Restrict Nearly Primary Submodules

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Abstract: In this work, all rings under consideration will be Supposedly to be commutative with nonzero identity and all modules will be nonzero unital. this work we investigate some properties of Restrict Nearly primary submodule as new generalization of primary sub module and strong form of Approximaitly primary submodule and stablished some sufficient conditions on Restrict Nearly primary to be primary. Also, we give necessarily condition on Approximaitly primary to be Restrict Nearly primary. In addition to we give several characterizations of Restrict Nearly primary submodule in class of multiplication modules.

Keywords: Restrict Nearly Primary submodules, Primary submodules, Approximaitly primary sub modules.

1. Introduction

A proper submodule $\mathcal{A}$ of an $\mathcal{R}$-module $\mathcal{M}$ is named a primary if whenever $\mathcal{S}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{M}$ it means that $\omega \in \mathcal{A}$ or $\mathcal{S} \in \sqrt{[\mathcal{A}:\mathcal{M}]}$**,** for some positive integerκ[8]. Newly primary submodules have been generalized to Nearly and Approximaitly primary submodule see [1], [6]. In this paper we say that $\mathcal{A}$ is Restrict Nearly Primary(RNPr), if whenever $\mathcal{S}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{M}$ it means that $\omega \in \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S} \in \sqrt{\left[\left(\mathcal{A} + \left(Soc(\mathcal{M}) \cap J(\mathcal{M})\right)\right) :_{\mathcal{R}} \mathcal{M}\right]}$, that is $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ for some positive integer κ. And an ideal $\mathcal{J}$ of a ring $\mathcal{R}$ is named a RNPr ideal of $\mathcal{R}$, iff $\mathcal{J}$ is a RNPr submodule of an $\mathcal{R}$-module $\mathcal{R}$.. Recall that an $\mathcal{R}$-module $\mathcal{M}$ is faithful if $ann(\mathcal{M}) = (0)$ [9], and an $\mathcal{R}$-module $\mathcal{M}$ is regular iff for each $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{M}$, $\exists\, t \in \mathcal{R}$ such that $\mathcal{S}t\mathcal{S}\omega = \mathcal{S}\omega$[9]. Recall that an $\mathcal{R}$-module $\mathcal{M}$ semisimple iff $Soc(\mathcal{M}) = \mathcal{M}$[2]. An $\mathcal{R}$-module $\mathcal{M}$ is multiplcation if every submodule $\mathcal{A}$ of $\mathcal{M}$ is of the form $\mathcal{A}=I\mathcal{M}$ for some ideal I of $\mathcal{R}$, this is equivalently to $\mathcal{M}$ is multiplcation if $\mathcal{A}$=[$\mathcal{A}$:$_{\mathcal{R}}$ $\mathcal{M}$]$\mathcal{M}$ [3]. An $\mathcal{R}$-module $\mathcal{M}$ is nonsingular if Z($\mathcal{M}$)=(0), where Z($\mathcal{M}$)={ ω ∈ $\mathcal{M}$: ωI=(0), for some essential ideal I of $\mathcal{R}$}[5]. Recall that an $\mathcal{R}$-module $\mathcal{M}$ is named to be content module if $(\bigcap_{i\in I} A_i)\,\mathcal{M} = \bigcap_{i\in I} A_i\,\mathcal{M}$ for each family of ideals A_i in $\mathcal{R}$ [8]. We denoted to the submodule by $\mathcal{S}$-$m$odule and $\mathcal{R}$-module by $\mathcal{R}$-modul.

2. Restrict Nearly primary $\mathcal{S}$-$m$odules

We introduce the following:

Definition 2.1

A proper $\mathcal{S}$-$m$odule $\mathcal{A}$ of an $\mathcal{R}$-modul $\mathcal{M}$ is named a Restrict Nearly Primary (RNPr) $\mathcal{S}$-$m$odule of $\mathcal{M}$, if whenever $\mathcal{S}$ω ∈ $\mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, ω ∈ $\mathcal{M}$ it means that ω ∈ $\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S} \in \sqrt{\left[\left(\mathcal{A} + \left(Soc(\mathcal{M}) \cap J(\mathcal{M})\right)\right) :_{\mathcal{R}} \mathcal{M}\right]}$,that is $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$, for some positive integer κ.And an ideal $\mathcal{J}$ of a ring $\mathcal{R}$ is named a RNPr ideal of $\mathcal{R}$, iff $\mathcal{J}$ is a RNPr $\mathcal{S}$-$m$odule of an $\mathcal{R}$-modul $\mathcal{R}$.

Remark 2.2

Every primary $\mathcal{S}$-$m$odule of an $\mathcal{R}$-modul $\mathcal{M}$ is a RNPr $\mathcal{S}$-$m$odule , but not conversely.

Proof

Assume that $\mathcal{S}$ω ∈ $\mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, ω ∈ $\mathcal{M}$, since $\mathcal{A}$ is a primary $\mathcal{S}$-$m$odule of $\mathcal{M}$, it follows that ω ∈ $\mathcal{A} \subseteq \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S} \in \sqrt{[\mathcal{A} :_{\mathcal{R}} \mathcal{M}]} \subseteq \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})) :_{\mathcal{R}} \mathcal{M}]}$ that is $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ for some positive integer κ. Hence $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

For the converse consider the following example:

