Paracompactly closed map

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Abstract— In this paper, we have introduced the definition of paracompactly closed set and paracompactly closed map. The relationship between the aforementioned map and different types of paracompact map has been proven under certain conditions. Finally, we discussed the composition of the paracompactly closed map.

Keywords— Paracompactly closed map, Compact map and Paracompact map.

1 Introduction

In 1944, Dieudonn´e [1] defined a new space called paracompact space, which is more general than the compact space. In 1957, Halfar [2] invented the concept of compact map. Also, in 1993, G. L. Garg and Asha Goel [3] had known the countably compact map. In 1997, D. Buhagiar [4] interodused the notion of paracompact map. Moreover, certain types of maps using the term of paracompact as the paracompact map introduced by Saad [5] in 2022. The aim of this work is to define a new type of map by the concept of paracompactly closed and to give some theories on the subject.

In this paper, a space $\mathbb{W}$ means a topological space $(\mathbb{W}, \tau)$, by a map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$, we mean continuous surjection map $\mathcal{L}$ of a space $\mathbb{W}$ into a space $\mathbb{M}$.

2 Preliminaries

2.1 Definition [2]: A space $\mathbb{W}$ is said to be compact if every open cover of it has a finite subcover.
2.2 Definition [1]: A Hausdorff space $\mathbb{W}$ is said to be paracompact if any open cover of it has a locally-finite open refinement.
2.3 Definition [6]: A space $\mathbb{W}$ is said to be a countably compact if every countable open cover of it has a finite subcover.
2.4 Definition [7]: A space $\mathbb{W}$ is said to be a Lindelöf space if every open cover of $\mathbb{W}$ has a countable subcover.

2.5 Definition [2]: Let $\mathbb{W}$ and $\mathbb{M}$ be two spaces. A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is known as compact providing the pre-image of any compact set in $\mathbb{M}$ is compact in $\mathbb{W}$.

2.6 Definition [8]: A space $\mathbb{W}$ is called countably paracompact (some time called binormal) if every open countable covering has a locally finite open refinement.

2.7 Definition [9]: Let $\mathbb{W}$ and $\mathbb{M}$ be two spaces. A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is known as a countably compact providing the pre-image of any closed and countably compact set in $\mathbb{M}$ is countably compact in $\mathbb{W}$.

2.8 Definition [5]: A surjective continuous map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is said to be paracompact if the inverse image for any paracompact set in $\mathbb{M}$ is paracompact set in $\mathbb{W}$.

2.9 Definition [9] : A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is called countably paracompact providing the pre-image of any closed and countably paracompact set in $\mathbb{M}$ is countably paracompact in $\mathbb{W}$.

2.10 Theorem [10]: A closed subset of a Lindelöf space, is a Lindelöf subspace.

2.11 Theorem [11]: Every countably comp-act space is a countably paracompact space.

2.12 Theorem [8]: Every closed subspace of compact (res. paracompact, countably compact, countably paracompact) space is compact (res. paracompact, countably compact, countably paracompact).

2.13 Theorem [8]: Every compact space is paracompact.

2.14 Theorem [12]: Every compact subset of Hausdorff space is closed.

2.15 Theorem [13]: Every compact space is a Lindelof.

2.16 Theorem [11]: If a space $\mathbb{W}$ is a countably compact and paracompact, then it is compact.

2.17 Theorem [11]: Every countably paracompact and Lindelöf space is paracom-pact.

2.18 Theorem [14]: Let $\mathbb{W}$ be a space and A be a subset of $\mathbb{W}$, $x \in \mathbb{W}$. Then $x \in \overline{\mathrm{A}}$ if and only if there exists a net in A which converges to x.

2.19 Theorem [5]: Every proper map of $\mathbb{W}$ onto M is paracompact.

2.20 Theorem [5]: Let the map $\mathcal{L}$ of $\mathbb{W}$ onto $\mathbb{M}$ be an open and proper, then the image for any paracompact set in $\mathbb{W}$ is paracompact set in $\mathbb{M}$.

2.21 Theorem [11]: Every paracompact space is a countably paracompact.

2.22 Theorem [11]: Any compact space is a countably compact space.

3 Main results

3.1 Definition: A space $\mathbb{W}$ is said to be Pa-closed if every paracompact subset of $\mathbb{W}$ is closed.

3.2 Definition: Let $\mathbb{W}$ be a space, then $A \subseteq \mathbb{W}$ is said to be a paracompactly closed set if $A \cap K$ is paracompact, for every paracompact set K in $\mathbb{W}$.

