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On Fuzzy Feebly Compact Space

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ABSTRACT: The object of this work to introduce a new form of fuzzy compact space in fuzzy topological space, named by fuzzy feebly compact space. It is stronger than a fuzzy compact space. We investigated the properties of fuzzy feebly compact in a fuzzy topological space. Important mathematical concepts were relied upon to prove the properties of fuzzy feebly compact such as the net, the sub net, the filter and the finite intersection property. The concept of fuzzy feebly density and its dependence in feebly compact space has also been introduced. Finally, the subspace of fuzzy feebly space was discussed.

Keywords: fuzzy feebly compact space, feebly open , feebly closed and feebly neighborhood.

1. INTRODUCTION

The concept of fuzzy has permeated all branches of mathematics after Zadeh introduced the concept of fuzzy set in 1965 [1]. Then fuzzy topological space was defined by Chang in 1968 [2]. Maheswari and Tapi in 1978 proposed the concepts feebly open set and feebly closed set [3]. In 1985 Lee, Je-Yoon and Gyu-Ihn introduced a weaker form of fuzzy open set, named by fuzzy feebly open set [4].

2. PRILIMINARIES

This section all the basic definition and Propositions necessary for our work are stated.

Definition 2.1 [5]. Let $\mathbb{W}$ be a non-empty set, a collection of maps from $\mathbb{W}$ to the closed interval [0,1] is a fuzzy set of $\mathbb{W}$. In other words, $A: \mathbb{W} \longrightarrow [0,1]$, then a fuzzy set $A = \{(x, A(x)) : x \in \mathbb{W}\}$, $\mathrm{I}^{\mathbb{W}}$ refer to the family of all fuzzy sets in $\mathbb{W}$.

Definition 2.2 [5]. Let $r \in [0,1]$, we define a fuzzy point x_r, as: $x_r(y) = \begin{cases} r & if \quad y = x \\ 0 & if \quad y \neq x \end{cases}$

We write FP ($\mathbb{W}$) to denote the set of all fuzzy points in a space $\mathbb{W}$. If $r \leq A(x)$, then we say x_r belong to a fuzzy set A.

Definition 2.3 [5]. Let ,$B \in \mathrm{I}^{\mathbb{W}}$, then $A$is called quasi-coincident with Bdenoted by AqBif and only if $A(x) + B(x) > 1$, for some $x \in \mathbb{W}$. If $A(x) + B(x) \leq 1$for every $x \in \mathbb{W}$, then $A$is not quasi-coincident with Bdenoted by$A\tilde{q}B$.

Proposition 2.4 [6]. Let $A, B \in \mathrm{I}^{\mathbb{W}}$. Then:

a) $A\tilde{q}B \Leftrightarrow A \leq B^c$;
b) $A \wedge B = 0 \Rightarrow A\tilde{q}B$;
c) $x_r \in A \Leftrightarrow x_r\tilde{q}A^c$

Definition 2.5 [6]. A fuzzy topology on $\mathbb{W}$ is family $\tau \leq \mathrm{I}^{\mathbb{W}}$ if we have the following conditions:

i) $1,0 \in \tau$;
ii) If $A, B \in \tau$, then $A \wedge B \in \tau$;

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iii) If $\{A_\alpha : \alpha \in \Delta(index\ set\}$is family of fuzzy sets in τ , then $\vee A_\alpha \in \tau$, where $\alpha \in \Delta$.

We say that the members of τ are open fuzzy sets and their complements are closed fuzzy sets. A fuzzy topological space on $\mathbb{W}$ is denoted by the pair ($\mathbb{W}$, τ), in short, we write (FTS).

Definition 2.6 [6]. Let ($\mathbb{W}$, τ) be a FTS and $A, B \in \mathrm{I}^{\mathbb{W}}$. Then:

a) The fuzzy interior of A defined as $A^\circ = \vee \{B: B \leq A, B \in \tau\}$

b) The fuzzy closure of A defined as $\overline{A} = \wedge \{B: A \leq B, B^c \in \tau\}$

Definition 2.7 [5]. Let $A \in \mathrm{I}^{\mathbb{W}}$. Then A is said to be Q-neighborhood (Q-nbd) of a fuzzy point x_r in $\mathbb{W}$ if there exists $B \in \tau$ such that $x_r q B \leq A$, all Q-neighborhood of a fuzzy point x_r we say the system of Q-neighborhood of x_r, write by $N^Q_{x_r}$.

