Restrict Nearly Primary Submodules

1 Haibat K. Mohammad ali

Department of Mathematics / College of Computer Science and Mathematics/ Tikrit University / Iraq
dr.mohammadali2013@gmail.com

2 Khudhayer O. Khadem

Department of Mathematics / College of Computer Science and Mathematics/ Tikrit University / Iraq
Khudhayer1981@gmail.com

Abstract: In this work, all rings under consideration will be Supposedly to be commutative with nonzero identity and all modules will be nonzero unital. this work we investigate some properties of Restrict Nearly primary submodule as new generalization of primary sub module and strong form of Approximaitly primary submodule and established some sufficient conditions on Restrict Nearly primary to be primary. Also, we give necessarily condition on Approximaitly primary to be Restrict Nearly primary. In addition to we give several characterizations of Restrict Nearly primary submodule in class of multiplication modules.

Keywords: Restrict Nearly Primary submodules, Primary submodules, Approximaitly primary sub modules.

1. Introduction

A proper submodule $\mathcal{A}$ of an $\mathcal{R}$-module $\mathcal{M}$ is named a primary if whenever $Sw \in \mathcal{A}$, for $S \in \mathcal{R}, w \in \mathcal{M}$ it means that $w \in \mathcal{A}$ or $S \in \sqrt{\mathcal{A}: \mathcal{M}}$, for some positive integer $\kappa$. Newly primary submodules have been generalized to Nearly and Approximaitly primary submodule see [1], [6]. In this paper we say that $\mathcal{A}$ is Restrict Nearly Primary(RNPr), if whenever $Sw \in \mathcal{A}$, for $S \in \mathcal{R}, w \in \mathcal{M}$ it means that $w \in \mathcal{A} + (\text{Soc} (\mathcal{M}) \cap J(\mathcal{M}))$ or $S \in \sqrt{\left(\mathcal{A} + (\text{Soc} (\mathcal{M}) \cap J(\mathcal{M}))\right) : \mathcal{R} \mathcal{M}}$, that is $\mathcal{A}^\kappa \mathcal{M} \subseteq \mathcal{A} + (\text{Soc} (\mathcal{M}) \cap J(\mathcal{M}))$ for some positive integer $\kappa$. And an ideal $J$ of a ring $\mathcal{R}$ is named a RNPr ideal of $\mathcal{R}$, iff $J$ is a RNPr submodule of an $\mathcal{R}$-module $\mathcal{R}$. Recall that an $\mathcal{R}$-module $\mathcal{M}$ is faithful if $\text{ann} (\mathcal{M}) = (0)$ [9], and an $\mathcal{R}$-module $\mathcal{M}$ is regular iff for each $S \in \mathcal{R}, w \in \mathcal{M}, \exists t \in \mathcal{R}$ such that $Stw = Sw[9]$. Recall that an $\mathcal{R}$-module $\mathcal{M}$ semisimple iff $\text{Soc} (\mathcal{M}) = \mathcal{M}$ [2]. An $\mathcal{R}$-module $\mathcal{M}$ such that every submodule $\mathcal{A}$ of $\mathcal{M}$ is of the form $\mathcal{A} = \mathcal{I} \mathcal{M}$ for some
ideal $I$ of $\mathcal{R}$, this is equivalently to $\mathcal{M}$ is multiplication if $\mathcal{A} = [\mathcal{A};_R \mathcal{M}]_R \mathcal{M}$ [3]. An $\mathcal{R}$-module $\mathcal{M}$ is nonsingular if $Z(\mathcal{M}) = (0)$, where $Z(\mathcal{M}) = \{w \in \mathcal{M} : wI = (0)\}$, for some essential ideal $I$ of $\mathcal{R}$ [5]. Recall that an $\mathcal{R}$-module $\mathcal{M}$ is named to be content module if $(\bigcap_{i \in I} A_i) \mathcal{M} = \bigcap_{i \in I} A_i \mathcal{M}$ for each family of ideals $A_i$ in $\mathcal{R}$ [8]. We denoted to the submodule by $S$-module and $\mathcal{R}$-module by $\mathcal{R}$-modul.

2. Restrict Nearly primary $\mathcal{S}$-modules

We introduce the following:

**Definition 2.1**

A proper $\mathcal{S}$-module $\mathcal{A}$ of an $\mathcal{M}$ is named a Restrict Nearly Primary (RNPr) $\mathcal{S}$-module of $\mathcal{M}$, if whenever $\mathcal{S}w \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $w \in \mathcal{M}$ it means that $\mathcal{A} = [\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M})) :_R \mathcal{M}] \subseteq \sqrt{[\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M})) :_R \mathcal{M}]}$, that is $\mathcal{S}^k \mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$, for some positive integer $k$. And an ideal $J$ of a ring $\mathcal{R}$ is named a RNPr ideal of $\mathcal{R}$, iff $J$ is a RNPr $\mathcal{S}$-module of an $\mathcal{R}$-modul.

**Remark 2.2**

Every primary $\mathcal{S}$-module of an $\mathcal{M}$ is a RNPr $\mathcal{S}$-module, but not conversely.