Example 2.3

Consider the Z-module $\mathcal{S}$-$m$odule Z_{50}. The essential $\mathcal{S}$-$m$odule s of Z_{50} are Z_{50} itself and the $\mathcal{S}$-$m$odule $<\bar{5}>$, so $Soc(Z_{50}) = Z_{50} \cap <\bar{5}> = <\bar{5}>$. And the only maximal $\mathcal{S}$-$m$odule s of Z_{50} are $<\bar{2}>$ and $<\bar{5}>$, so $J(Z_{50}) = <\bar{2}> \cap <\bar{5}> = \langle\overline{10}\rangle$. Hence $Soc(Z_{50}) \cap J(Z_{50}) = <\bar{5}> \cap \langle\overline{10}\rangle = \langle\overline{10}\rangle$. The $\mathcal{S}$-$m$odule $\langle\bar{0}\rangle$ is not primary $\mathcal{S}$-$m$odule since, $5 \cdot \overline{10} \in \langle\bar{0}\rangle$ for $5 \in Z$, $\overline{10} \in Z_{50}$, but $\overline{10} \notin \langle\bar{0}\rangle$ and $\bar{5} \notin \sqrt{[\langle\bar{0}\rangle :_Z Z_{50}]} = \sqrt{50Z} = 10Z$.

On the other hand $\langle\bar{0}\rangle$ is RNPr $\mathcal{S}$-$m$odule since, $5 \cdot \overline{10} \in \langle\bar{0}\rangle$ for $5 \in Z$, $\overline{10} \in Z_{50}$, implies that either $\overline{10} \in \langle\bar{0}\rangle + (Soc(Z_{50}) \cap J(Z_{50})) = \langle\bar{0}\rangle + \langle\overline{10}\rangle = \langle\overline{10}\rangle$ or $\bar{5} \notin \sqrt{[\langle\bar{0}\rangle + (Soc(Z_{50}) \cap J(Z_{50})) :_Z Z_{50}]} = \sqrt{[\langle\bar{0}\rangle + \langle\overline{10}\rangle :_Z Z_{50}]} = \sqrt{[\langle\overline{10}\rangle :_Z Z_{50}]} = \sqrt{10Z} = 10Z$.

Propostion 2.4

Let $\mathcal{M}$ be an $\mathcal{R}$-modul, and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. with $(\mathrm{Soc}(\mathcal{M}) \cap \mathrm{J}(\mathcal{M})) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a primary $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Proof

$\Rightarrow$ Direct

$\Leftarrow$ Assume that $\mathcal{S}\mathrm{w} \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\mathrm{w} \in \mathcal{M}$. Since $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule , it follows that $\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S}^{\kappa} \in \left[\left(\mathcal{A} + \left(Soc(\mathcal{M}) \cap J(\mathcal{M})\right)\right) :_{\mathcal{R}} \mathcal{M}\right]$ for some positive integer κ. But we have $(\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M})) \subseteq \mathcal{A}$, then $\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})) = \mathcal{A}$, therefore it means that $\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})) \subseteq \mathcal{A}$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + \left(Soc(\mathcal{M}) \cap J(\mathcal{M})\right) \subseteq \mathcal{A}$, that is $\mathrm{w} \in \mathcal{A}$ or $\mathcal{S} \in \sqrt{[\mathcal{A} :_{\mathcal{R}} \mathcal{M}]}$. Hence $\mathcal{A}$ is a primary $\mathcal{S}$-$m$odule of $\mathcal{M}$.

We need to remember the next lemma before we present the following results:

Lemma 2.5 $[10, Lemma(2.3)]$

Let $\mathcal{M}$ be injective $\mathcal{R}$-modul, then $J(\mathcal{M}) = \mathcal{M}$.

Propostion 2.6

Let $\mathcal{M}$ be injective $\mathcal{R}$-modul, and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$ with $\mathrm{Soc}(\mathcal{M}) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a primary $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Proof

$\Rightarrow$ Direct

$\Leftarrow$ Assume that $\mathcal{S}\mathrm{w} \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\mathrm{w} \in \mathcal{M}$. Since $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule , it follows that $\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S}^{\kappa} \in \left[\left(\mathcal{A} + \left(Soc(\mathcal{M}) \cap J(\mathcal{M})\right)\right) :_{\mathcal{R}} \mathcal{M}\right]$ for some positive integer κ . But we have $\mathrm{Soc}(\mathcal{M}) \subseteq \mathcal{A}$, thus $\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})) \subseteq \mathcal{A} + (\mathcal{A} \cap J(\mathcal{M}))$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \left[\left(\mathcal{A} + \left(Soc(\mathcal{M}) \cap J(\mathcal{M})\right)\right)\right] \subseteq \mathcal{A} + (\mathcal{A} \cap J(\mathcal{M}))$, now $\mathcal{M}$ is injective $\mathcal{S}$-$m$odule then by lemma (**2.5**) $J(\mathcal{M}) = \mathcal{M}$, thus $\mathrm{w} \in \mathcal{A} + (\mathcal{A} \cap J(\mathcal{M})) = \mathcal{A} + (\mathcal{A} \cap \mathcal{M}) = \mathcal{A} + \mathcal{A} = \mathcal{A}$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (\mathcal{A} \cap J(\mathcal{M})) = \mathcal{A} + (\mathcal{A} \cap \mathcal{M}) = \mathcal{A} + \mathcal{A} = \mathcal{A}$, implies that $\mathcal{S} \in \sqrt{[\mathcal{A} :_{\mathcal{R}} \mathcal{M}]}$. Hence $\mathcal{A}$ is a primary $\mathcal{S}$-$m$odule of $\mathcal{M}$.

We need to remember the following lemma:

Lemma 2.7 $[7, Theo(9.2.1)(a)(h)]$

If $\mathcal{M}$ is a semi simple then $J(\mathcal{M}) = 0$.