Remark 2.8[2]. Let $A \in \mathrm{I}^{\mathbb{W}}$. Then A is open if and only if A is Q-neighborhood of every fuzzy point belong to A.

Theorem 2.9 [8]. Let $\mathbb{W}$ be a FTS and $B, A \in \mathrm{I}^{\mathbb{W}}$. Then:

i) $0 = \overline{0}$;

ii) $A^\circ = 1 - \overline{(1 - A)}$;

iii) $\overline{(A \vee B)} = \overline{A} \vee \overline{B}$, $\overline{(A \wedge B)} \leq \overline{A} \wedge \overline{B}$;

iv) $(A \wedge B)^\circ = A^\circ \wedge B^\circ$, $(A \vee B)^\circ = A^\circ \vee B^\circ$;

v) $\overline{\overline{A}} = \overline{A}$, $(A^\circ)^\circ = A^\circ$;

vi) $A^\circ \leq A \leq \overline{A}$;

vii) If $A \leq B$, then $A^\circ \leq B^\circ$ and $\overline{A} \leq \overline{B}$

Definition 2.10 [9]. Let $\mathbb{W}$ be a FTS and $A \in \mathrm{I}^{\mathbb{W}}$. Then:

a) A is called fuzzy feebly open (f-open) if $A \leq \overline{A^\circ}^\circ$;

b) A is called fuzzy feebly closed (f-closed) if $\overline{\overline{A}^\circ} \leq A$.

The set of all f-open sets denoted by $FFO(\mathbb{W})$.

Remark 2.12 [9]. Let $A, B \in \mathrm{I}^{\mathbb{W}}$. Then $A \wedge B$ is f-open if A and B are f-open in $\mathbb{W}$.

Definition 2.13. Let ($\mathbb{W}$, τ) be a FTS and $A, B \in \mathrm{I}^{\mathbb{W}}$. Then:

a) The fuzzy feebly interior ($A^{\circ f}$) of A defined as $A^{\circ f} = \vee \{B: B \leq A, B \in FFO(\mathbb{W})\}$

b) The fuzzy feebly closure ($\overline{A}^f$) of A defined as $\overline{A} = \wedge \{B: A \leq B, B^c \in FFO(\mathbb{W})\}$

Proposition 2.14. Let $\mathbb{W}$ be a FTS and $B, A \in \mathrm{I}^{\mathbb{W}}$. Then:

i) $A^\circ \leq A^{\circ f} \leq A$;

ii) $A \leq \overline{A}^f \leq \overline{A}$;

iii) $x_r \in \overline{A}^f \Leftrightarrow B \wedge A \neq 0, \forall B \in FFO(\mathbb{W}), x_r \in B$;

iv) A is fuzzy f-closed$\Leftrightarrow \overline{A}^f = A$;

v) $\overline{\overline{A}^f}^f = \overline{A}^f$;

vi) If $A \leq B$, then $\overline{A}^f \leq \overline{B}^f$;

vii) $\vee \overline{A_\alpha}^f \leq \overline{\vee A_\alpha}^f$;

viii) $A^{\circ f} = 1 - \overline{(1 - A)}^f$.

Proof. From Theorm 2.9**.**

Definition 2.15. Let $A \in \mathrm{I}^{\mathbb{W}}$. Then A is said to be feebly Q-neighborhood (feebly Q-nbd) of a fuzzy point x_r in $\mathbb{W}$ if there exists $B \in FFO(\mathbb{W})$ such that $x_r q B \leq A$.

All feebly Q-neighborhood of a fuzzy point x_r, we say the system of feebly Q-neighborhood of x_r, write by $N^{fQ}_{x_r}$.

Definition 2.16 [1]. Let $(\mathbb{W}, \tau)$ be a FTS and $A \in I^{\mathbb{W}}$. A fuzzy point x_r, we say a fuzzy limit point of A, if for every fuzzy open set G, then $(G - x_r) \wedge A \neq 0$.