3.3 Example: Every subset of space $\mathbb{W}$ under discrete topology is a paracompactly closed.

3.4 Theorem: Every closed subset of a compact and Pa-closed space $\mathbb{W}$ is a paracompactly closed.

Proof: Let A be a closed subset of $\mathbb{W}$ and let K be a paracompact in $\mathbb{W}$. K is closed due to $\mathbb{W}$ is Pa-closed space, thus $A \cap K$ is a closed set, Theorem 2.12 insists that A is a compact set in $\mathbb{W}$. This implies that $A \cap K$ is a paracompact set owing to Theorem 2.13. Hence, A is a paracompactly closed.

3.5 Theorem: Let $\mathbb{W}$ be a Pa-closed space. If $A \subseteq \mathbb{W}$ is paracompactly closed then it is a closed set.

Proof: Let A be a paracompactly closed set in $\mathbb{W}$ and $x \in \bar{A}$. By Theorem 2.18, there exist a net $(\chi_d)_{d \in D}$ in A, such that $\chi_d \to x$. Since The set $F = \{\chi_d, x\}$ is paracompact and A is paracompactly closed, then $A \cap F$ is paracompact set in $\mathbb{W}$. We have $\mathbb{W}$ is a Pa-closed space, then $A \cap F$ is closed due to Definition 3.1. Because $\chi_d \to x$ and $\chi_d \in A \cap F = \overline{A \cap F}$, therefore $x \in A \cap F$ by Theorem 2.18, so, $x \in A$, as result $\bar{A} \subseteq A$ and $A = \bar{A}$. Hence, A is a closed set. ■

3.6 Theorem: Let $\mathbb{W}$ be a compact and Pa-closed space. Then $A \subseteq \mathbb{W}$ is a paracompact set if and only if it is paracompactly closed.

Proof: Let A be a paracompact subset of $\mathbb{W}$. Since $\mathbb{W}$ is a Pa-closed space, then A is a closed subset of $\mathbb{W}$. From Theorem 3.4, A is a paracompactly closed. Conversely, assume that A is a paracompactly closed set. Theorem 3.5 insists that A is a closed subset of $\mathbb{W}$. Therefore, A is a compact set due to Theorem 2.12. Hence, By Theorem 2.13 A is a paracompact subspace of $\mathbb{W}$.■

3.7 Corollary: Let $\mathbb{W}$ be a compact and Pa-closed space. Then $A \subseteq \mathbb{W}$ is a compact if and only if it is paracompactly closed.

3.8 Theorem: If the injective map $\mathcal{L}$ of $\mathbb{W}$ onto $\mathbb{M}$ is an open and proper. Then A is a paracompactly closed set in $\mathbb{W}$ if and only if $\mathcal{L}(\mathrm{A})$ is a paracompactly closed set in $\mathbb{M}$.

Proof: Assume that A is a paracompactly closed subset of $\mathbb{W}$ and K is a paracompact in $\mathbb{M}$. Since $\mathcal{L}$ is proper map, then $\mathcal{L}$ is paracompact due to Theorem 2.19. So, $\mathcal{L}^{-1}(\mathrm{K})$ is a paracompact set in $\mathbb{W}$, which implies $A \cap \mathcal{L}^{-1}(\mathrm{K})$ is a paracompact set. Theorem 2.20 asserts that $\mathcal{L}(A \cap \mathcal{L}^{-1}(\mathrm{K})) = \mathcal{L}(\mathrm{A}) \cap K$ is a paracompact set. Therefore, $\mathcal{L}(\mathrm{A})$ is a paracompactly closed subset of $\mathbb{M}$. Conversely, assume that $\mathcal{L}(\mathrm{A})$ is a paracompactly closed set. To show that A is a paracompactly closed subset of $\mathbb{W}$. Let K be a paracompact in $\mathbb{W}$, so $\mathcal{L}(\mathrm{K})$ is a paracompact set in $\mathbb{M}$ owing to theorem 2.20 Then, $\mathcal{L}(\mathrm{A}) \cap \mathcal{L}(\mathrm{K})$ is paracompact because that $\mathcal{L}(\mathrm{A})$ is a paracompactly closed. So, $\mathcal{L}^{-1}(\mathcal{L}(\mathrm{A}) \cap \mathcal{L}(\mathrm{K})) = A \cap K$ is a paracom-pact subspace of $\mathbb{W}$ due to $\mathcal{L}$ is a paracompact map. Hence, A is a paraco-mpactly closed. ■

3.9 Definition: A map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is called paracompactly closed providing the pre-image of any paracompactly closed set in $\mathbb{M}$ is para-compactly closed in $\mathbb{W}$.

3.10 Example: Let $\mathbb{W}$ be any finite space and let $\mathbb{M}$ any space, then the map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is paracompactly closed.

