On Set-Texture Spaces and Degree-Texture Spaces

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Abstract— A general theory of fuzzy sets was described employing texture spaces. This work aimed to introduce two concepts in texture theory, namely set-texture and degree-texture spaces using topological spaces and new properties. Moreover, many examples were studied to investigate new texture spaces. The new concepts will use in other topological spaces and find another application on it.

Keywords— Degree- topology, Set-T_0 Space, Texture space, Set-Texture Space, Degree-Texture Space.

1   Introduction

The concepts of texture space and ditopological texture space as fuzzy structures and ditopological fuzzy structures were introduced by Brown at the 2nd BUFSA Conference on Fuzzy Systems and Artificial Intelligence held at Trabzon University in 1992. In 2000, Lawrence and Erturk investigated the complement products of texture and sums [2]. While the concept of subtexture by comparing to quotient texture in a complemented texture space defined by the same authors [3]. Brown et.al. extended the bitopological notion of sequential normality to ditopological texture spaces. They also defined (pseudo-) dimetrizability [4].

The concept of paracompactness was discussed in ditopological texture spaces [5]. In 2019, Kule and Dost [6] used the concepts of semi-open and semi-closed sets to introduce the semi-separation axioms in ditopological texture spaces. In addition, they had been establishing the properties of these concepts and the relations between them. In [7], the ideal texture space and the di-local function were introduced by using the families of neighborhood structure for a ditopological texture space.

Currently, Kunduraci et. al. established texture graphs from texture spaces, known as ditopological texture graphs, via corresponding *n*-point graphs with n -point dito-pologies [8], for more details see [9-13]. Hameed and Kadhem [14] introduced a new topology on a graph, namely the degree topology that is defined by the degree of the vertices of the graphs. Moreover, they initiated a new property, namely set-T_0 space and discussed the relationship between this property and T_0-space.

This paper aims to relocate the topological space of the graph to the texture space. Set-texture space and degree-texture spaces that are based on topological spaces and set-T_0-spaces are presented and discussed. By using the relevant examples, we have furthered the new concepts.

2   Preliminaries

This section includes the basic ideas regarding the texture space, set-T_0 space and degree- topology are specified which will benefit in the rest of sections.

Definition 2.1 [2]: Let A be any set and let ψ be a subset of $P(A)$, then ψ is called a texture of A, if it is satisfied the following:

- $(\psi, \subseteq)$ is a complete lattice including A and $\emptyset$ besides the meet and join operations in $(\psi, \subseteq)$ are correlated alongside the intersection and union operations.
- ψ is completely distributive.
- ψ divides the points of A. Specifically, for any distinct elements in A there exists U in ψ contains one element and not the other.

Definition 2.2 [14]: Let A be any set and let ψ be a subset of $P(A)$, then ψ is called a texture of A, if it is satisfied Let $G(V, E)$ be a simple graph and K be the max degree of all vertices in G. Then, the topology that defines on vertex set V and generated by a basis B is called degree-topology and denoted by T_{deg} where $B_{deg} = \{A_i : i = 0, \dots, k\}$, where A_i is the set of all vertices that have a degree i, and k is the maximum degree of all vertices in G.

Definition 2.3 [14]: Let X be a non-empty set and τ be a topology on X, then τ is called set- T_0 space which is denoted by $T_{0(s)}$ if there exist non-empty sets $M_1, M_2, M_3, \ldots M_K \in \tau$ with $M_i \neq X$ for all $i = 1,2, \ldots, k$ and k is any natural number such that $\bigcap_{i=1}^{k} M_i = \emptyset$ and, $\bigcup_{i=1}^{k} M_i = X$.

Remark 2.4 [14]:

1. The set-T_0 space is not necessarily satisfied T_0 space

2. The T_0-space is not necessarily satisfied the set-T_0 space

Theorem 2.5 [14]: Every degree- topology is a set-T_0 space

3   Set-Texture Space

Definition 3.1: Let S be a non-empty set and τ be a topology on S, then τ is called a Set-texture of S and S is called a Set-textured by τ, if there is a relation $\wp$ satisfies the following conditions:

- $(\tau, \wp)$ is a complete lattice.
- $(\tau, \cap, \cup)$ is a distributive lattice.
- τ is Set-T_0 space.