Theorem 2.17. Let $(\mathbb{W}, \tau)$ be a FTS and $A \in I^{\mathbb{W}}$. Then a fuzzy point $x_r \in \overline{A}^f$ if and only if every fuzzy f-open B such that $x_r q B$ then AqB.

Proof. Let us assume that $x_r \in \overline{B}^f$ and B is a fuzzy f-open set of $I^{\mathbb{W}}$ such that $x_r qB$, if $A\tilde{q}B$, then $x_r \notin 1 - B$ and $A \leq 1 - B$ owing to proposition 2.4 (b). Since $1 - B$ is f-closed set, then $\overline{A}^f \leq 1 - B$. Hence $x_r \notin \overline{A}^f$, but this contradiction. Conversely, suppose that $x_r \notin \overline{A}^f$, thus we have fuzzy f-closed B in $\mathbb{W}$ such that $x_r \notin B$ and $A \leq B$, this implies $x_r q 1 - B$ from proposition 2.4 (c) and $A\tilde{q}1 - B$ from Proposition 2.4 (b). Hence $x_r \in \overline{A}^f$ if $x_r qB$ and AqB.

Definition 2.18 [10]. Let $\mathcal{S}: D \longrightarrow$FP$(\mathbb{W})$ be a mapping when D is a discrete set. Then $\mathcal{S}$ is called a fuzzy net in $\mathbb{W}$ denoted by $\{\mathcal{S}(n) \text{ or } x^n_{\alpha_n}: n \in D, x \in \mathbb{W}, \alpha_n \in (0,1]\}$ or simply $\{x^n_{\alpha_n}\}$.

Definition 2.19 [10]. Let $\mathcal{P} = \{y^m_{\alpha_m}: m \in E\}$ be a fuzzy net and $\mathcal{S} = \{x^n_{\alpha_n}: n \in D\}$is another fuzzy net. Then $\mathcal{P}$ is called fuzzy subnet of $\mathcal{S}$ if there exists a mapping $f: E \longrightarrow D$ such that:

a) $\mathcal{P} = f \circ \mathcal{S}$, (i.e. $y^i_{\alpha_i} = x^{f(i)}_{\alpha_{f(i)}}, \forall i \in E$);
b) For each $n \in D$, there exists $m \in E$ such that $f(m) \geq n$.

We shall denote a fuzzy subnet of a fuzzy net $\{x^n_{\alpha_n}: n \in D\}$by $\{x^{f(m)}_{\alpha_{f(m)}}: m \in E\}$.

Definition 2.20. Let $(\mathbb{W}, \tau)$ be a FTS and $\{x^n_{\alpha_n}\}$ is a fuzzy net in $\mathbb{W}$. If $A \in I^{\mathbb{W}}$, then:

i) $\{x^n_{\alpha_n}\}$ is called eventually with A if $\exists m \in D$ such that $x^n_{\alpha_n} qA$, $\forall n \geq m$.
ii) $\{x^n_{\alpha_n}\}$ is called frequently with A if $\forall n \in D, \exists m \in D$ such that $m \geq n$ and $\{x^n_{\alpha_n}\}qA$.

Definition 2.21. Let $(\mathbb{W}, \tau)$ be a FTS and $\{x^n_{\alpha_n}\}$ is a fuzzy net in $\mathbb{W}$. If $x_\alpha \in AFP(\mathbb{W})I^{\mathbb{W}}$, then:

i) $\{x^n_{\alpha_n}\}$ is called f-convergent to x_α dented by $\{x^n_{\alpha_n} \xrightarrow{f} x_\alpha\}$, if it is eventually with A, $\forall A \in N^{fQ}_{x_\alpha}$, x_α is said to be feebly limit point.
ii) $\{x^n_{\alpha_n}\}$ is called has f-ccluster point x_α dented by $\{x^n_{\alpha_n} \overset{f}{\propto} x_\alpha\}$, if it is frequently with A, $\forall A \in N^{fQ}_{x_\alpha}$.

Definition 2.22[7]. Let $(\mathbb{W}, \tau)$ be an FTS and the family $\mathcal{F} \leq I^{\mathbb{W}}$. Then $\mathcal{F}$ is called fuzzy filter base on $\mathbb{W}$ if:

1) $0 \in \mathcal{F}$;
2) There exists $A_3 \in \mathcal{F}$ s.t $A_3 \leq A_1 \wedge A_2$ $\forall A_1, A_2 \in \mathcal{F}$.