3.11 Theorem: Every paracompact subset of Pa-closed compact space is compact.

Proof: From Definition 3.1 and Theorem 2. 12.

3.12 Theorem: Let $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed compact space, then the map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is paracompactly closed if and only if it is para-compact.

Proof: Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a paracompact (res. paracompactly closed) map and let K be a paracompactly closed (res. paracompact) subset of $\mathbb{M}$. Because $\mathbb{M}$ Pa-closed and compact space, K is paracompact (res. paracompactly closed) set in $\mathbb{M}$ due to Theorem 3.6. So, $\mathcal{L}^{-1}(K)$ is a paracompact (res. paracompactly closed) set in $\mathbb{W}$. Since $\mathbb{W}$ is Pa-closed and compact space, then $\mathcal{L}^{-1}(K)$ is a paracompactly closed (res. paracompact). Hence, $\mathcal{L}$ is a paracompactly closed (res. paracompact) map. ■

3.13 Corollary: Let $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed and compact space, then the map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is paracompactly closed if and only if it is compact.
Proof: By Corollary 3.7.

3.14 Corollary: Let $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed and compact space. If the map $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is paracompactly closed then it is a closed.
Proof: By Theorem 3.4 and Theorem 3.5.

3.15 Theorem: Let $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed and compact space, then the continuous image of any paracompactly closed in $\mathbb{W}$ is para-compactly closed in $\mathbb{M}$.
Proof: Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a continuous map where $\mathbb{W}$ and $\mathbb{M}$ be a Pa-closed and compact spaces. Assume that K is a paracompactly closed in $\mathbb{W}$. By Corollary 3.7 K is compact thus, $\mathcal{L}(K)$ is a compact set in $\mathbb{M}$ due to $\mathcal{L}$ is a continuous map. From Corollary 3.7 $\mathcal{L}(K)$ is a para-compactly closed. ■

3.16 Theorem: Every countably paracompact map of a Pa-closed com-pact space onto a Pa-closed compact space is paracompactly closed map.
Proof: Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a countably paracompact map such that $\mathbb{W}$ and $\mathbb{M}$ are a Pa-closed compact space. Assume that K is a paracompactly closed set in $\mathbb{M}$. So, K is a paracompact subspace of $\mathbb{M}$ due to Theorem 3.6 which implies K is a countably paraco-mpact subset of $\mathbb{M}$ by Theo-rem 2.21. Since $\mathbb{M}$ is a Pa-closed, then K is a closed subset of $\mathbb{M}$. Thus, $\mathcal{L}^{-1}(K)$ is a countably paracompact set in $\mathbb{W}$ because $\mathcal{L}$ is a countably

paracompact map. In addition $\mathcal{L}^{-1}(K)$ is closed in $\mathbb{W}$ owing to continuity of $\mathcal{L}$. Since $\mathbb{W}$ is a Lindelöf space, so Theorem 2.10 asserts that $\mathcal{L}^{-1}(K)$ is a Lindelöf subspace of $\mathbb{W}$. Then, Theorem 2.17 implies that $\mathcal{L}^{-1}(K)$ is a paracompact set thus, $\mathcal{L}^{-1}(K)$ is a paracompactly closed set in $\mathbb{W}$ by Theorem 3.6. Hence, $\mathcal{L}$ is a paracompactly closed map. ∎

3.17 Theorem: Every compact map of a Pa-closed compact space onto a Pa-closed compact space is paracompactly closed.

Proof: Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a compact map such that $\mathbb{W}$ and $\mathbb{M}$ are a Pa-closed compact space. Assume that K is a paracompactly closed set in $\mathbb{M}$. Consequently, K is a paracompactly closed subsets of $\mathbb{M}$ due to Corollary 3.7. Now, $\mathcal{L}^{-1}(K)$ is a compact set in $\mathbb{W}$ due to $\mathcal{L}$ is a compact map. Thus, $\mathcal{L}^{-1}(K)$ is a paracompactly closed set in $\mathbb{W}$ by Corollary 3.7. Hence, $\mathcal{L}$ is a paracom-pactly closed map. ∎

3.18 Theorem: Every countably compact map onto a Lindelöf and countably compact space is countably paracompact map.

Proof: Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a countably compact map. Assume that K is a closed and countably paracompact set in $\mathbb{M}$. Since $\mathbb{M}$ is a Lindelöf space, then K is a Lindelöf subspace of $\mathbb{M}$ by Theorem 2.10. So, K is a paracompact subset of $\mathbb{M}$ owing to Theorem 2.17. Since $\mathbb{M}$ is a countably compact space and K is closed in $\mathbb{M}$. Then, K is a countably compact by Theorem 2.12. Now, Theorem 2.16 asserts that K is compact, and so, Theorem 2.22 justifies that K is a countably compact subset of $\mathbb{M}$. Thus, $\mathcal{L}^{-1}(K)$ is a countably compact set in $\mathbb{W}$ because $\mathcal{L}$ is a countably compact map, and so $\mathcal{L}^{-1}(K)$ is contably paracompact in $\mathbb{W}$ due to Theorem 2.11. Hence, $\mathcal{L}$ is a countably paracompact map. ∎