Propostion 2.8

Let $\mathcal{M}$ be a semi simple $\mathcal{R}$-modul, and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a primary $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Proof

$\Longrightarrow$ Direct

$\Longleftarrow$ Assume that $\mathcal{S}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{M}$. Since $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule , then $\omega \in \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S}^{\kappa} \in [\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{R} \mathcal{M}]$ for some positive integer κ. But $\mathcal{M}$ be a semi simple then by lemma (**2.7**) $J(\mathcal{M}) = 0$ implies that $\omega \in \mathcal{A} + (Soc(\mathcal{M}) \cap 0\ ) = \mathcal{A} + 0 = \mathcal{A}$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap 0) = \mathcal{A} + 0 = \mathcal{A}$, then $\omega \in \mathcal{A}$ or $\mathcal{S} \in \sqrt{[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]}$. Hence $\mathcal{A}$ is a primary $\mathcal{S}$-$m$odule of $\mathcal{M}$.

It is well known if $\mathcal{M}$ is regular $\mathcal{R}$-modul then $J(\mathcal{M}) = 0$[9, prop.(9.3)].

Corollary 2.9

Let $\mathcal{M}$ be a regular $\mathcal{R}$-modul, and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a primary $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

3. RNPr $\mathcal{S}$-$m$odule and Approximaitly Primary $\mathcal{S}$-$m$odule

In this part of the note we study the relationships of RNPr $\mathcal{S}$-$m$odule with Approximaitly primary $\mathcal{S}$-$m$odule . Recall that a proper $\mathcal{S}$-$m$odule $\mathcal{A}$ of an $\mathcal{R}$-modul $\mathcal{M}$ is named an Approximaitly primary $\mathcal{S}$-$m$odule of $\mathcal{M}$, if whenever $\mathcal{S}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{M}$, it means that either $\omega \in \mathcal{A} + Soc(\mathcal{M})$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + Soc(\mathcal{M})$ for some $\kappa \in Z^{+}$) [3].

Remark 3.1

Every RNPr $\mathcal{S}$-$m$odule of an $\mathcal{R}$-modul $\mathcal{M}$ is an Approximaitly primary $\mathcal{S}$-$m$odule but not conversely.

Proof

Assume that $\mathcal{S}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{M}$. Since $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule , it follows that $\omega \in \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M})\ ) \subseteq \mathcal{A} + Soc(\mathcal{M})$ it means that $\omega \in \mathcal{A}+Soc(\mathcal{M})$ or $\mathcal{S}^{\kappa} \in \left[\left(\mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))\right):_{\mathcal{R}} \mathcal{M}\right]$ for some positive integer κ, that is $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M})) \subseteq \mathcal{A} + Soc(\mathcal{M})$, implies that $\mathcal{S} \in \sqrt{\mathcal{A} + (\ Soc(\mathcal{M}) \cap J(\mathcal{M})}$. Hence $\mathcal{A}$ is Approximaitly primary $\mathcal{S}$-$m$odule of $\mathcal{M}$.

The following example is the opposite:

Example 3.2

Consider the Z-module $\mathcal{S}$-$m$odule Z_{20}. The essential $\mathcal{S}$-$m$odule s of Z_{20} are Z_{20} itself and the $\mathcal{S}$-$m$odule s $< \bar{2} >$, so $Soc(Z_{20}) = Z_{20} \cap < \bar{2} > = < \bar{2} >$. And the only maximal $\mathcal{S}$-$m$odule s of Z_{20} is $<\bar{2}>$ and $< \bar{5} >$, so $J(Z_{20}) = < \bar{2} > \cap < \bar{5} > = <\overline{10}>$. Hence $Soc(Z_{20}) \cap J(Z_{20}) = < \bar{2} > \cap < \bar{5} > = <\overline{10}>$. The $\mathcal{S}$-$m$odule $< \overline{10} >$ is not RNPr $\mathcal{S}$-$m$odule since $Soc(Z_{20}) \cap J(Z_{20}) = < \overline{10} >$ and $\mathcal{A} + (Soc(Z_{20}) \cap J(Z_{20})) =$

$<\overline{10}> + <\overline{10}> = <\overline{10}>$ it follows that for $\mathcal{S}\overline{\omega} \in <\overline{10}>$ for $\mathcal{S} \in Z$, $\overline{\omega} \in Z_{20}$ implies that
$\overline{\omega} \in <\overline{10}> + (Soc(Z_{20}) \cap J(Z_{20})) = <\overline{10}>$ and
$\mathcal{S} \in \sqrt{[(<\overline{10}> + (Soc(Z_{20}) \cap J(Z_{20})): Z_{20}]} = \sqrt{10Z} = 10Z$. For example whenever $2 \cdot \overline{5} \in <\overline{10}>$, for $2 \in Z$, $\overline{5} \in Z_{20}$, but $\overline{5} \notin <\overline{10}> + <\overline{10}> = <\overline{10}>$ and $2 \notin \sqrt{[<\overline{10}> + <\overline{10}>:_Z Z_{20}]} = \sqrt{10Z} = 10Z$ Similarly for the other elements in $<\overline{10}>$.

On the other hand $<\overline{10}>$ is an Approximaitly primary since for $\mathcal{S} \in Z$, $\overline{\omega} \in Z_{20}$, $\mathcal{S}\overline{\omega} \in <\overline{10}>$ implies that $\overline{\omega} \in <\overline{10}> + Soc(Z_{20}) = <\overline{10}> + <\overline{2}> = <\overline{2}>$ or $\mathcal{S} \in \sqrt{[(<\overline{10}> + Soc(Z_{20}): Z_{20}]} = \sqrt{[(<\overline{10}> + <\overline{2}>: Z_{20}]} = \sqrt{2Z} = 2Z$. That is whenever $2 \cdot \overline{5} \in <\overline{10}>$, for $2 \in Z$, $\overline{5} \in Z_{20}$, implies that $\overline{5} \notin <\overline{10}> + Soc(Z_{20}) = <\overline{10}> + <\overline{2}> = <\overline{2}>$ or $2 \in \sqrt{[(<\overline{10}> + Soc(Z_{20}): Z_{20}]} = \sqrt{[(<\overline{10}> + <\overline{2}>: Z_{20}]} = \sqrt{2Z} = 2Z$ Similarly for the other elements in $<\overline{10}>$.