Definition 2.23. Let $(\mathbb{W}, \tau)$ be a FTS and $\mathcal{F}$ is a fuzzy filter base on $\mathbb{W}$, then $x_\alpha \in FP(\mathbb{W})$, we say fuzzy f-cluster of $\mathcal{F}$ if $x_\alpha \in \overline{F}^f$ for all $F \in \mathcal{F}$.

Definition 2.24. Let $(\mathbb{W}, \tau)$ be a FTS. We say that $\mathbb{W}$ is a fuzzy feebly Hausdorff (T_2-space) if every pair of distinct fuzzy points $x_r, y_s \in FP(\mathbb{W})$, there exists $A \in N^{fQ}_{x_r}$ and $B \in N^{fQ}_{y_s}$ such that $A \wedge B = 0$.

Proposition 2.25 [11],[12]. Let $(\mathbb{W}, \tau)$ be a FTS and $A, B \in I^{\mathbb{W}}$. Then $A \wedge \bar{B} \leq \overline{A \wedge B}$, $\forall A \in \tau$.

Definition 2.26. Let $(\mathbb{W}, \tau)$ be a FTS and $A, B \in I^{\mathbb{W}}$ when B is subspace of $\mathbb{W}$. Then A is called f-open in B if there exists f-open H in $\mathbb{W}$ such that $A = H \wedge B$.

Proposition 2.27. Let $(\mathbb{W}, \tau)$ be a FTS and $A, B \in I^{\mathbb{W}}$ when B is subspace of $\mathbb{W}$. Then A is f-open in B if it is f-open in $\mathbb{W}$.
Proof. Clear.

Proposition 2.28. Let $A, B \in I^{\mathbb{W}}$ when $C \leq B$ and B is f-open in $\mathbb{W}$. Then A is f-open in B if and only if $A = E \wedge B$, $E \in \tau$.

Proof. Since $E \wedge B \le E \wedge \overline{B^{\circ}}^{\circ} = E^{\circ\circ} \wedge \overline{B^{\circ}}^{\circ} = (E^{\circ} \wedge \overline{B^{\circ}})^{\circ} \le (\overline{E^{\circ} \wedge B^{\circ}})^{\circ} \le (\overline{(E \wedge B)^{\circ}})^{\circ}$. Thus $E \wedge B$ is f-open in $\mathbb{W}$. Hence A is f-open in B from Proposition 2.27. Conversely, since $A = E \wedge B$ and E is a fuzzy open in $\mathbb{W}$, when E is f-open. Hence A is f-open in B.

Definition 2.29. Let $(\mathbb{W}, \tau)$ be a FTS and $\mho \le\in I^{\mathbb{W}}$. Then $\mho$ is called has finite intersection property (f.i.p.) if the intersection of finite members of $\mho$ is non-empty.

Definition 2.30. Let $(\mathbb{W}, \tau)$ be a FTS and $C \in I^{\mathbb{W}}$. Then C is called feebly dense if there exists no f-closed set B in $\mathbb{W}$ such that $C < B < 1$ ($\overline{C}^{f} = 1$).

Theorem 2.31. Let $(\mathbb{W}, \tau)$ be a FTS and $C \in I^{\mathbb{W}}$. Then C is dense if and only if it is feebly dense, with $C^{\circ} \neq 0$.

Proof. To show that C is fuzzy feebly dense in $\mathbb{W}$. Let $x_\alpha \in \bar{C}$ but $x_\alpha \notin \overline{C}^{f}$, then $x_\alpha \in 1 - \overline{C}^{f}$. Thus $x_\alpha \in (1 - C)^{\circ f}$, this implies $x_\alpha \in \overline{(1 - C)}$. Since $C^{\circ} \le (C)^{\circ f}$, then $x_\alpha \in (1 - C^{\circ})$, so $x_\alpha \notin C^{\circ}$. Therefore, we have no fuzzy open A_{x_α} such that $A \le C$ and $A \wedge C = 0$ and this C contradiction with is dense set. Hence $x_\alpha \in \overline{C}^{f}$ and $1 \le \overline{C}^{f}$ which implies $\overline{C}^{f} = 1$. Conversely, suppose that C is a fuzzy f-closed, then $\overline{C}^{f} = 1$, but $\overline{C}^{f} \le \overline{C}$ by Proposition 2.14(ii). Hence $\overline{C} = 1$.

