate in Fig 1.

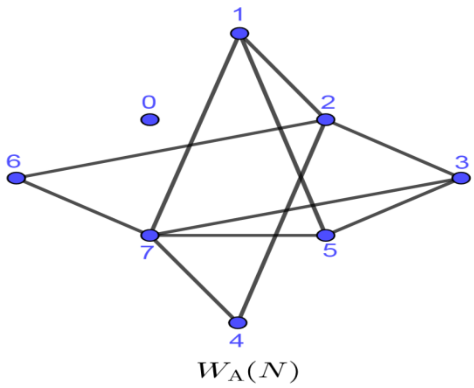


Figure 1. Graph of $W_{\mathcal{A}}(N)$

Proposition 2.5. Let the ideal $\mathcal{A} \neq \{0\}$ be a w- c-semiprime ideal of N and the ideal $\mathcal{A}$ is a vertex cover of $W_{\mathcal{A}}(N)$, then $\mathcal{A}$ is a w-comp ideal of N.

Proof.

Let x,y$\in$N and $0 \neq x.y \in \mathcal{A}$, therefore if x=y so we get $x \in \mathcal{A}$ as $\mathcal{A}$ is w- c-semiprime ideal of N. Now if $x \neq y$ with $x,y \in N \setminus \mathcal{A}$ as $\mathcal{A}$ is a vertex cover of $W_{\mathcal{A}}(N)$, therefore $\overline{xy} \notin E(W_{\mathcal{A}}(N))$, then $x.y \notin \mathcal{A}$ and $y.x \notin \mathcal{A}$, a contradiction as $x.y \in \mathcal{A}$. Then $\mathcal{A}$ is a w-comp ideal of N. □

Definition 2.6. The element $a \in N$ is called nilpotent if $a^n = 0$ and the set of all nilpotent in N is denoted by nilp(N).

Proposition 2.7. Let the ideal $\mathcal{A} \neq \{0\}$ be a w-comp ideal of a commutative near ring N and $a \in nilp(N)$ then either $deg(a) \leq deg(b) \leq n-1$ for $b \in \mathcal{A}$ or a is isolated vertex in $W_{\mathcal{A}}(N)$.

Proof.

From Theorem(2.2), we get $a \in \mathcal{A}$ or $a.\mathcal{A} = \{0\}$, then if $a \in \mathcal{A}$ and from(Proposition (2.13),[4]) we get $deg(a)=\Delta\big(W_I(N)\big) \leq n-1$ or $a.\mathcal{A} = \{0\}$ then from definition of $W_{\mathcal{A}}(N)$, we get a is isolated vertex. Now if $a \notin \mathcal{A}$ and $a.\mathcal{A} \neq \{0\}$ from this we get $a \neq 0$, then from (Proposition (2.13),[4]),we get $deg(a) \leq deg(b) \leq n-1$ for $b \in \mathcal{A}$. □

3.Homomorphism of $W_I(N)$ and Homomorphism near ring

Theorem 3.1. Let the ideal $\mathcal{A} \neq \{0\}$ of N_1 with $\psi: N_1 \to N_2$ be a epimorphism .Then $\psi: W_{\mathcal{A}}(N_1) \to W_{\psi(\mathcal{A})}(N_2)$ is a graph homomorphism .

Proof.

Let $x,y \in N_1$ with $\overline{xy} \in E(W_{\mathcal{A}}(N_1))$, therefor x and y are adjacent in $W_{\mathcal{A}}(N)$, thus $0 \neq x.y \in \mathcal{A}$, that mean $0_{N_1} \neq x.y \in \mathcal{A}$ so $\psi(0_{N_1}) \neq \psi(x.y) \in \psi(\mathcal{A})$ then $\psi(0_{N_1}) \neq \psi(x).\psi(y) \in \psi(\mathcal{A})$, as ψ is a surjective near ring homomorphism so that $\psi(0_{N_1}) = 0_{N_2}$, for this we get , $0_{N_2} \neq \psi(x).\psi(y) \in \psi(\mathcal{A})$, this mean $\overline{\psi(x)\psi(y)} \in E\left(W_{\psi(I)}(N_2)\right)$.

Therefore ψ is a graph homomorphism. □

Remark 3.2. It is not necessary the homomorphism image of a w-comp (respect. c-prime) ideal is a w-comp (respect. c-prime) ideal.