3.19 Theorem: let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be closed map where $\mathbb{W}$ is a Pa-closed space and $\mathbb{M}$ is a Pa-closed compact space. Then, the image of any paracompactly closed set in $\mathbb{W}$ is parcompactly closed in $\mathbb{M}$.

Proof: Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a closed map where $\mathbb{W}$ is a Pa-closed space and $\mathbb{M}$ is a Pa-closed compact space. Suppose that K is a paracompactly closed set in $\mathbb{W}$. By Theorem 3.5 K is closed, thus, $\mathcal{L}(K)$ is a closed set in $\mathbb{M}$ due to $\mathcal{L}$ is a closed map. From Theorem 3.4 $\mathcal{L}(K)$ is a para-compactly closed. ■

3.20 Theorem: The composition of paraco-mpactly closed maps is also a paracompactly closed map.
Proof: Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ and $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ be two paracompactly closed maps. To show that $\mathcal{J} \circ \mathcal{L}$ is also a paracompactly map. Assume that K is a paracompactly closed set in $\mathbb{E}$, For demonstrating that $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a paracompactly closed set in $\mathbb{W}$. We have $\mathcal{J}^{-1}(\mathrm{K})$ is a paracompactly closed set in $\mathbb{M}$ since $\mathcal{J}$ is a paracompact map. Thus, $\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathrm{K}))$ is a paracompactly closed set in $\mathbb{W}$ due to, $\mathcal{L}$ is a para-compactly closed map, but $\mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathrm{K})) = (\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$. So, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is a paracompactly closed set in $\mathbb{W}$. Hence, $\mathcal{J} \circ \mathcal{L}$ is par-acompactly closed. ■

3.21 Theorem: Let $\mathbb{M}$ be a Pa-closed space and $\mathbb{E}$ is a Pa-closed com-pact space. If $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \rightarrow \mathbb{E}$ is a paracompactly closed map and $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is a closed injective map, then $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a paracompactly closed map.
Proof: Assume that K a paracompactly closed set in $\mathbb{M}$. Since $\mathbb{M}$ is a Pa-closed space and $\mathbb{E}$ is a Pa-closed compact space , then $\mathcal{J}(\mathrm{K})$ is par-acompactly closed subspace of $\mathbb{E}$ due to Theorem 3.19. Thus, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathcal{J}(\mathrm{K}))$ is a paracompactly closed set in $\mathbb{W}$ because of $\mathcal{J} \circ \mathcal{L}$ is a paracompactly closed map. Therefore, $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathcal{J}(\mathrm{K})) = \mathcal{L}^{-1}(\mathcal{J}^{-1}(\mathcal{J}(\mathrm{K}))) = \mathcal{L}^{-1}(\mathrm{K})$ is a paracompactly closed subspace of $\mathbb{W}$. Hence, $\mathcal{L}$ is a paracompactly closed map. ■

3.22 Theorem: Let $\mathbb{W}$ be a Pa-closed space and $\mathbb{M}$ a Pa-closed com-pact space. If $\mathcal{J} \circ \mathcal{L} : \mathbb{W} \rightarrow \mathbb{E}$ is a paracompactly closed map and $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ is a closed surjective map, then $\mathcal{J}: \mathbb{M} \rightarrow \mathbb{E}$ is a paracompactly closed map.
Proof: Suppose that K a paracompactly closed set in $\mathbb{E}$. Since $\mathcal{J} \circ \mathcal{L}$ is a paracompactly closed map then $(\mathcal{J} \circ \mathcal{L})^{-1}(\mathrm{K})$ is paracompactly closed subspace of $\mathbb{W}$. But $\mathcal{L}$ is a surjective closed map then, $\mathcal{L}(\mathcal{J} \circ$

$\mathcal{L}\,)^{-1}(\mathrm{K}) = \mathcal{L}\left(\mathcal{L}^{-1}\left(\mathcal{J}^{-1}(\mathrm{K})\right)\right) = \mathcal{J}^{-1}(\mathrm{K})$ is paracompactly closed in $\mathbb{M}$ by Theorem 3.19. Hence, $\mathcal{J}$ is a paracompactly closed map. ■

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