Then, (S, τ), is called Set-texture space

Example 3.2: Let $S = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$, and $\wp$ be the inclusion relation defined on τ. Then, τ is a Set-texture of S because $\wp = \{(\emptyset, \emptyset),\ (\emptyset, S),\ (\emptyset, \{a\}),\ (\emptyset, \{b, c\}),\ (\{a\}, S), (\{a\}, \{a\}\ ), (S, S), (\{b, c\}, S),\ (\{b, c\}, \{b, c\})\}$. We have $\wp$ is a reflexive, anti-symmetric, and transitive relation. Since for every pair of elements in X that the supremum is S and the infimum is $\emptyset$. Thus, $(\tau, \wp)$ is a lattice, and so it is a complete lattice due to it is finite. Now, for all $A, B, C \in \tau$, that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ implies $(\tau, \cap, \cup)$ is a distributive lattice. Now, take $M_1 = \{a\}$, and $M_2 = \{b, c\}$, we notice that $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = S$. Therefore, τ is a Set-texture of S.

Example 3.3: Let $E = \{e_1, e_2, e_3, e_4\}$, $\tau = \{E, \emptyset, \{e_1, e_3\}, \{e_2, e_4\}\}$, and $\wp$ be the inclusion relation defined on τ. Then, τ is a Set-texture of S because $\wp = \{(\emptyset, \emptyset), (\emptyset, E), (\emptyset, \{e_1, e_3\}), (\emptyset, \{e_2, e_4\}), (\{e_1, e_3\}, E), (\{e_1, e_3\}, \{e_1, e_3\}), (E, E), (\{e_2, e_4\}, E), (\{e_1, e_3\}, \{e_1, e_3\})\}$ and $\wp$ is a reflexive, anti-symmetric, and transitive relation. Since for every pair of elements, the supremum is E and the infimum is $\emptyset$. Then, $(\tau, \wp)$ is a lattice and so it is a complete lattice since it is a finite set. Now, for all $A, B, C \in \tau$, that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Then, $(\tau, \cap, \cup)$ is a distributive lattice. Take $M_1 = \{e_1, e_3\}$, and $M_2 = \{e_2, e_4\}$ we notice that $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = S$. Therefore, τ is a Set-texture of S.

Example 3.4: Let $X = \{y, w, z\}$ and $\tau = \{\emptyset, X, \{y\}, \{z\}, \{y, z\}\}$ be a topology on X. Then, τ is not a Set-texture of X since it is not a set-T_0 space because there are no open subsets of X which satisfy the conditions of a set-T_0 space.

Remark 3.5: Any two sets A and B in degree-topology are satisfied one of the following:

1. $A \cap B = \emptyset$.
2. $A \subseteq B$, in this case, $A \cap B \neq \emptyset$.

Remark 3.6: Note that the topology τ in Example 3.3 with the inclusion relation ξ define on τ is a texture space cause $\xi = \{(\emptyset, \emptyset), (\emptyset, X), (\emptyset, \{y\}), (\emptyset, \{z\}), (\emptyset, \{y, z\}), (\{y\}, X), (\{y\}, \{y, z\}), (\{z\}, \{y, z\}), (X, X), (\{z\}, X), (\{y, z\}, X), (\{y\}, \{y\}), (\{z\}, \{z\}), (\{y, z\}, \{y, z\})\}$. And, ξ is a reflexive, anti-symmetric, and transitive relation. Since for every pair of elements, the supremum is X and the infimum is $\emptyset$. Then, (τ, ξ) is a lattice and it is complete for lattice due to it being finite. Now, for all $A, B, C \in \tau$, that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Then, $(\tau, \cap, \cup)$ is a distributive lattice. Since for any two distinct elements in X, there is an open set containing one of them and not both. Hence, τ is a texture space.

Remark 3.7:

1. A Set-texture space is not necessarily a texture space as in Example 3.2 which is a Set-texture space but not a texture space. Since for $e_1, e_3 \in V$ where $e_1 \neq e_3$, there is no open set containing either e_1 or e_3 but not both.