Theorem 3.3. Let the ideal $\mathcal{A} \neq \{0\}$ of N_1 with $\psi: N_1 \to N_2$ be a epimorphism, a mapping $\psi: W_{\mathcal{A}}(N_1) \to W_{\psi(\mathcal{A})}(N_2)$ is a graph homomorphism if and only if

1- P_k is a path in $W_{\mathcal{A}}(N_1)$ then $\psi(0), \psi(1), \ldots \psi(k)$ is a walk in $W_{\psi(\mathcal{A})}(N_2)$.

2-C_k is a cyclic in $W_{\mathcal{A}}(N_1)$ then $\psi(0), \psi(1), \ldots \psi(k)$ is a closed walk in $W_{\psi(\mathcal{A})}(N_2)$.

Proof.

1-Suppose ψ is a graph homomorphism with the path $0,1,2,\ldots,k$ in $W_{\mathcal{A}}(N_1)$, then from Theorem(3.1), as every edges in the path is same as in $W_{\psi(\mathcal{A})}(N_2)$, so that $\psi(0), \psi(1), \ldots \psi(k)$is a walk(with repetition the vertices or not) in $W_{\psi(\mathcal{A})}(N_2)$.

Conversely, same proof as every edges in the path is same as in $W_{\psi(\mathcal{A})}(N_2)$ so we get $W_{\mathcal{A}}(N_1)$ is homomorphism with $W_{\psi(\mathcal{A})}(N_2)$.

2- Same proof (1) . □

Remark 3.4.
1-The Path P_L has L vertices and L-1 edges.
2-The Cyclic C_L has L vertices and L edges

Corollary 3.5. Let the ideal $\mathcal{A} \neq \{0\}$ of N_1 with $\psi:N_1 \to N_2$ be a epimorphism, if $\psi:W_{\mathcal{A}}(N_1) \to W_{\psi(\mathcal{A})}(N_2)$ is a graph homomorphism , then for any two vertices $d_{W_{\psi(\mathcal{A})}(N_2)}(\psi(a), \psi(b)) \leq d_{W_{\mathcal{A}}(N_1)}(a, b)$, $a,b \in V(W_{\mathcal{A}}(N_1))$.

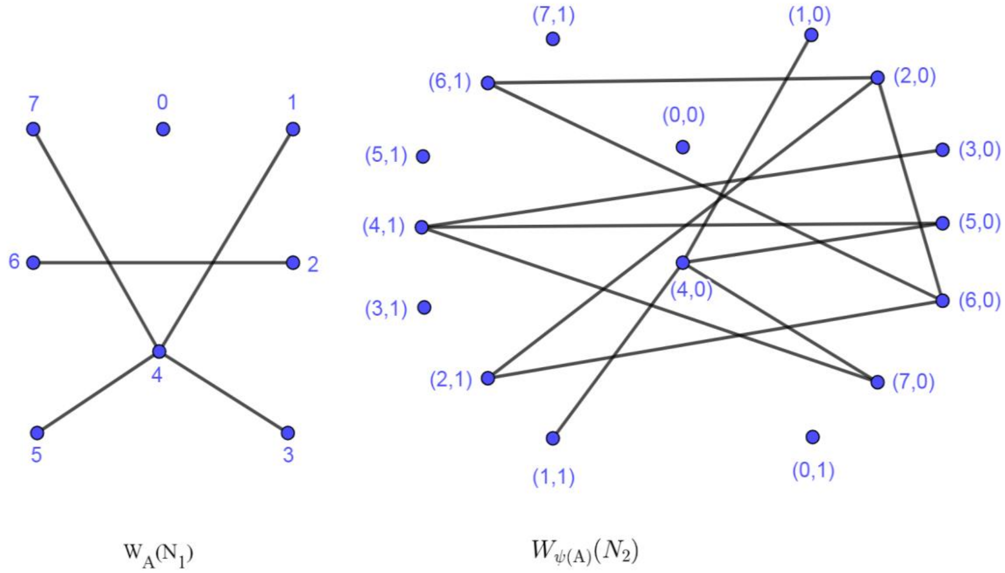
Proof.
If a=0,1,….k=b is a path in $W_{\mathcal{A}}(N_1)$ then by Theorem(3.3), we get ψ(0), ψ(1),…. ψ(k)is a walk in $W_{\psi(\mathcal{A})}(N_2)$ with same length k. Since every walk from ψ(a) to ψ(b) contains a path from ψ(a) to ψ(b), so that $d_{W_{\psi(\mathcal{A})}(N_2)}(\psi(a), \psi(b)) \leq d_{W_{\mathcal{A}}(N_1)}(a, b)$.
□

Observation 3.6. The chromatic graphs of a graph homomorphism for: $\psi:W_{\mathcal{A}}(N) \to K_n$,are precisely n colors.