Propostion 3.3

Let $\mathcal{M}$ be an $\mathcal{R}$-modul, and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$ with $\mathcal{A} \subseteq J(\mathcal{M})$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{A}$ is a Approximaitly primary $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Proof

$\Rightarrow$ Direct

$\Leftarrow$ Since $\mathcal{A} \subseteq J(\mathcal{M})$, then there exist $\omega \in J(\mathcal{M})$ but $\omega \notin \mathcal{A}$, assume that $\mathcal{S}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{M}$, since $\mathcal{A}$ is Approximaitly primary $\mathcal{S}$-$m$odule of $\mathcal{M}$ then $\omega \in \mathcal{A} + Soc(\mathcal{M})$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + Soc(\mathcal{M})$ for some positive integer κ. Now we have $\omega \in J(\mathcal{M})$, hence $\omega \in (\mathcal{A} + Soc(\mathcal{M})) \cap J(\mathcal{M})$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq (\mathcal{A} + Soc(\mathcal{M})) \cap J(\mathcal{M})$. But $\mathcal{A} \subseteq J(\mathcal{M})$ and $\mathcal{A} + Soc(\mathcal{M}) = Soc(\mathcal{M}) + \mathcal{A}$, therefore $\omega \in (Soc(\mathcal{M}) + \mathcal{A}) \cap J(\mathcal{M})$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq (Soc(\mathcal{M}) + \mathcal{A}) \cap J(\mathcal{M})$, then by modular law $\omega \in \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ implies that $\mathcal{S} \in \sqrt{\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})}$. Hence $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Propostion 3.4

Let $\mathcal{M}$ be an $\mathcal{R}$-modul, and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$ with $Soc(\mathcal{M}) \subseteq J(\mathcal{M})$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{A}$ is a Approximaitly primary $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Proof

$\Rightarrow$ Direct

$\Leftarrow$ Assume that $\mathcal{S}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{M}$, since $\mathcal{A}$ is Approximaitly primry $\mathcal{S}$-$m$odule of $\mathcal{M}$ then $\omega \in \mathcal{A} + Soc(\mathcal{M})$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + Soc(\mathcal{M})$ for some positive integer κ. Now we have $Soc(\mathcal{M}) \subseteq J(\mathcal{M})$ then $Soc(\mathcal{M}) \cap J(\mathcal{M}) = Soc(\mathcal{M})$, hence $\omega \in \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ implies that $\mathcal{S} \in \sqrt{\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})}$. Thus $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

4. Characterizations of RNPr $\mathcal{S}$-$m$odule s in Multiplication Modules

We mention this part through the following Characterizations:

Propostion 4.1

Let $\mathcal{M}$ be an $\mathcal{R}$-modul, and $\mathcal{A}$ be a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{J}\beta \subseteq \mathcal{A}$ for $\mathcal{J}$ is an ideal of $\mathcal{R}$ and β is a $\mathcal{S}$-$m$odule of $\mathcal{M}$, it means that $\beta \subseteq \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ $\mathcal{J} \subseteq \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$.

Proof

$\Longrightarrow$ Assume that $\mathcal{J}\beta \subseteq \mathcal{A}$ with $\beta \nsubseteq \mathcal{A} + \big(Soc(\mathcal{M}) \cap J(\mathcal{M})\big)$, it follows that there exists $w \in \beta$ and $w \notin \mathcal{A} + \big(Soc(\mathcal{M}) \cap J(\mathcal{M})\big)$. We must show that $\mathcal{J} \subseteq \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$. Now, assume that $\mathcal{S} \in \mathcal{J}$, if $\mathcal{S}w \in \mathcal{A}$ and $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule , then $\mathcal{S} \in \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$, that is $\mathcal{J} \subseteq \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$.

$\Longleftarrow$ Assume that $\mathcal{S}w \in \mathcal{A}$ for $\mathcal{S} \in \mathcal{R}$, $w \in \mathcal{M}$, then $< \mathcal{S} > <w> \subseteq \mathcal{A}$, it follows by hypothesis either $< w > \subseteq \mathcal{A} + \big(Soc(\mathcal{M}) \cap J(\mathcal{M})\big)$ or $< \mathcal{S} > \subseteq \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$. Hence either $w \in \mathcal{A} + \big(Soc(\mathcal{M}) \cap J(\mathcal{M})\big)$ or $\mathcal{S} \subseteq \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$. That is $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

The following corollaries are a direct consequence of the propostion (**4.1**).

Corollary 4.2

Let $\mathcal{M}$ be an $\mathcal{R}$-modul, and $\mathcal{A}$ be a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{S}\beta \subseteq \mathcal{A}$ for $\mathcal{S} \in \mathcal{R}$ and β is a $\mathcal{S}$-$m$odule of $\mathcal{M}$, it means that $\beta \subseteq \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or

$\mathcal{S} \in \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$.

Corollary 4.3

Let $\mathcal{M}$ be an $\mathcal{R}$-modul, and $\mathcal{A}$ be a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $\mathcal{J}w \subseteq \mathcal{A}$ for $\mathcal{J}$ is an ideal of $\mathcal{R}$, $w \in \mathcal{M}$, it means that $w \in \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or

$\mathcal{J} \subseteq \sqrt{[\mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$.