**Proof**

Assume that $\mathcal{S}w \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}, w \in \mathcal{M}$, since $\mathcal{A}$ is a primary $\mathcal{S}$-module of $\mathcal{M}$, it follows that $\mathcal{A} = [\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M})) :_R \mathcal{M}] \subseteq \sqrt{[\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M})) :_R \mathcal{M}]}$, that is $\mathcal{S}^k \mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ for some positive integer $k$. Hence $\mathcal{A}$ is a RNPr $\mathcal{S}$-module of $\mathcal{M}$.

For the converse consider the following example:

**Example 2.3**

Consider the $\mathcal{Z}$-module $\mathcal{S}$-module $\mathcal{Z}_{50}$. The essential $\mathcal{S}$-module $s$ of $\mathcal{Z}_{50}$ are $\mathcal{Z}_{50}$ itself and the $\mathcal{S}$-module $<5>$, so $\text{Soc}(\mathcal{Z}_{50}) = \mathcal{Z}_{50} \cap <5> = <5>$. And the only maximal $\mathcal{S}$-module $s$ of $\mathcal{Z}_{50}$ are $<2>$ and $<5>$, so $J(\mathcal{Z}_{50}) = <2> \cap <5> = (10)$. Hence $\text{Soc}(\mathcal{Z}_{50}) \cap J(\mathcal{Z}_{50}) = <5> \cap (10) = (10)$. The $\mathcal{S}$-module $(0)$ is not primary $\mathcal{S}$-module since, $5 \cdot (10) \in (0)$ for $5 \in \mathcal{Z}$, $(10) \in \mathcal{Z}_{50}$, but $(10) \notin (0)$ and $5 \notin \sqrt{(0);_2 \mathcal{Z}_{50}}$

= $\sqrt{50Z} = 10Z$.

On the other hand $(0)$ is RNPr $\mathcal{S}$-module since, $5 \cdot (10) \in (0)$ for $5 \in \mathcal{Z}$, $(10) \in \mathcal{Z}_{50}$ implies that either $\sqrt{(0)} + (\text{Soc}(\mathcal{Z}_{50}) \cap J(\mathcal{Z}_{50})) = (0) + (10) = (10) \text{ or } 5 \notin \sqrt{(0) + (\text{Soc}(\mathcal{Z}_{50}) \cap J(\mathcal{Z}_{50})):_{_2} \mathcal{Z}_{50}} = \sqrt{(0) + (10):_{_2} \mathcal{Z}_{50}} = \sqrt{10Z} = 10Z$.
Proposition 2.4
Let $\mathcal{M}$ be an $R$-modul, and $\mathcal{A}$ is a proper $S$-module of $\mathcal{M}$. with $(\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\}) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a primary $S$-module of $\mathcal{M}$ if and only if $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}$.

Proof

$\Rightarrow$ Assume that $S\mathcal{A} \subseteq \mathcal{A}$, for $S \in R$, $w \in \mathcal{M}$. Since $\mathcal{A}$ is a RNPr $S$-module, it follows that $w \in \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\})$ or $S^k \in \left(\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\})\right)_{[\mathcal{M}]}$ for some positive integer $k$. But we have $(\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\}) \subseteq \mathcal{A}$, thus $\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\}) \subseteq \mathcal{A} + (\mathcal{A} \cap \{\mathcal{M}\})$ or $S^k \mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\})$, that is $w \in \mathcal{A}$ or $S \in \sqrt{[\mathcal{A}:S/R]}$. Hence $\mathcal{A}$ is a primary $S$-module of $\mathcal{M}$.

We need to remember the next lemma before we present the following results:

Lemma 2.5 [10, Lemma 2.3]
Let $\mathcal{M}$ be injective $R$-modul, then $J(\mathcal{M}) = \mathcal{M}$.

Proposition 2.6
Let $\mathcal{M}$ be injective $R$-modul, and $\mathcal{A}$ is a proper $S$-module of $\mathcal{M}$ with $\text{Soc}(\mathcal{M}) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a primary $S$-module of $\mathcal{M}$ if and only if $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}$.

Proof

$\Rightarrow$ Assume that $S\mathcal{A} \subseteq \mathcal{A}$, for $S \in R$, $w \in \mathcal{M}$. Since $\mathcal{A}$ is a RNPr $S$-module, it follows that $w \in \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\})$ or $S^k \in \left(\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\})\right)_{[\mathcal{M}]}$ for some positive integer $k$. But we have $(\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\}) \subseteq \mathcal{A}$, thus $\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\}) \subseteq \mathcal{A} + (\mathcal{A} \cap \{\mathcal{M}\})$ or $S^k \mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{\mathcal{M}\})$, that is $w \in \mathcal{A}$ or $S \in \sqrt{[\mathcal{A}:S/R]}$. Hence $\mathcal{A}$ is a primary $S$-module of $\mathcal{M}$.

We need to remember the following lemma:

Lemma 2.7 [7, Theo 9.2.1 (a) (h)]
If $\mathcal{M}$ is a semi simple then $J(\mathcal{M}) = 0$.