Definition 2.32[7]. Let $(\mathbb{W}, \tau)$ be a FTS. Then $\mathbb{W}$ is called fuzzy feebly regular if for any fuzzy point x_t and $U \in N_{x_t}^{fQ}$ of x_t, there exists a fuzzy f-open set V in $\mathbb{W}$ such that $x_t q V \le \overline{V}^{f} \le U$.

Theorem 2.33. Let $(\mathbb{W}, \tau)$ be a FTS and $A \in I^{\mathbb{W}}$. Then a fuzzy point x_α belong to $\overline{A}^{f}$ if and only if we have a net $\{x_{\alpha_n}^{n}\}$ in A and $x_{\alpha_n}^{n} \xrightarrow{f} x_\alpha$.

Proof. To prove $x_{\alpha_n}^{n} \xrightarrow{f} x_\alpha$. Let $x_\alpha \in FP(\mathbb{W})$ and then BqA for each $B$is f-open set in $\mathbb{W}$ such that $x_\alpha q B$. Thus there exists $x_B \in (0,1]$ such that $x_{\alpha_B}^{B} \in A$ and $x_{\alpha_B}^{B} qB$.Assume that $D = N_{x_\alpha}^{fQ}$ and $\geqslant$ is the inclusion relation, then $(D, \geqslant)$ is a direct set. Now, it is possible to define a fuzzy net $\mathcal{S}: D \longrightarrow FP(\mathbb{W})$ such that $\mathcal{S} = x_{\alpha_B}^{B}$, $\forall B \in D$. If we take $P \in D$ and $P \geqslant B$, therefore $P \le B$, so there exists a fuzzy net $\{x_{\alpha_P}^{P}\}_{P \in D}$such that $x_{\alpha_P}^{P} qP$. Hence $x_{\alpha_B}^{B} \xrightarrow{f} x_\alpha$.

Conversely, let us assume that a fuzzy net $\{x_{\alpha_n}^{n}\}$ with the direct set $(D, \geqslant)$, and $x_{\alpha_n}^{n} \xrightarrow{f} x_\alpha$. So, there is $m \in D$ for every $P \in N_{x_\alpha}^{fQ}$, achieves $x_{\alpha_n}^{n} qP$, $\forall n \ge m$. But $x_{\alpha_n}^{n} \in A$, thus $x_{\alpha_n}^{n} \tilde{q} A^{c}$ by Proposition. Hence AqP and $x_\alpha \in \overline{A}^{f}$.

Proposition 2.34. A fuzzy net $\mathcal{S} = \{x_{\alpha_n}^{n} : n \in D\}$has a fuzzy f-cluster $x_\alpha$if and only if it is possible to obtain fuzzy subnet of $\mathcal{S}$ which feebly converes to x_α, where $(D, \geqslant)$is direct set in a FTS$(\mathbb{W}, \tau)$.

Proof. Assume that x_α is f-cluster point of $\{x_{\alpha_n}^{n} : n \in D\}$and let $N_{x_\alpha}^{fQ}$be the collection of all feebly Q-nbds of x_α. Therefore, for each $P \in N_{x_\alpha}^{fQ}$there exists$\{x_{\alpha_n}^{n}\}$ such that $\{x_{\alpha_n}^{n}\} qP$. Let Zbe the set of all ordered pairs (n, P) with the above character, i.e $n \in D$, $P \in N_{x_\alpha}^{fQ}$ and $\{x_{\alpha_n}^{n}\} qP$. Let us define a relation $\mathfrak{Z}$ on Z given by $(m, U)\mathfrak{Z}(n, V)$ if $m \ge n$ in D and $U \le V$, then $(Z, \mathfrak{Z})$ is adirect set and it is easy to see that $S: Z \to FP(\mathbb{W})$ given by $S(m.U) = \{x_{\alpha_m}^{m}\}$ is a fuzzy sub net of the assumed fuzzy net. Now, let P be any feebly of Q-nbds of x_α, then there exists $n \in D$ such that$(n, P) \in Z$ and therefor$\{x_{\alpha_n}^{n}\} qP$ .So$(n, P) \in Z$, and $(m, U)\mathfrak{Z}(n, P) \Rightarrow S(m, U) = \{x_{\alpha_m}^{m}\} qU$ and $U \le P \Rightarrow S(m, U) qP$. Hence S, is a feebly converges to x_α.