Propostion 4.4

Let $\mathcal{M}$ be a multiplcation $\mathcal{R}$-modul, and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff, whenever $C\beta \subseteq \mathcal{A}$ for C, β are $\mathcal{S}$-$m$odule s of $\mathcal{M}$, it means that either $\beta \subseteq \mathcal{A}$+(Soc($\mathcal{M}$) $\cap J(\mathcal{M})$) or $C^{\kappa} \subseteq \mathcal{A}$+(Soc($\mathcal{M}$) $\cap$ J($\mathcal{M}$)) for some positive integer κ.

Proof

$\Longrightarrow$ Assume that $C\,\beta \subseteq \mathcal{A}$ for K, β are $\mathcal{S}$-$m$odules of $\mathcal{M}$. Since $\mathcal{M}$ is a multiplication, then $C = I\mathcal{M}$, $\beta = \mathcal{J}\mathcal{M}$ for some ideals I, $\mathcal{J}$ of $\mathcal{R}$. That is $I(\mathcal{J}\mathcal{M}) \subseteq \mathcal{A}$. But $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$, then by propostion (**4.1**) either $\mathcal{J}\mathcal{M} \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)) or $I^{\kappa}\mathcal{M} \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ $J(\mathcal{M})$) for some positive integer κ. It follows that either $\beta \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)) or $C^{\kappa} \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)).

$\Longleftarrow$ Assume that $I\,C \subseteq \mathcal{A}$ for I is an ideal of $\mathcal{R}$, C is $\mathcal{S}$-$m$odule of $\mathcal{M}$, Since $\mathcal{M}$ is a multiplcation $C = \mathcal{J}\mathcal{M}$ for $\mathcal{J}$ is an ideal of $\mathcal{R}$. thus $I(\mathcal{J}\mathcal{M}) \subseteq \mathcal{A}$ it follows that $LC \subseteq \mathcal{A}$ where $L = I\mathcal{M}$, then by hypothesis we have $C \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)) or $L^{\kappa} \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)) , that is $I^{\kappa}\mathcal{M} \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)) . Thus $C \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)) or $I^{\kappa} \subseteq \sqrt{[\mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap \mathrm{J}(\mathcal{M})):_{\mathcal{R}} \mathcal{M}]}$. Hence by proposition (**4.1**) $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

As direct application of propostion (**4.4**) we get the following corollary:

Corollary 4.5

Let $\mathcal{M}$ be a multiplcation $\mathcal{R}$-modul, and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr iff wherever $mm' \subseteq \mathcal{A}$, for $m, m' \in \mathcal{M}$, it means that $m' \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)) or $m^{\kappa} \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ $J(\mathcal{M})$) for some positive integer κ.

Remark 4.6

If $\mathcal{A}$ is an RNPr $\mathcal{S}$-$m$odule of an $\mathcal{R}$-module $\mathcal{M}$, then $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ need not to be an RNPr ideal of $\mathcal{R}$.

The following example illustrates that:

Example 4.7

Consider the Z-module Z_{50}, the $\mathcal{S}$-$m$odule $\mathcal{A} = \langle \bar{0} \rangle$. $\mathcal{A}$ is an RNPr $\mathcal{S}$-$m$odule of Z_{50} by remark (**2.2**). But $[\mathcal{A}:_{Z} \mathcal{M}] = [\langle \bar{0} \rangle:_{Z} Z_{50}] = 50Z$ is not an RNPr ideal of Z because $5.\overline{10} \in 50Z$ for $5, 10 \in Z$ but $\overline{10} \notin 50Z + (soc(Z) \cap \mathrm{J}(Z)) = 50Z + (0) = 50Z$ and $5 \notin \sqrt{[50Z + (soc(Z) \cap \mathrm{J}(Z)):_{Z} Z]} = \sqrt{[50Z + (0):_{Z} Z]} = \langle \bar{0} \rangle$.

The following propostion shows that the resudule of an RNPr $\mathcal{S}$-$m$odule is an RNPr ideal of $\mathcal{R}$ under certain condition.

Before that, we must mention the following:

Corollray 4.8 [4, Coro. (2.14)(i)]

If $\mathcal{M}$ is a faithful multiplication $\mathcal{R}$-modul, then $Soc(\mathcal{R})\mathcal{M} = Soc(\mathcal{M})$.

Remark 4.9 [11, Remark, P14]

If $\mathcal{M}$ is a faithful multiplcation $\mathcal{R}$-modul, then $J(\mathcal{M}) = J(\mathcal{R})\,\mathcal{M}$.

Propostion 4.10

Let $\mathcal{M}$ be a faithful multiplcation $\mathcal{R}$-modul and $\mathcal{A}$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$.

Proof

$\Rightarrow$ Assume that $\mathcal{S}\mathcal{J} \in [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ for $\mathcal{S} \in \mathcal{R}$, $\mathcal{J}$ is an ideal of $\mathcal{R}$, implies that $\mathcal{S}(\mathcal{J}\mathcal{M}) \subseteq \mathcal{A}$. But $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$, then by corollary (**4.2**) either $\mathcal{J}\mathcal{M} \subseteq \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$. Since $\mathcal{M}$ is multiplication, then $\mathcal{A} = [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M}$, and $\mathcal{M}$ is faithful multiplcation, then by corollary (**4.8**) and remark (**4.9**) $\mathrm{Soc}(\mathcal{R})\mathcal{M} = \mathrm{Soc}(\mathcal{M})$ and $J(\mathcal{R})\mathcal{M} = J(\mathcal{M})$, therefore $\big(\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R})\big)\mathcal{M} = (\mathrm{Soc}(\mathcal{R})\mathcal{M} \cap J(\mathcal{R})\mathcal{M}) = (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. Thus either $\mathcal{J}\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$,thus $\mathcal{J} \in [\mathcal{A}:_{\mathcal{R}} \mathcal{M}] + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R}))$ or $\mathcal{S}^{\kappa} \in [\mathcal{A}:_{\mathcal{R}} \mathcal{M}] + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R})) = \big[[\mathcal{A}:_{\mathcal{R}} \mathcal{M}] + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R})):_{\mathcal{R}} \mathcal{R}\big]$. Hence by corolary (**4.2**) $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ is RNPr ideal of $\mathcal{R}$.