Proposition 2.8
Let $\mathcal{M}$ be a semi simple $R$-modul, and $\mathcal{A}$ is a proper $S$-module of $\mathcal{M}$. Then $\mathcal{A}$ is a primary $S$-module of $\mathcal{M}$ if and only if $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}$.
Proof
⇒ Direct
⇔ Assume that $S \omega \in A$, for $S \in R$, $\omega \in M$. Since $A$ is a RNPr $S$-module, then $w \in A + (\text{Soc}(M) \cap J(M))$ or $S^\kappa \in \{A + (\text{Soc}(M) \cap J(M)) : R \}$. Hence $A$ is a primary $S$-module of $M$.

It is well known if $M$ is regular $R$-modul then $J(M) = 0$ implies that $A$ is a RNPr $S$-module of $M$.

Corollary 2.9
Let $M$ be a regular $R$-modul, and $A$ is a proper $S$-module of $M$. Then $A$ is a primary $S$-module of $M$ iff $A$ is a RNPr $S$-module of $M$.

3. RNPr $S$-module and Approximaitlty Primary $S$-module

In this part of the note we study the relationships of RNPr $S$-module with Approximaitlty primary $S$-module. Recall that a proper $S$-module $A$ of an $R$-modul $M$ is named an Approximaitlty primary $S$-module of $M$, if whenever $S \omega \in A$, for $S \in R$, $\omega \in M$, it means that either $w \in A + \text{Soc}(M)$ or $S^\kappa M \subseteq A + \text{Soc}(M)$ for some $\kappa \in Z^+$) [3].

Remark 3.1
Every RNPr $S$-module of an $R$-modul $M$ is an Approximaitlty primary $S$-module but not conversely.

Proof
Assume that $S \omega \in A$, for $S \in R$, $\omega \in M$. Since $A$ is a RNPr $S$-module, it follows that $w \in A + (\text{Soc}(M) \cap J(M)) \subseteq A + \text{Soc}(M)$ it means that $w \in \{A + (\text{Soc}(M) \cap J(M)) : R \}$ for some positive integer $\kappa$, that is $S^\kappa M \subseteq A + (\text{Soc}(M) \cap J(M)) \subseteq A + \text{Soc}(M)$, implies that $S \in \sqrt{\{A + (\text{Soc}(M) \cap J(M)) : R \}}$. Hence $A$ is Approximaitlty primary $S$-module of $M$.

The following example is the opposite:

Example 3.2
Consider the $Z$-module $S$-module $Z_{20}$. The essential $S$-module $s$ of $Z_{20}$ are $Z_{20}$ itself and the $S$-module $s < \bar{2} >$, so $\text{Soc}(Z_{20}) = Z_{20} \cap < \bar{2} > = < \bar{2} >$. And the only maximal $S$-module $s$ of $Z_{20}$ is $< \bar{2} >$ and $< 5 >$, so $J(Z_{20}) = < \bar{2} > \cap < 5 > = < 10 >$. Hence $\text{Soc}(Z_{20}) \cap J(Z_{20}) = < \bar{2} > \cap < 5 > = < 10 >$. The $S$-module $< 10 >$ is not RNPr $S$-module since $\text{Soc}(Z_{20}) \cap J(Z_{20}) = < 10 >$ and $A + (\text{Soc}(Z_{20}) \cap J(Z_{20})) = \ldots$
< 10 > + < 10 > = < 10 > it follows that for \( \overline{w} \in < 10 > \) for \( \overline{s} \in \mathbb{Z}, \overline{w} \in \mathbb{Z}_{20} \) implies that 
\[
\overline{w} \in \sqrt{\left[ ( < 10 > + \langle \text{Soc}(\mathbb{Z}_{20}) \cap J(\mathbb{Z}_{20}) \rangle : \mathbb{Z}_{20} \right]} = \sqrt{10Z} = 10Z.
\]
For example whenever 
\[
2 \cdot 5 \in < 10 >, \quad 5 \in \mathbb{Z}, 5 \in \mathbb{Z}_{20}, \quad 5 \notin < 10 > + < 10 > = < 10 > \text{ and } 2 \notin \sqrt{\left[ ( < 10 > + < 2 > : \mathbb{Z}_{20} \right]} = \sqrt{2Z} = 2Z. \]
Similarly for the other elements in < 10 >.

On the other hand < 10 > is an Approximately primary since for \( \overline{s} \in \mathbb{Z}, \overline{w} \in \mathbb{Z}_{20}, \overline{sw} \in < 10 > \) implies that \( \overline{w} \in < 10 > + \langle \text{Soc}(\mathbb{Z}_{20}) \rangle = < 10 > + < 2 > = < 2 > \) or \( \overline{s} \in \sqrt{\left[ ( < 10 > + \langle \text{Soc}(\mathbb{Z}_{20}) \rangle : \mathbb{Z}_{20} \right]} = \sqrt{2Z} = 2Z. \) That is whenever \( 2 \cdot 5 \in < 10 > \), for \( 2 \in \mathbb{Z}, 5 \in \mathbb{Z}_{20} \), implies that \( 5 \notin < 10 > + \langle \text{Soc}(\mathbb{Z}_{20}) \rangle = < 10 > + < 2 > = < 2 > \) or \( 2 \in \sqrt{\left[ ( < 10 > + \langle \text{Soc}(\mathbb{Z}_{20}) \rangle : \mathbb{Z}_{20} \right]} = \sqrt{2Z} = 2Z. \)

Proposition 3.3

Let \( \mathcal{M} \) be an \( \mathcal{R} \)-modul, and \( \mathcal{A} \) is a proper \( \mathcal{S} \)-module of \( \mathcal{M} \) with \( \mathcal{A} \subseteq J(\mathcal{M}) \). Then \( \mathcal{A} \) is a RNPr \( \mathcal{S} \)-module of \( \mathcal{M} \) if \( \mathcal{A} \) is an Approximately primary \( \mathcal{S} \)-module of \( \mathcal{M} \).