Theorem 3.7. Let the ideal $\mathcal{A} \neq \{0\}$ of N_1 with $\psi:N_1 \to N_2$ be a near ring homomorphism .Then $\chi(W_{\mathcal{A}}(N_1)) \leq \chi(W_{\psi(\mathcal{A})}(N_2))$.If one of the following holds:
1-ψ is inclusion mapping.
2-$W_{\psi(\mathcal{A})}(N_2)$ is a homomorphism to a complete graph.

Proof.
1-As ψ is inclusion mapping then N_1 is a subset of N_2 and ψ(x)=x for every $x \in N_1$,let $x \neq y \in N_1$ then by Theorem(3.1), so we get $\overline{\psi(x)\psi(y)} \in E(W_{\psi(\mathcal{A})}(N_2))$,whenever $\overline{xy} \in E(W_{\mathcal{A}}(N_1))$ and $y \in N_1$, so that every vertices are adjacent in $W_{\mathcal{A}}(N_1)$ the images are adjacent in $W_{\psi(\mathcal{A})}(N_2)$ as well as the vertices are in N_2 but not in N_1 may give extra adjacent, therefore $\chi(W_{\mathcal{A}}(N_1)) \leq \chi(W_{\psi(\mathcal{A})}(N_2))$.
2-Let g: $W_{\psi(\mathcal{A})}(N_2) \to K_n$ as $\psi:W_{\mathcal{A}}(N_1) \to W_{\psi(I)}(N_2)$, so by composition of homomorphism we get $\psi:W_{\mathcal{A}}(N_1) \to K_n$, therefore $\chi(W_{\mathcal{A}}(N_1)) \leq \chi(W_{\psi(\mathcal{A})}(N_2))$. □

Example 3.8. Let ψ be a near ring homomorphism(inclusion mapping) from $N_1 = Z_8$ to $N_2 = Z_8 \times Z_2$ with ψ(a)=(a,0) then ψ is a graph homomorphism from $W_{\mathcal{A}}(N_1)$ to $W_{\psi(\mathcal{A})}(N_2)$, let $\mathcal{A}$ ={0,4} be an ideal of N_1 then ψ(I)={(0,0),(4,0} is an ideal of N_2 . The chromatic of graphs are $\chi(W_{\mathcal{A}}(Z_8)) = 2$ and $\chi(W_{\psi(\mathcal{A})}(Z_8 \times Z_2))$ =3 are shown in Fig 2.



from $N_1=Z_8$ to $N_2 = Z_8 \times Z_2$ Figure 2. Graph homomorphism of $W_{\mathcal{A}}(N)$

Remark 3.9. If ψ is endomorphism in above theorems, then

$$\chi\big(W_{\mathcal{A}}(N_1)\big) = \chi\left(W_{\psi(\mathcal{A})}(N_2)\right)$$

Definition 3.10. Let the ideal $\mathcal{A} \neq \{0\}$ of N_1 with $\psi: N_1 \rightarrow N_2$ be a near ring homomorphism is called preserve vertex cover of weakly completely prime ideal graph if $\mathcal{A}$ is a vertex cover of $W_{\mathcal{A}}(N_1)$ then $\psi(\mathcal{A})$ is a vertex cover of $W_{\psi(\mathcal{A})}(N_2)$.

Theorem 3.11. Let the ideal $\mathcal{A} \neq \{0\}$ be a w-comp ideal of N_1 with $\psi: N_1 \rightarrow N_2$ be an epimorphism. If ψ is preserve vertex cover of w-comp ideal graph and $\mathcal{A}$ is w-c-prime ideal of N_1 with $\psi(\mathcal{A})$ is w-c-semiprime ideal of N_2 then $\psi(\mathcal{A})$ is a w- c-prime ideal of N_2 .

Proof.