$\Leftarrow$ Assume that $\omega H \subseteq \mathcal{A}$ for $\omega \in \mathcal{M}$ and H is a $\mathcal{S}$-$m$odule of $\mathcal{M}$. Since $\mathcal{M}$ is a multiplcation, then $\omega = \mathcal{R}\omega = \mathcal{J}\mathcal{M}$, $H = I\mathcal{M}$ for some ideals $I, \mathcal{J}$ of $\mathcal{R}$, that is $\mathcal{J}I\mathcal{M} \subseteq \mathcal{A}$, implies that $\mathcal{J}I \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$, but $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$, then by proposition (**4.1**) either $I \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}] + (\mathrm{Soc}(\mathcal{R}) \cap \mathrm{J}(\mathcal{R}))$ or $\mathcal{J}^{\kappa} \subseteq \big[[\mathcal{A}:_{\mathcal{R}} \mathcal{M}] + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R})):_{\mathcal{R}} \mathcal{R}\big] = [\mathcal{A}:_{\mathcal{R}} \mathcal{M}] + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R}))$. Hence either $I\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$ or $\mathcal{J}^{\kappa}\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} + (\mathrm{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$. Hence by corollary (**4.8**) and remark (**4.9**) $\mathrm{Soc}(\mathcal{R})\mathcal{M} = \mathrm{Soc}(\mathcal{M})$ and $J(\mathcal{R})\mathcal{M} = J(\mathcal{M})$, therefore either $H\mathcal{M} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $\mathcal{J}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. That is $H \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $\omega^{\kappa} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. Thus by corollary (**4.2**) $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Before this proposition, we should mention the following two propositions:

<u>Propostion 4.11</u> $[2, Prop.(17.10)]$

If $\mathcal{M}$ is projective $\mathcal{R}$-modul then $J(\mathcal{M}) = J(\mathcal{R})\,\mathcal{M}$.

<u>Propostion 4.12</u> [11]

If $\mathcal{M}$ is projective $\mathcal{R}$ -modul then $Soc(\mathcal{M}) = Soc(\mathcal{R})\,\mathcal{M}$.

<u>Propostion 4.13</u>

Let $\mathcal{M}$ be a projective multiplcation $\mathcal{R}$-modul and $\mathcal{A}$ be a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ if and only if $[\mathcal{A}:_{R} \mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$.

Proof

$\Rightarrow$ Assume that $\mathcal{S}\mathcal{J} \subseteq [\mathcal{A}:_{R} \mathcal{M}]$, for $\mathcal{J}$ is an ideal of $\mathcal{R}$, $\mathcal{S} \in \mathcal{R}$, it follows that $\mathcal{S}\mathcal{J}\mathcal{M} \subseteq \mathcal{A}$, since $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$, then by proposition (**4.1**) either $\mathcal{S}\mathcal{M} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap \mathrm{J}(\mathcal{M}))$ or $\mathcal{J}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{M}) \cap \mathrm{J}(\mathcal{M}))$ for some positive integer κ. But $\mathcal{M}$ is a projective $\mathcal{R}$-module, then by propostions (**4.11**) and (**4.12**) $\mathrm{Soc}(\mathcal{M}) = \mathrm{Soc}(\mathcal{R})\mathcal{M}$, and $J(\mathcal{M}) = J(\mathcal{R}).\mathcal{M}$, and $\mathcal{M}$ is multiplcation $\mathcal{R}$-module then $\mathcal{A}=[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\,\mathcal{M}$. Hence $\mathcal{S}\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} + (\mathrm{Soc}(\mathcal{R})\mathcal{M} \cap J(\mathcal{R})\mathcal{M}) = [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M}$

+ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) $\mathcal{M}$ or $\mathcal{J}^{\kappa}\mathcal{M} \subseteq [\mathcal{A} :_{\mathcal{R}} \mathcal{M}]\mathcal{M}$ + (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$)$\mathcal{M}$. It follows that $\mathcal{S} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ + (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) or $\mathcal{J}^{\kappa} \in [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ + (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) = [[$\mathcal{A}:_{\mathcal{R}} \mathcal{M}$] + (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) $:_{\mathcal{R}} \mathcal{R}$]. Hence by propostion (**4.1**) $[\mathcal{A} :_{\mathcal{R}} \mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$.