Proof

\[ \Rightarrow \text{Direct} \]

\[ \Leftarrow \text{Since } \mathcal{A} \subseteq J(\mathcal{M}), \text{ then there exist } w \in J(\mathcal{M}) \text{ but } w \notin \mathcal{A}, \text{ assume that } \overline{sw} \in \mathcal{A}, \text{ for } \overline{s} \in \mathcal{R}, \overline{w} \in \mathcal{M}, \text{ since } \mathcal{A} \text{ is an Approximately primary } \mathcal{S} \text{-module of } \mathcal{M} \text{ then } w \in \mathcal{A} + \langle \text{Soc}(\mathcal{M}) \rangle \text{ or } \mathcal{S}^k \mathcal{M} \subseteq \mathcal{A} + \langle \text{Soc}(\mathcal{M}) \rangle \text{ for some positive integer } k. \text{ Now we have } w \in J(\mathcal{M}), \text{ hence } w \in (\mathcal{A} + \langle \text{Soc}(\mathcal{M}) \rangle) \cap J(\mathcal{M}) \text{ or } \mathcal{S}^k \mathcal{M} \subseteq (\mathcal{A} + \langle \text{Soc}(\mathcal{M}) \rangle) \cap J(\mathcal{M}). \text{ But } \mathcal{A} \subseteq J(\mathcal{M}) \text{ and } \mathcal{A} + \langle \text{Soc}(\mathcal{M}) \rangle = \langle \text{Soc}(\mathcal{M}) \rangle + \mathcal{A}, \text{ therefore } w \in (\langle \text{Soc}(\mathcal{M}) \rangle + \mathcal{A}) \cap J(\mathcal{M}) \text{ or } \mathcal{S}^k \mathcal{M} \subseteq (\langle \text{Soc}(\mathcal{M}) \rangle + \mathcal{A}) \cap J(\mathcal{M}), \text{ then by modular law } w \in \mathcal{A} + (\langle \text{Soc}(\mathcal{M}) \rangle \cap J(\mathcal{M})) \text{ or } \mathcal{S}^k \mathcal{M} \subseteq \mathcal{A} + (\langle \text{Soc}(\mathcal{M}) \rangle \cap J(\mathcal{M})) \text{ implies that } s \in \sqrt{\mathcal{A} + (\langle \text{Soc}(\mathcal{M}) \rangle \cap J(\mathcal{M}))}. \text{ Hence } \mathcal{A} \text{ is a RNPr } \mathcal{S} \text{-module of } \mathcal{M}. \]

Proposition 3.4

Let \( \mathcal{M} \) be an \( \mathcal{R} \)-modul, and \( \mathcal{A} \) is a proper \( \mathcal{S} \)-module of \( \mathcal{M} \) with \( \text{Soc}(\mathcal{M}) \subseteq J(\mathcal{M}) \). Then \( \mathcal{A} \) is a RNPr \( \mathcal{S} \)-module of \( \mathcal{M} \) if \( \mathcal{A} \) is an Approximately primary \( \mathcal{S} \)-module of \( \mathcal{M} \).

Proof

\[ \Rightarrow \text{Direct} \]

\[ \Leftarrow \text{Assume that } \overline{sw} \in \mathcal{A}, \text{ for } \overline{s} \in \mathcal{R}, \overline{w} \in \mathcal{M}, \text{ since } \mathcal{A} \text{ is an Approximately primary } \mathcal{S} \text{-module of } \mathcal{M} \text{ then } w \in \mathcal{A} + \langle \text{Soc}(\mathcal{M}) \rangle \text{ or } \mathcal{S}^k \mathcal{M} \subseteq \mathcal{A} + \langle \text{Soc}(\mathcal{M}) \rangle \text{ for some positive integer } k. \text{ Now we have } \text{Soc}(\mathcal{M}) \subseteq J(\mathcal{M}) \text{ then } \text{Soc}(\mathcal{M}) \cap J(\mathcal{M}) = \text{Soc}(\mathcal{M}), \text{ hence } w \in \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M})) \text{ or } \mathcal{S}^k \mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M})) \text{ implies that } s \in \sqrt{\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))}. \text{ Thus } \mathcal{A} \text{ is a RNPr } \mathcal{S} \text{-module of } \mathcal{M}. \]
4. Characterizations of RNPr $S$-module s in Multiplication Modules

We mention this part through the following Characterizations:

**Proposition 4.1**

Let $M$ be an $R$-modul, and $A$ be a proper $S$-module of $M$. Then $A$ is a RNPr $S$-module of $M$ iff $J\beta \subseteq A$ for $J$ is an ideal of $R$ and $\beta$ is a $S$-module of $M$, it means that $\beta \subseteq A + (\text{Soc}(M) \cap J(M))$ $J \subseteq \sqrt{[A + (\text{Soc}(M) \cap J(M));R M]}$.