2. Texture space is not necessarily satisfied a Set-texture space as in Example 3.4 which is not a Set-texture space and it is texture space.

4   Degree-Texture Space

Definition 4.1: Let S be a non-empty set and T_{deg} be a degree-topology on S. Then, T_{deg} is called a degree-texture of S and S is called a degree-textured which is denoted by T_{deg} provided satisfying the following conditions:

- $(T_{deg}, \Im)$ is a complete lattice.
- $(T_{deg}, \cap, \cup)$ is a distributive lattice.

Then, (S, T_{deg}) is called degree-texture space. The relation $\Im$ is defined by $\Im = \{A \times B \subseteq T_{deg} \times T_{deg} : A = B \text{ or} |A| < |B|\}\}$, where $|A|$ and $|B|$ are the numbers of vertices in A and B, respectively. And, $\Im$ is called a degree- relation.

Remark 4.2: Let $\Im$ be the relation $\Im$ which is defined in Definition 4.1 with degree-topology T_{deg}. Then, $(T_{deg}, \Im)$ is a complete lattice and $(T_{deg}, \cap, \cup)$ is a distributive lattice because if we permit $G(V, E)$ as a graph with vertex set V and E as an edge set. Firstly, we must prove that $(T_{deg}, \Im)$ is a lattice where $\Im = \{A \times B \subseteq T_{deg} \times T_{deg} : A = B \text{ or} |A| < |B| \}$, where $|A|$ and $|B|$ the numbers of vertices in A and B, respectively. For any $A, B \in T_{deg}$, that $A = B$ or $|A| < |B|$, for $A = A$. Then, $\Im$ is a reflexive relation.

If $(A, B) \in \Im$, then $A = B$ or $|A| < |B|$. And, if $(B, A) \in \Im$, then $B = A$ or $|B| < |A|$. Consequently, there are many cases:

- If $A = B$ and $B = A$, then $\Im$ is anti-symmetric
- If $A = B$ and $|B| < |A|$, then it is impossible for if $A = B$ then $|A| = |B|$ similarly when $B = A$ and $|A| < |B|$.
- If $|A| < |B|$ and $|B| < |A|$, then $|B| = |A|$, so $A = B$, and so $\Im$ is anti-symmetric.

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So, $(T_{deg}, \Im)$ is an anti-symmetric relation.

For any $A, B, C \in T_{deg}$, with $(A, B) \in \Im$, then $A = B$ or $|A| < |B|$. If $(B, C) \in \Im$, then $B = C$ or $|B| < |C|$. Either $A = B = C$, then $A \times A \in \Im$. So, $\Im$ is a transitive relation. Or $|A| < |B| < |C|$, then $|A| < |C|$ that is $(A, C) \in \Im$. So, $\Im$ is a transitive relation. Hence, $(T_{deg}, \subseteq)$ is a partial order set. For any $A, B \in T_{deg}$, there is $V \in T_{deg}$ such that $A \times V \in \Im$ and $B \times V \in \Im$. Since no element W in T_{deg} less than V such that $A \times W \in \Im$ and $B \times W$. So A and B have a supremum V in T_{deg}, there is $\emptyset \in T_{deg}$ such that $\emptyset \times A \in \Im$ and $\emptyset \times B \in \Im$. Since no element M in T_{deg} greater than $\emptyset$ such that $\emptyset \times A \in \Im$ and $\emptyset \times B \in \Im$. Then, A and B have the infimum $\emptyset$ in T_{deg}. Hence, $(T_{deg}, \Im)$ is a lattice, and it is a complete lattice due to it is a finite set. To investigate the distribution $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, for all $A, B, C \in T_{deg}$, there are many cases:

1. If $A \subseteq B$, $A \subseteq C$, and $B \subseteq C$, then $A \cap (B \cup C) = A \cap C = A$ and $(A \cap B) \cup (A \cap C) = A \cup A = A$. Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2. If $A \subseteq B$, $A \subseteq C$, and $B \cap C = \emptyset$, then $A \cap (B \cup C) = A$ and $(A \cap B) \cup (A \cap C) = A$. Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