⇐ Assume that $mm' \subseteq \mathcal{A}$, for $m, m' \in \mathcal{M}$. Since $\mathcal{M}$ is a multiplcation $\mathcal{R}$-module, then $m = \mathcal{R}m = I\,\mathcal{M}$ $m'= \mathcal{R}m'=\mathcal{J}\,\mathcal{M}$ for some ideals I, $\mathcal{J}$ of $\mathcal{R}$, it follows that $I\mathcal{J}\mathcal{M} \subseteq \mathcal{A}$, hence $I\mathcal{J} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$. But $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$, then by propostion (**4.1**) either $\mathcal{J} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ + (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) or $I^{\kappa} \in$ [$[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ + (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) $:_{\mathcal{R}} \mathcal{R}$] = $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ + (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$). Hence either $\mathcal{J}\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\,\mathcal{M}$ + (Soc($\mathcal{R}$) ∩ J($\mathcal{R}$))$\mathcal{M}$ = $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\ \mathcal{M}$ + (Soc($\mathcal{R}$)$\mathcal{M} \cap J(\mathcal{R})\mathcal{M}$) or $I^{\kappa}\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\,\mathcal{M}$ +(Soc($\mathcal{R}$)$\mathcal{M} \cap J(\mathcal{R})\mathcal{M}$). Since $\mathcal{M}$ is a projective $\mathcal{R}$-module, then by propostions (**4.11**) and (**4.12**) Soc($\mathcal{M}$) = Soc($\mathcal{R}$)$\mathcal{M}$ and $J(\mathcal{M}) = J(\mathcal{R})\mathcal{M}$. Hence $m' \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ $J(\mathcal{M})$) or $m'^{\kappa}\mathcal{M} \subseteq \mathcal{A}$ + (Soc($\mathcal{M}$) ∩ J($\mathcal{M}$)). So by corollary (**4.5**) $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule $\mathcal{M}$.

Before this proposition, we should mention the following two corollary and proposition:

Corollary 4.14 [5, Coro. (1.26)]

Let $\mathcal{M}$ be a non-singular $\mathcal{R}$-modul, then $Soc(\mathcal{R})\ \mathcal{M} = soc(\mathcal{M})$.

Propostion 4.15 [11, $prop$(1.11)]

If $\mathcal{M}$ is content module, then $J(\mathcal{M}) = J(\mathcal{R})\mathcal{M}$.

Propostion 4.16

Let $\mathcal{M}$ be a non-singular content multiplication $\mathcal{R}$-modul. $\mathcal{A}$ a proper $\mathcal{S}$-$m$odule of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ iff $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$.

Proof

⟹ Assume that $\mathcal{S}\omega \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ for $\mathcal{S}, \omega \in \mathcal{R}$, implies that $\mathcal{S}(\omega\mathcal{M}) \subseteq \mathcal{A}$. But $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$, then by corollary (**4.2**) either $\omega\mathcal{M} \subseteq \mathcal{A} +$ (Soc($\mathcal{M}$) ∩ $J(\mathcal{M})$) or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} +$ (Soc($\mathcal{M}$) ∩ $J(\mathcal{M})$). Now, since $\mathcal{M}$ is non-singular, then by corollary (**4.14**) Soc($\mathcal{R}$)$\mathcal{M}$ = Soc($\mathcal{M}$) and $\mathcal{M}$ is content then by propostion (**4.15**) then $J(\mathcal{R})\mathcal{M} = J(\mathcal{M})$ and $\mathcal{M}$ is multiplication, then $\mathcal{A} = [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M}$. Thus either $\omega\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$)$\mathcal{M}$ or $\mathcal{S}^{\kappa}\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$)$\mathcal{M}$, it follows that $\omega \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}] +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) or $\mathcal{S}^{\kappa} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}] +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) $= \big[[\mathcal{A}:_{\mathcal{R}} \mathcal{M}] +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$)$:_{\mathcal{R}} \mathcal{R}\big]$. Thus by corollary (**4.2**) $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$.

⇐ Assume that $H\omega \subseteq \mathcal{A}$, where H is a $\mathcal{S}$-$m$odule of $\mathcal{M}$, $\omega \in \mathcal{M}$. Since $\mathcal{M}$ is a multiplication, then $H = \mathcal{J}\mathcal{M}$, $\omega = \mathcal{R}\omega = I\mathcal{M}$ for some ideals $I, \mathcal{J}$ of $\mathcal{R}$, it follows that $\mathcal{J}I\mathcal{M} \subseteq \mathcal{A}$, implies that $\mathcal{J}I \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$, but $[\mathcal{A}:_{\mathcal{R}} \mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$, then by propostion (**4.1**) either $I \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}] +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) or $\mathcal{J}^{\kappa} \subseteq \big[[\mathcal{A}:_{R} \mathcal{M}] +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) $:_{\mathcal{R}} \mathcal{R}\big] = [\mathcal{A}:_{\mathcal{R}} \mathcal{M}] +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$) , that is $I\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$)$\mathcal{M}$ or $\mathcal{J}^{\kappa}\mathcal{M} \subseteq [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M} +$ (Soc($\mathcal{R}$) ∩ $J(\mathcal{R})$)$\mathcal{M}$. Since $\mathcal{M}$ is non-singular, then by corollary (**4.14**) Soc($\mathcal{R}$)$\mathcal{M}$ = Soc($\mathcal{M}$) and $\mathcal{M}$ is content then by propostion (**4.15**) then $J(\mathcal{R})\mathcal{M} = J(\mathcal{M})$ and $\mathcal{M}$ is multiplication, then $\mathcal{A} = [\mathcal{A}:_{\mathcal{R}} \mathcal{M}]\mathcal{M}$. Hence either $\text{ɯ} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $H^{\kappa} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. Thus by corollary **(4.2)** $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Propostion 4.17

Let Φ: $\mathcal{M} \longrightarrow \mathcal{M}'$ be an $\mathcal{R}$-monomorphism and $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$ and $Ker\ \Phi \subseteq \mathcal{A}$ with $ker\ \Phi$ is a small, $\Phi(\mathcal{A})$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}'$.