**Proof**

$\Rightarrow$ Assume that $J\beta \subseteq A$ with $\beta \subseteq A + (\text{Soc}(M) \cap J(M))$, it follows that there exists $w \in \beta$ and $w \notin A + (\text{Soc}(M) \cap J(M))$. We must show that $J \subseteq \sqrt{[A + (\text{Soc}(M) \cap J(M));R M]}$. Now, assume that $S \in J$, if $Sw \in A$ and $A$ is a RNPr $S$-module, then $S \in \sqrt{[A + (\text{Soc}(M) \cap J(M));R M]}$, that is $J \subseteq \sqrt{[A + (\text{Soc}(M) \cap J(M));R M]}$.

$\Leftarrow$ Assume that $Sw \in A$ for $S \in R$, $w \in M$, then either $w \in A + (\text{Soc}(M) \cap J(M))$ or $w \notin A + (\text{Soc}(M) \cap J(M))$, that is $w \notin \sqrt{[A + (\text{Soc}(M) \cap J(M));R M]}$. Hence either $w \in A + (\text{Soc}(M) \cap J(M))$ or $w \notin \sqrt{[A + (\text{Soc}(M) \cap J(M));R M]}$. That is $A$ is a RNPr $S$-module of $M$.

The following corollaries are a direct consequence of the proposition (4.1).

**Corollary 4.2**

Let $M$ be an $R$-modul, and $A$ be a proper $S$-module of $M$. Then $A$ is a RNPr $S$-module of $M$ iff $S\beta \subseteq A$ for $S \in R$ and $\beta$ is a $S$-module of $M$, it means that $\beta \subseteq A + (\text{Soc}(M) \cap J(M))$ or $S \subseteq \sqrt{[A + (\text{Soc}(M) \cap J(M));R M]}$.

**Corollary 4.3**

Let $M$ be an $R$-modul, and $A$ be a proper $S$-module of $M$. Then $A$ is a RNPr $S$-module of $M$ iff $J w \subseteq A$ for $J$ is an ideal of $R$, $w \in M$, it means that $w \in A + (\text{Soc}(M) \cap J(M))$ or $J \subseteq \sqrt{[A + (\text{Soc}(M) \cap J(M));R M]}$.

**Proposition 4.4**

Let $M$ be a multiplication $R$-modul, and $A$ is a proper $S$-module of $M$. Then $A$ is a RNPr $S$-module of $M$ iff, whenever $C \beta \subseteq A$ for $C$, $\beta$ are $S$-module $s$ of $M$, it means that either $\beta \subseteq A + (\text{Soc}(M) \cap J(M))$ or $C^k \subseteq A + (\text{Soc}(M) \cap J(M))$ for some positive integer $k$.

**Proof**
Assume that $\beta \subseteq \mathcal{A}$ for $K, \beta$ are $S$-modules of $\mathcal{M}$. Since $\mathcal{M}$ is a multiplication, then $C = \mathcal{M}$, $\beta = \mathcal{M}$ for some ideals $I, J$ of $\mathcal{R}$. That is $I(J) \subseteq \mathcal{A}$. But $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}$, then by proposition (4.1) either $\mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$ or $\mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$ for some positive integer $\kappa$. It follows that either $\beta \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$ or $\beta \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$.

$\implies$ Assume that $IC \subseteq \mathcal{A}$ for $I$ is an ideal of $\mathcal{R}, C$ is a $S$-module of $\mathcal{M}$. Since $\mathcal{M}$ is a multiplication $C = \mathcal{M}$ for $J$ is an ideal of $\mathcal{R}$. Thus $I(J) \subseteq \mathcal{A}$ it follows that $LC \subseteq \mathcal{A}$ where $L = \mathcal{M}$. Then by hypothesis we have $C \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$ or $L^\kappa \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$ that is $I^\kappa \mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$. Thus $C \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$ or $I^\kappa \subseteq [\mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})]_{\mathcal{M}}$. Hence by proposition (4.1) $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}$.

As direct application of proposition (4.4) we get the following corollary:

**Corollary 4.5**

Let $\mathcal{M}$ be a multiplication $\mathcal{R}$-module, and $\mathcal{A}$ is a proper $S$-module of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr iff wherever $mm' \subseteq \mathcal{A}$, for $m, m' \in \mathcal{M}$, it means that $m' \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$ or $m^\kappa \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap \{J\})$ for some positive integer $\kappa$.

**Remark 4.6**

If $\mathcal{A}$ is a faithful multiplication $\mathcal{R}$-module of an $\mathcal{R}$-module $\mathcal{M}$, then $[\mathcal{A} : \mathcal{M}]$ need not to be an RNPr ideal of $\mathcal{R}$.