3. If $A \subseteq B$, $B \subseteq C$, and $A \cap C = \emptyset$, then $A \cap (B \cup C) = A \cap B = A$ and $(A \cap B) \cup (A \cap C) = A \cup \emptyset = A$. So, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. If $A \subseteq C$, $B \subseteq C$, and $A \cap B = \emptyset$, then $A \cap (B \cup C) = A \cap C = A$ and $(A \cap B) \cup (A \cap C) = A \cup \emptyset = A$. Consequently, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

5. If $A \subseteq C$, $B \cap C = \emptyset$, and $A \cap B = \emptyset$, then $A \cap (B \cup C) = A \cap A = A$ and $(A \cap B) \cup (A \cap C) = A \cup \emptyset = A$. So, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

6. If $B \subseteq C$, $A \cap B = \emptyset$, and $A \cap C = \emptyset$, then $A \cap (B \cup C) = A \cap C = \emptyset$ and $(A \cap B) \cup (A \cap C) = \emptyset \cup \emptyset = \emptyset$. Accordingly, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. If $B \cap C = \emptyset$, $A \cap B = \emptyset$, and $A \cap C = \emptyset$, then $A \cap (B \cup C) = \emptyset$ and $(A \cap B) \cup (A \cap C) = \emptyset$. Subsequently, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

8. If $A \subseteq B$, $A \cap C = \emptyset$, and $B \cap C = \emptyset$, then $A \cap (B \cup C) = A$ and $(A \cap B) \cup (A \cap C) = A$. So, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Therefore, $(T_{deg}, \cap, \cup)$ is distributive.

Example 4.3: Let $P_4(V,E)$ be a path graph of order four with the first vertex v_1 and the last vertex v_4. Then, $V = \{v_1, v_2, v_3, v_4\}$ we have $A_0 = \emptyset, A_1 = \{v_1, v_4\}$, and $A_2 = \{v_2, v_3\}$. Accordingly, the basis for T_{deg} is $\{\emptyset, \{v_1, v_4\}, \{v_2, v_3\}\}$ and by taking all unions, the degree-topology can be written as $\{\emptyset, V, \{v_1, v_4\}, \{v_2, v_3\}\}$.

Then, $T_{deg}(P_4)$ is a degree texture of V with the degree-relation $\Im$ which is defined on $T_{\deg}(P_4)$ as

$\Im = \{(\emptyset, \emptyset), (\emptyset, V), (\emptyset, \{v_1, v_4\}), (\emptyset, \{v_2, v_3\}), (\{v_1, v_4\}, V), (\{v_1, v_4\}, \{v_1, v_4\}),$
$(V, V), (\{v_2, v_3\}, V), (\{v_2, v_3\}, \{v_2, v_3\})\}$. And, $\Im$ is a reflexive, anti-symmetric, and transitive relation. Since every pair of an element has the supremum is V and the infimum is $\emptyset$. Then, $(T_{deg}(P_n), \Im)$ be lattice and it is a complete lattice being finite. For all $A, B, C \in \tau$, we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Then $(T_{deg}(P_n), \cap, \cup)$ is a distributive lattice. Note that $T_{deg}(P_n)$ is the Set-T_0 space for if the permit $M_1 = \{v_1, v_4\}$, and $M_2 = \{v_2, v_3\}$ such that $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = V$. Hence, $T_{deg}(P_n)$ is the set texture of V.

Theorem 4.4: Every degree-texture space is a Set-texture space.

Proof: Since degree-topology is a topology and by Remark 3.7 part (2) it is satisfied the complete lattice and distributive lattice. Theorem 2.5 implies that every degree-topology is satisfied set-T_0 space. Thus, it is a Set-texture space as desired. ∎

Remark 4.5:

1) The converse of Theorem 4.4 is not true in general as in Example 3.2 which is a set-texture space but not a degree-texture space. Since the inclusion relation is different from the degree relation.

2) Degree-texture space is not necessarily satisfied texture space and the converse for the inclusion relation is different from the degree-relation. Remark 2.4 asserts that not every T_0 space is satisfied Set-T_0 space and vice versa.

6   References

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