Conversely, assume that $\{x_{\alpha_n}^{n} : n \in D\}$has no f-cluster point. So, for every $x_\alpha \in FP(\mathbb{W})$ there is feebly Q-nbds of x_α and $n \in D$ such that $\{x_{\alpha_m}^{m}\} \tilde{q} U$, for all $m \ge n$. Hence, no fuzzy net feebly converges to x_α.

Proposition 2.35. Every feebly convergent fuzzy net in a fuzzy feebly Hausdorff space $(\mathbb{W}, \tau)$, has a unique limit point.

Proof. Assume that $x_{\alpha_n}^{n}$ is a fuzzy net on $\mathbb{W}$ with directe set D, such that $x_{\alpha_n}^{n} \xrightarrow{f} x_\alpha$, $x_{\alpha_n}^{n} \xrightarrow{f} y_\beta$ and $x \neq y$. Since $x_{\alpha_n}^{n} \xrightarrow{f} x_\alpha$, then $\forall P \in N_{x_\alpha}^{fQ}$, $\exists\, m_1 \in D$, such that $\{x_{\alpha_n}^{n}\}\, q\, P$, $\forall n \ge m_1$. Also, $x_{\alpha_n}^{n} \xrightarrow{f} y_\beta$ we have $\forall \acute{P} \in N_{y_\beta}^{fQ}$,

$\exists$ $m_2 \in D$, such that $\{x^n_{\alpha_n}\}$ q$\acute{P}$, $\forall n \geq m_2$. Thus, there is $m \in D$, such that , $m \geq m_1$ and $m \geq m_2$ then $\{x^n_{\alpha_n}\}$ q(P$\wedge$ $\acute{P}$) , $\forall n \geq m$. Therefore P$\wedge$ $\acute{P}$ $\neq 0$. Hence $\mathbb{W}$ is not fuzzy feebly Hausdorff. Conversely, consider that $\mathbb{W}$ be a not fuzzy feebly Hausdorff, then there is $x_\alpha , y_\beta \in FF(\mathbb{W})$, such that $x \neq y$ and $\acute{P} \wedge \acute{\acute{P}} \neq 0$, $\forall \acute{P} \in N^{fQ}_{x_\alpha}$, $\forall \acute{\acute{P}} \in N^{fQ}_{y_\beta}$. Put $N^{fQ}_{x_\alpha, y_\beta} = \left\{\acute{P} \wedge \acute{\acute{P}} : \acute{P} \in N^{fQ}_{x_\alpha}, \acute{\acute{P}} \in N^{fQ}_{y_\beta}\right\}$. Therefore $\forall$ P $\in N^{fQ}_{x_\alpha, y_\beta}$, there exists x_Pq P, then $\{x_P\}_{P \in N^{y_\beta}_{x_\alpha}}$ is a fuzzy net in $\mathbb{W}$. Now, lets show $x_P \xrightarrow{f} x_\alpha$ *and* $x_P \xrightarrow{f} y_\beta$. Take A $\in N^{fQ}_{x_\alpha}$, thus A$\in N^{fQ}_{x_\alpha, y_\beta}$ (since A =A $\wedge$ $\mathbb{W} \neq 0$. Thus x_pq A, $\forall P \geq A$, $x_P \xrightarrow{f} x_\alpha$ *and* $x_P \xrightarrow{f} y_\beta$. Hence $\{x_P\}_{P \in N^{y_\beta}_{x_\alpha}}$ has two limit point.

3. MAIN RESULTS

Definition 3.1. A FTS $(\mathbb{W}, \tau)$ is called feebly compact (for short f-compact) if every cover of has finite sucover.