The following example illustrates that:

**Example 4.7**

Consider the $Z$-module $\mathcal{Z}$, the $S$-module $\mathcal{A} = 1$. $\mathcal{A}$ is an RNPr $S$-module of $\mathcal{Z}$ by remark (2.2). But $[\mathcal{A} : \mathcal{Z}] = [\mathcal{Z}] = 50Z$ is not an RNPr ideal of $\mathcal{Z}$ because $5, 10 \in 50Z$ for $5, 10 \in Z$ but $10 \in 50Z + (soc(Z) \cap \{J\}) = 50Z + (0) = 50Z$ and $5 \in \sqrt{[50Z + (soc(Z) \cap \{J\})]} = \sqrt{50Z + (0) + 1} = \sqrt{50Z + (0)}$.

The following proposition shows that the residue of an RNPr $S$-module is an RNPr ideal of $\mathcal{R}$ under certain condition.

Before that, we must mention the following:

**Corollary 4.8**[4, Coro. (2.14)(i)]

If $\mathcal{M}$ is a faithful multiplication $\mathcal{R}$-modul, then $\text{Soc} (\mathcal{R}) \mathcal{M} = \text{Soc}(\mathcal{M})$.

**Remark 4.9**[11, Remark, P14]

If $\mathcal{M}$ is a faithful multiplication $\mathcal{R}$-modul, then $J(\mathcal{M}) = J(\mathcal{R}) \mathcal{M}$.

**Proposition 4.10**

...
Let $\mathcal{M}$ be a faithful multiplication $\mathcal{R}$-modul and $\mathcal{A}$ is a proper $\mathcal{S}$-module of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-module of $\mathcal{M}$ iff $[\mathcal{A};\mathcal{R}\mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$.

**Proof**

$\Rightarrow$ Assume that $SJ \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]$ for $S \in \mathcal{R}$, $J$ is an ideal of $\mathcal{R}$, implies that $S(J\mathcal{M}) \subseteq \mathcal{A}$. But $\mathcal{A}$ is a RNPr $\mathcal{S}$-module of $\mathcal{M}$, then by corollary (4.2) either $JM \subseteq A + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $S^k\mathcal{M} \subseteq A + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. Since $\mathcal{M}$ is multiplication, then $\mathcal{A} = [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M}$, and $\mathcal{M}$ is faithful multiplication, then by corollary (4.8) and remark (4.9) $\text{Soc}(\mathcal{R})M = \text{Soc}(\mathcal{M})$ and $J(\mathcal{M}) = J(\mathcal{M})$, therefore $(\text{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M} = (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. Thus either $JM \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M} + (\text{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$ or $S^k\mathcal{M} \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M} + (\text{Soc}(\mathcal{R}) \cap J(\mathcal{M}))$. Hence either $JM \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M} + (\text{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$ or $S^k\mathcal{M} \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M} + (\text{Soc}(\mathcal{R}) \cap J(\mathcal{M}))$. Also by corollary (4.2) $[\mathcal{A};\mathcal{R}\mathcal{M}]$ is RNPr ideal of $\mathcal{R}$.

$\Leftarrow$ Assume that $wH \subseteq \mathcal{A}$ for $w \in \mathcal{M}$ and $H$ is a $\mathcal{S}$-module of $\mathcal{M}$. Since $\mathcal{M}$ is a multiplication, then $w = RH = J\mathcal{M}$, for some ideals $I, J$ of $\mathcal{R}$, that is $J\mathcal{M} \subseteq \mathcal{A}$, implies that $JI \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]$, but $[\mathcal{A};\mathcal{R}\mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$, then by proposition (4.1) either $I \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}] + (\text{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$ or $S^k\mathcal{M} \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}] + (\text{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$. Hence either $IM \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M} + (\text{Soc}(\mathcal{R}) \cap J(\mathcal{R}))\mathcal{M}$ or $S^kIM \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M} + (\text{Soc}(\mathcal{R}) \cap J(\mathcal{M}))\mathcal{M}$. Hence by corollary (4.8) and remark (4.9) $\text{Soc}(\mathcal{R})\mathcal{M} = \text{Soc}(\mathcal{M})$ and $J(\mathcal{R})\mathcal{M} = J(\mathcal{M})$, therefore either $IM \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $S^kIM \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. Then $H \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $S^kH \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. Also by corollary (4.2) $\mathcal{A}$ is a RNPr $\mathcal{S}$-module of $\mathcal{M}$.

Before this proposition, we should mention the following two propositions:

**Proposition 4.11** [2, Prop. (17.10)]

If $\mathcal{M}$ is projective $\mathcal{R}$-modul then $J(\mathcal{M}) = J(\mathcal{M})\mathcal{M}$.

**Proposition 4.12** [11]

If $\mathcal{M}$ is projective $\mathcal{R}$-modul then $\text{Soc}(\mathcal{M}) = \text{Soc}(\mathcal{R})\mathcal{M}$.

**Proposition 4.13**

Let $\mathcal{M}$ be a projective multiplication $\mathcal{R}$-modul and $\mathcal{A}$ be a proper $\mathcal{S}$-module of $\mathcal{M}$. Then $\mathcal{A}$ is a RNPr $\mathcal{S}$-module of $\mathcal{M}$ if and only if $[\mathcal{A};\mathcal{R}\mathcal{M}]$ is a RNPr ideal of $\mathcal{R}$.