Proof

$\Phi(\mathcal{A})$ is a proper $\mathcal{S}$-$m$odule of $\mathcal{M}'$. If not, that is $\Phi(\mathcal{A}) = \mathcal{M}'$. Assume that $\text{ɯ} \in \mathcal{M}$, then $\Phi(\text{ɯ}) \in \mathcal{M}' = \Phi(\mathcal{A})$, so there exists $x \in \mathcal{A}$ such that $\Phi(\text{ɯ}) = \Phi(x)$, implies that $\Phi(\text{ɯ} - x) = 0$, that is $\text{ɯ} - x \in Ker\ \Phi \subseteq \mathcal{A}$, it follows that $\text{ɯ} \in \mathcal{A}$. Thus $\mathcal{A} = \mathcal{M}$ contradiction. Assume that $\mathcal{S}\text{ɯ} \in \Phi(\mathcal{A})$, where $\mathcal{S} \in \mathcal{R}$, $\text{ɯ} \in \mathcal{M}'$ with $\text{ɯ} \notin \Phi(\mathcal{A}) + (Soc(\mathcal{M}') \cap J(\mathcal{M}'))$, that is $\Phi^{-1}(\mathcal{S}\text{ɯ}) = \mathcal{S}\Phi^{-1}(\text{ɯ}) \in \Phi^{-1}(\Phi(\mathcal{A})) = \mathcal{A}$, implies that $\mathcal{S}\Phi^{-1}(\text{ɯ}) \in \mathcal{A}$, with $\Phi^{-1}(\text{ɯ}) \notin \Phi^{-1}(\Phi(\mathcal{A})) + \Phi^{-1}(Soc(\mathcal{M}') \cap J(\mathcal{M}')) = \mathcal{A} + \big(Soc(\mathcal{M}) \cap J(\mathcal{M})\big)$ and $\Phi^{-1}(\text{ɯ}) \in \mathcal{M}$. But $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}'$, it follows that $\mathcal{S}^{\kappa}\mathcal{M} \subseteq \mathcal{A} + \big(Soc(\mathcal{M}) \cap J(\mathcal{M})\big)$ for some positive integer κ. To show that $\mathcal{S}^{\kappa}\Phi(\mathcal{M}) \subseteq \Phi(\mathcal{A}) + (Soc(\mathcal{M}') \cap J(\mathcal{M}'))$, if $\text{ɯ} \in \Phi(\mathcal{M})$, then $\Phi^{-1}(\text{ɯ}) \in \mathcal{M}$. Thus $\Phi^{-1}(\text{ɯ}) \in \Phi^{-1}(\mathcal{S}^{\kappa}\text{ɯ}) \in \mathcal{A} + \big(Soc(\mathcal{M}) \cap J(\mathcal{M})\big)$, implies that $\mathcal{S}^{\kappa}\text{ɯ} \in \Phi(\mathcal{A})) +$ $\Phi^{-1}\big(Soc(\mathcal{M}) \cap J(\mathcal{M})\big) = \Phi(\mathcal{A}) + (Soc(\mathcal{M}') \cap J(\mathcal{M}'))$, that is $\mathcal{S}^{\kappa}\Phi(\mathcal{M}) \subseteq \Phi(\mathcal{A}) + (Soc(\mathcal{M}') \cap J(\mathcal{M}'))$, so $\Phi(\mathcal{A})$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Propostion 4.18

Let Φ:$\mathcal{M} \longrightarrow \mathcal{M}'$ be an $\mathcal{R}$-monomorphism and $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}'$ and $ker\ \Phi$ is a small, $\Phi^{-1}(\mathcal{A})$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

Proof

Assume that $\mathcal{S}\text{ɯ} \in \Phi^{-1}(\mathcal{A})$, where $\mathcal{S} \in \mathcal{R}$, $\text{ɯ} \in \mathcal{M}$ with $\text{ɯ} \notin \Phi^{-1}(\mathcal{A}) + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$, that is $\Phi(\mathcal{S}\text{ɯ}) = \mathcal{S}\Phi(\text{ɯ}) \in \Phi(\Phi^{-1}(\mathcal{A})) = \mathcal{A}$, implies that $\mathcal{S}\Phi(\text{ɯ}) \in \mathcal{A}$, with $\Phi(\text{ɯ}) \notin \Phi(\Phi^{-1}(\mathcal{A})) + \Phi(Soc(\mathcal{M}) \cap J(\mathcal{M})) = \mathcal{A} + \big(Soc(\mathcal{M}') \cap J(\mathcal{M}')\big)$ and $\Phi(\text{ɯ}) \in \mathcal{M}'$. But $\mathcal{A}$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}'$, it follows that $\mathcal{S}^{\kappa}\mathcal{M}' \subseteq \mathcal{A} + \big(Soc(\mathcal{M}') \cap J(\mathcal{M}')\big)$ for some positive integer κ. To show that $\mathcal{S}^{\kappa}\Phi^{-1}(\mathcal{M}') \subseteq \Phi^{-1}(\mathcal{A}) + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$, if $\text{ɯ} \in \Phi^{-1}(\mathcal{M}')$, then $\Phi(\text{ɯ}) \in \mathcal{M}'$. Thus an $\Phi(\text{ɯ}) \in \Phi(\mathcal{S}^{\kappa}\text{ɯ}) \in \mathcal{A} + \big(Soc(\mathcal{M}') \cap J(\mathcal{M}')\big)$, it means that $\mathcal{S}^{\kappa}\text{ɯ} \in \Phi^{-1}(\mathcal{A})) + \Phi\big(Soc(\mathcal{M}') \cap J(\mathcal{M}')\big) =$ $\Phi^{-1}(\mathcal{A}) + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$, that is $\mathcal{S}^{\kappa}\Phi^{-1}(\mathcal{M}') \subseteq \Phi^{-1}(\mathcal{A}) + (Soc(\mathcal{M}) \cap J(\mathcal{M}))$, so $\Phi^{-1}(\mathcal{A})$ is a RNPr $\mathcal{S}$-$m$odule of $\mathcal{M}$.

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