Theorem 3.2. A FTS $(\mathbb{W}, \tau)$ is f-compact if and only if $\{B_i : i \in I\}$ has f. i. p. when B_i is f $-$ closed, then $\wedge_{i \in I} B_i \neq 0$.
Proof. Assume that $\mathbb{W}$ is fuzzy f-compact space and $\{B_i : i \in I\}$is collection of fuzzy f-closed sets of $\mathbb{W}$ has f.i.p. but $\wedge_{i \in I} B_i \neq 0$. Thus $\vee_{i \in I} B^c_i = 1$ and each B^c_i is fuzzy f-open set, thus there exist $i_1, i_2, \ldots, i_n$ such that $\vee^n_{k=1} B^c_{i_k} = 1$, therefor $\wedge^n_{k=1} B_{i_k} = 0$ which is contradiction and thus $\wedge_{i \in I} B_i \neq 0$. Conversely, let $\{A_i : i \in I\}$ be a fuzzy f-open cover of $\mathbb{W}$ and every collection of fuzzy f-closed sets in $\mathbb{W}$ with f.i.p. has a non $-$ empty intersection. To show that $\mathbb{W}$ is a fuzzy f-compact space. Since $\vee_{i \in I} A_i = 1$, then $\wedge_{i \in I} A^c_i = 0$ and each A^c_i is fuzzy f-closed set which implies that $\{A^c_i : i \in I\}$*is* collection of fuzzy f-closed sets with empty intersection and so by hypothesis this collection does not have the finite intersection property. So, there exist a finite member of fuzzy sets A^c_{ji}, k = 1,2, ..., n, such that $\wedge^n_{k=1} A^c_{ik} = 0$, which implies $\vee^n_{k=1} A_{ik} = 1$ and $\{A_{ik} : k = 1,2, \ldots, n\}$is finite sub cover of the space $\mathbb{W}$ belong to a fuzzy f-open cover $\{A_i : i \in I\}$. Hence, $\mathbb{W}$ is f-compact space.

Theorem 3.3. A FTS $(\mathbb{W}, \tau)$ is a fuzzy f-compact if and only if for every fuzzy filter base on $\mathbb{W}$ has a fuzzy f-cluster point.
Proof. Suppose that $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filter base on $\mathbb{W}$ having no fuzzy f- cluster point, where $\mathbb{W}$ is a fuzzy f-compact. Let $x \in \mathbb{W}$, then for each $n \in \mathbb{N}$, there exists a feebly q-nbd U^n_x of the fuzzy point $x_{1/n}$ and $F^n_x \in \mathcal{F}$ s.t $U^n_x \tilde{q} F^n_x$, and so, $U^n_x(x) > 1 - 1/n$. Since we have $U_x(x) = 1$, where $U_x = \vee\{U^n_x : n \in \mathbb{N}\}$. Therefore $\mathcal{O} = \{U^n_x : n \in \mathbb{N}, x \in \mathbb{W}\}$ is a fuzzy f-open cover of $\mathbb{W}$. But $\mathbb{W}$ is fuzzy f-compact, then there is finitely many members $U^{n_1}_{x_1}, U^{n_2}_{x_2}, \ldots, U^{n_k}_{x_k}$ of $\mathcal{O}$ such that $\vee^k_{i=1} U^{n_i}_{x_i} = 1$. Take $F \in \mathcal{F}$ such that $F \leq F^{n_1}_{x_1} \wedge F^{n_2}_{x_2} \wedge \ldots \wedge F^{n_k}_{x_k}$, then $F \tilde{q} 1$, and so $F = 0$ which is contradict the definition of a fuzzy filter base.
Conversely, Assume that every a fuzzy filter base have a fuzzy f-cluster point. Let $A = \{F_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy f-closed sets with f.i.p. and so the set of finite intersections of members of A then forms a fuzzy filter base $\mathcal{F}$ on $\mathbb{W}$. So by the given condition $\mathcal{F}$ has a fuzzy f-cluster point. Let x_α be a fuzzy feebly cluster point of $\mathcal{F}$. So, $x_\alpha \leq \wedge_{\alpha \in \Lambda} \overline{F_\alpha}^f = \wedge_{\alpha \in \Lambda} F_\alpha$. Thus $\wedge\{F : F \in \mathcal{F}\} \neq 0$. Hence by Theorem 3.2, $\mathbb{W}$ is a f-compact.