**Proof**

$\Rightarrow$ Assume that $S\mathcal{J} \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]$, for $\mathcal{J}$ is an ideal of $\mathcal{R}$, $S \in \mathcal{R}$, it follows that $S\mathcal{J}\mathcal{M} \subseteq \mathcal{A}$, since $\mathcal{A}$ is a RNPr $\mathcal{S}$-module of $\mathcal{M}$, then by proposition (4.1) either $S\mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $S^k\mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ for some positive integer $k$. But $\mathcal{M}$ is a projective $\mathcal{R}$-modul, then by propositions (4.11) and (4.12) $\text{Soc}(\mathcal{M}) = \text{Soc}(\mathcal{R})\mathcal{M}$, and $J(\mathcal{M}) = J(\mathcal{M})\mathcal{M}$, and $\mathcal{M}$ is multiplication $\mathcal{R}$-modul then $\mathcal{A} = [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M}$. Hence $S\mathcal{M} \subseteq [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M} + (\text{Soc}(\mathcal{R})\mathcal{M} \cap J(\mathcal{M})) = [\mathcal{A};\mathcal{R}\mathcal{M}]\mathcal{M}$.
\( (\text{Soc}(R) \cap J(R)) M + J^k M \subseteq [A :_R M] M + (\text{Soc}(R) \cap J(R)) M \). It follows that 
\( S \subseteq [A :_R M] + (\text{Soc}(R) \cap J(R)) \) or \( J^k \in [A :_R M] + (\text{Soc}(R) \cap J(R)) = [[A :_R M] + (\text{Soc}(R) \cap J(R))] :_R R] \). Hence by proposition (4.1) \([A :_R M] \) is a RNPr ideal of \( R \).

\( \iff \) Assume that \( mm' \subseteq A \), for \( m, m' \in M \). Since \( M \) is a multiplication \( R \)-module, then \( m = Rm = lM \) \( m' = Rm' = JM \) for some ideals \( l, J \) of \( R \), it follows that \( lJM \subseteq A \), hence \( lJ \subseteq [A :_R M] \). But \([A :_R M] \) is a RNPr ideal of \( R \), then by proposition (4.1) either \( J \subseteq [A :_R M] + (\text{Soc}(R) \cap J(R)) \) or \( J^k \in [A :_R M] + (\text{Soc}(R) \cap J(R)) :_R R] \). Hence either \( J M \subseteq [A :_R M] M + (\text{Soc}(R) \cap J(R)) M \) or \( J^k M \subseteq [A :_R M] M + (\text{Soc}(R) \cap J(R)) M \). Since \( M \) is a projective \( R \)-module, then by propositions (4.11) and (4.12) \( \text{Soc}(M) = \text{Soc}(R)M \) and \( J(M) = J(R)M \). Hence \( m' \subseteq A + (\text{Soc}(M) \cap J(M)) \) or \( m'^k \subseteq A + (\text{Soc}(M) \cap J(M)) \). So by corollary (4.5) \( A \) is a RNPr \( \delta \)-module \( M \).

Before this proposition, we should mention the following two corollary and proposition:

**Corollary 4.14** [5, Coro. (1.26)]
Let \( M \) be a non-singular \( R \)-modul, then \( \text{Soc}(R) M = \text{soc}(M) \).

**Proposition 4.15** [11, prop (1.11)]
If \( M \) is content module, then \( J(M) = J(R)M \).

**Proposition 4.16**
Let \( M \) be a non-singular content multiplication \( R \)-modul, \( A \) a proper \( S \)-module of \( M \). Then \( A \) is a RNPr \( \delta \)-module of \( M \) if \([A :_R M] \) is a RNPr ideal of \( R \).

**Proof**
\( \Rightarrow \) Assume that \( \delta w \subseteq [A :_R M] \) for \( \delta, w \in R \), implies that \( \delta(wM) \subseteq A \). But \( A \) is a RNPr \( \delta \)-module of \( M \), then by corollary (4.2) either \( wM \subseteq A + (\text{Soc}(M) \cap J(M)) \) or \( \delta^k M \subseteq A + (\text{Soc}(M) \cap J(M)) \). Now, since \( M \) is non-singular, then by corollary (4.14) \( \text{Soc}(R)M = \text{Soc}(M) \) and \( M \) is content then by proposition (4.15) \( J(R)M = J(M) \) and \( M \) is multiplication, then \( A = [A :_R M] M \). Thus either \( wM \subseteq [A :_R M] M + (\text{Soc}(R) \cap J(R)) M \) or \( \delta^k M \subseteq [A :_R M] M + (\text{Soc}(R) \cap J(R)) M \). It follows that \( w \subseteq [A :_R M] + (\text{Soc}(R) \cap J(R)) \) or \( \delta^k \subseteq [A :_R M] + (\text{Soc}(R) \cap J(R)) :_R R \). Thus by corollary (4.2) \([A :_R M] \) is a RNPr ideal of \( R \).