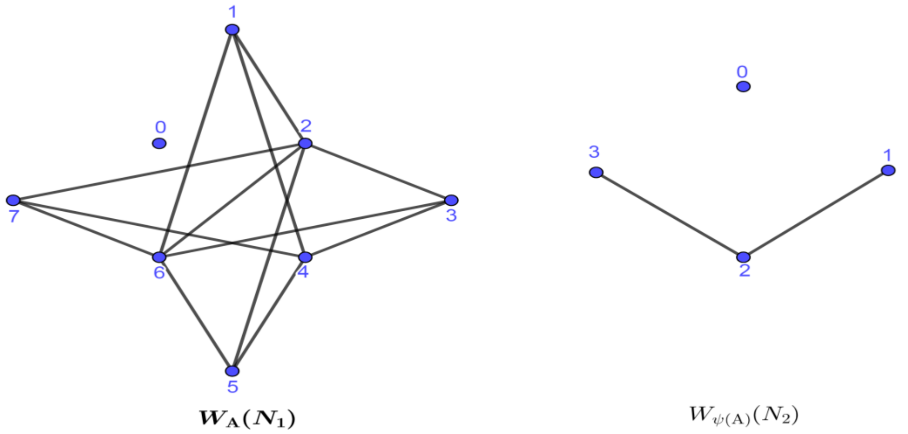
From Theorem(3.1), we get ψ is a graph homomorphism from $W_{\mathcal{A}}(N_1)$ to $W_{\psi(\mathcal{A})}(N_2)$. Let $\mathcal{A} = N_1$ be a w-comp ideal, therefore $\psi(\mathcal{A}) = \psi(N_1) = N_2$ be a w-comp ideal of N_2. Let $\mathcal{A} \subset N_1$ and $\overline{xy} \in E(W_{\psi(\mathcal{A})}(N_2))$. Then , $0 \neq x.y \in \psi(\mathcal{A})$, for every $x,y \in N_2$

Case1:if x=y , then $0 \neq x.x \in \psi(\mathcal{A})$ therefore $x \in \psi(\mathcal{A})$, so we get the required.

Case2:if $x \neq y$, as ψ is preserve vertex cover, therefore $\psi(\mathcal{A})$ is a vertex cover of N_2, then $x \in \psi(\mathcal{A})$ or $y \in \psi(\mathcal{A})$, so we get the required.

Now, as ψ is a surjective. Let x=ψ(x_1) for some $x_1 \in N_1$ and $0 \neq x_1.y_1 \in \mathcal{A}$ for every $y_1 \in N_1$, choose $y_1 \in N_1 \setminus \mathcal{A}$, as $\mathcal{A}$ is a w-comp ideal then $x_1 \in \mathcal{A}$, so we get ψ(x_1)∈ψ($\mathcal{A}$) therefore x∈ψ($\mathcal{A}$) and ψ($\mathcal{A}$) is a w-comp ideal of N_2. $\square$

Example 3.12. . let ψ be a near ring homomorphism from $N_1 = \frac{Z}{8Z}$ to $N_2 = \frac{Z}{4Z}$ with ψ(a+8Z)=a+4Z and ψ is preserve vertex cover of weakly completely prime ideal graph from $W_{\mathcal{A}}(N_1)$ to $W_{\psi(\mathcal{A})}(N_2)$,let $\mathcal{A}$ ={0+8Z,2+8Z,4+8Z,6+8Z}} is a w-comp ideal of N_1 so that ψ($\mathcal{A}$)={0+4Z,2+6Z} is a w-comp ideal of N_2.The graph of $W_{\mathcal{A}}(N_1)$ to $W_{\psi(\mathcal{A})}(N_2)$ are shown in Fig 3.



from $N_1 = \frac{Z}{8Z}$ to $N_2 = \frac{Z}{4Z}$ Figure 3. Graph homomorphism of $W_{\mathcal{A}}(N)$

4.Conclusion

We concluded through this paper that the minimum degree in graph $W_{\mathcal{A}}(N)$ is for elements belonging to nilpotent, also in the case of the presence of homomorphism between N_1 and N_2, the distance between any two elements in the image N_2 is less than in N_1 unlike the chromatic.

5.References

[1] Lin, Y. F., & Ratti, J. S. (1970). The graphs of semirings. Journal of Algebra, 14(1), 73-82.

[2] Ratti, J. S., & Lin, Y. F. (1971). The graphs of semirings II. Proceedings of the American Mathematical Society, 30(3), 473-478.

[3] Neuberger, J. W. (1978). Gunter Pilz, Near-rings, the theory and its applications. Bulletin of the American Mathematical Society, 84(5), 934-937.

[4] AL-Swidi, A. A., & Omran, A. (2022). New type graph with respect to ideal near ring. International Journal of Nonlinear Analysis and Applications.

[5] D. D. Anderson and E. Smith.(2003) .Weakly prime ideals, Houston J. no. 4, 831–840.

[6] N. J. Groenwald.(1988).The completely prime radical in near- rings. Acta Mathematica Hungarica, 51(3-4), 301-305.
[7] Loomis, L. H., & Sternberg, S. (1968). Advanced calculus. Department of Mathematics, Harvard University.
[8] C. Vasudev.(2006).Graph theory with applications, New Age International.
[9] N. Deo.(2017). Graph theory with applications to engineering and computer science,Courier Dover Publications.

Article submitted 1 February 2023. Published as resubmitted by the authors 12 March 2023.