\( \Leftarrow \) Assume that \( Aw \subseteq A \), where \( H \) is a \( S \)-module of \( M \), \( w \in M \). Since \( M \) is a multiplication, then \( H = J M, w = Rw = IM \) for some ideals \( I, J \) of \( R \), it follows that \( JIM \subseteq A \), implies that \( JI \subseteq [A :_R M] \), but \([A :_R M] \) is a RNPr ideal of \( R \), then by proposition (4.1) either \( I \subseteq [A :_R M] + (\text{Soc}(R) \cap J(R)) \) or \( J^k \subseteq [A :_R M] + (\text{Soc}(R) \cap J(R)) :_R R \) = \([A :_R M] + (\text{Soc}(R) \cap J(R)) \), that is \( IM \subseteq [A :_R M] M + (\text{Soc}(R) \cap J(R)) M \) or \( I^k M \subseteq [A :_R M] M + (\text{Soc}(R) \cap J(R)) M \). Since \( M \) is non-singular, then by corollary (4.14) \( \text{Soc}(R)M = \text{Soc}(M) \) and \( M \) is content then by proposition (4.15) \( J(R)M = J(M) \) and \( M \) is multili-
cATION, THEN $\mathcal{A} = [\mathcal{A}: \mathcal{M}]\mathcal{M}$. Hence either $w \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ or $H^k \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$. Thus by corollary (4.2) $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}$.

**Proposition 4.17**

Let $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ be an $R$-monomorphism and $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}$ and $\text{Ker} \ \Phi \subseteq \mathcal{A}$ with $\text{Ker} \ \Phi$ is a small, $\Phi(\mathcal{A})$ is a RNPr $S$-module of $\mathcal{M}'$.

**Proof**

$\Phi(\mathcal{A})$ is a proper $S$-module of $\mathcal{M}'$. If not, that is $\Phi(\mathcal{A}) = \mathcal{M}'$. Assume that $w \in \mathcal{M}'$, then $\Phi(w) \in \mathcal{M}' = \Phi(\mathcal{A})$, so there exists $x \in \mathcal{A}$ such that $\Phi(w) = \Phi(x)$, implies that $\Phi(w - x) = 0$, that is $w - x \in \text{Ker} \ \Phi \subseteq \mathcal{A}$, it follows that $w \in \mathcal{A}$. Thus $\mathcal{A} = \mathcal{M}$ contradiction. Assume that $\mathcal{S}w \in \Phi(\mathcal{A})$, where $\mathcal{S} \in R$, $w \in \mathcal{M}'$ with $w \notin \Phi(\mathcal{A}) + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$, that is $\Phi(\mathcal{S}w) = \mathcal{S}\Phi(w) \in \Phi(\mathcal{A})$ implies that $\mathcal{S}\Phi(w) \in \mathcal{A}$, with $\Phi(w) \in (\Phi(\mathcal{A}) + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}')) = \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ and $\Phi(w) \in \mathcal{M}$. But $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}'$, it follows that $\mathcal{S}^k \mathcal{M} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}) \cap J(\mathcal{M}))$ for some positive integer $k$. To show that $\mathcal{S}^k \Phi(\mathcal{M}) \subseteq \Phi(\mathcal{A}) + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$, if $w \in \Phi(\mathcal{M})$, then $\Phi^{-1}(w) \in \mathcal{M}$. Thus $\Phi^{-1}(w) \in \Phi^{-1}(\mathcal{S}^k w) \in \mathcal{A} + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$, that is $\mathcal{S}^k \Phi(\mathcal{M}) \subseteq \Phi(\mathcal{A}) + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$, so $\Phi(\mathcal{A})$ is a RNPr $S$-module of $\mathcal{M}$.

**Proposition 4.18**

Let $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ be an $R$-monomorphism and $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}'$ and $\text{Ker} \ \Phi$ is a small, $\Phi^{-1}(\mathcal{A})$ is a RNPr $S$-module of $\mathcal{M}$.

**Proof**

Assume that $\mathcal{S}w \in \Phi^{-1}(\mathcal{A})$, where $\mathcal{S} \in R$, $w \in \mathcal{M}'$ with $w \notin \Phi^{-1}(\mathcal{A}) + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$, that is $\Phi(\mathcal{S}w) = \mathcal{S}\Phi(w) \in \Phi^{-1}(\mathcal{A}) = \mathcal{A}$, implies that $\mathcal{S}\Phi(w) \in \mathcal{A}$, with $\Phi(w) \in (\Phi^{-1}(\mathcal{A}) + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}')) = \mathcal{A} + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$ and $\Phi(w) \in \mathcal{M}'$. But $\mathcal{A}$ is a RNPr $S$-module of $\mathcal{M}'$, it follows that $\mathcal{S}^k \mathcal{M}' \subseteq \mathcal{A} + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$ for some positive integer $k$. To show that $\mathcal{S}^k \Phi^{-1}(\mathcal{M}') \subseteq \Phi^{-1}(\mathcal{A}) + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$, if $w \in \Phi^{-1}(\mathcal{M}')$, then $\Phi(w) \in \mathcal{M}$. Thus $\Phi(w) \in \Phi(\mathcal{S}^k w) \in \mathcal{A} + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$, that is $\mathcal{S}^k \Phi^{-1}(\mathcal{M}') \subseteq \Phi^{-1}(\mathcal{A}) + (\text{Soc}(\mathcal{M}') \cap J(\mathcal{M}'))$, so $\Phi^{-1}(\mathcal{A})$ is a RNPr $S$-module of $\mathcal{M}$.
References


Article submitted 16 August 2022. Published as resubmitted by the authors 30 Dec. 2022.