Theorem 3.4. A FTS $(\mathbb{W}, T)$ is a fuzzy f-compact if and only if for every fuzzy net in $\mathbb{W}$ has a fuzzy f-cluster point.
Proof. Let $\{S(n) : n \in D\}$ be a fuzzy net in $\mathbb{W}$ that doesn't have a feebly cluster point, where $\mathbb{W}$ be a fuzzy f- compact. Then for every fuzzy point x_α, there is a fuzzy feebly q-nbds U_{x_α} of x_α and an $n_{U_{x_\alpha}} \in D$ such that $S_m \tilde{q} U_{x_\alpha}$ for all $m \in D$ with $m \geq n_{U_{x_\alpha}}$. Now, $x_\alpha q U_{x_\alpha}$ and so, $S_m \neq 0$, $\forall$ $m \geq n_{U_{x_\alpha}}$ and let $\mathcal{U}$ denoted the collection of all U_{x_α}, where x_α includes all fuzzy points in $\mathbb{W}$. Let us prove that the collection $V = \{1 - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$ is a family of fuzzy f-closed sets in $\mathbb{W}$ having finite intersection property. At first notice that there exists $k \geq U_{x_{\alpha_1}}, U_{x_{\alpha_2}}, \ldots, U_{x_{\alpha_m}}$ such that $S_p \tilde{q} U_{x_{\alpha_i}}$ for i = 1,2, ..., m and for all $p \geq k$ ($p \in D$), that means $S_p \in 1 - \vee^m_{i=1} U_{x_{\alpha_i}} = \wedge^m_{i=1}(1 - U_{x_{\alpha_i}})$ for all $p \geq k$. Hence $\wedge\{1 - U_{x_{\alpha_i}} : i = 1,2, \ldots, m\} \neq 0$. Since $\mathbb{W}$ is a fuzzy f-compact, by Theorem 3.2, then there exists a fuzzy point y_β in $\mathbb{W}$ such that $y_\beta \in \wedge\{1 - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\} = 1 - \vee\{U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$. Thus, $y_\beta \in 1 - U_{x_\alpha}$, for all $U_{x_\alpha} \in \mathcal{U}$ and hence in particular $y_\beta \in 1 - U_{y_\beta}$, i.e., $y_\beta \tilde{q} U_{y_\beta}$. But by construction, there exists $U_{x_\alpha} \in \mathcal{U}$ for every fuzzy point x_α , that is $x_\alpha q U_{x_\alpha}$ which is a contradiction. Conversely, To prove that converse by Theorem 3.2, shows that every fuzzy filter base on $\mathbb{W}$ has a fuzzy f-cluster point, which shows that the opposite is true. Let $\mathcal{F}$ be a fuzzy filter base on $\mathbb{W}$, then each $F \in \mathcal{F}$ is non empty set, we select a fuzzy point $x_F \in \mathcal{F}$. Let $S = \{x_F : F \in \mathcal{F}\}$ with the relation " $\succcurlyeq$ " be defined as follows $F_\alpha \succcurlyeq F_\beta$ if and only if $F_\alpha \leq F_\beta$ in $\mathbb{W}$ for $F_\alpha, F_\beta \in \mathcal{F}$. Thus $(F, \succcurlyeq)$ is directed set. Now, S is a fuzzy net with the directed set $(F, \succcurlyeq)$ as domain. The fuzzy net S has a cluster point x_t, which is a given. Then for every fuzzy feebly q–nbd N of x_t and for each $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ with $G \geq F$ such that x_GqN. As $x_G \leq G \leq F$. This means that FqN for each $F \in \mathcal{F}$, then from Proposition 2.18, $x_t \in \overline{F}^f$. Hence x_t is a fuzzy f-cluster point of $\mathcal{F}$.

Corollary 3.5. A FTS $(\mathbb{W}, \tau)$ is a fuzzy f-compact if and only if for every fuzzy net in $\mathbb{W}$ has a feebly convergent fuzzy subnet.
Proof. From Theorem 3.4 and Proposition 2.34.

Proposition 3.6. For any two fuzzy f-compact set G and H in a FTS $(\mathbb{W}, T)$, then $G \vee H$ is also fuzzy f-compact.
Proof. Let $\mathbb{A} = \{A_\alpha : \alpha \in \Lambda\}$ is a fuzzy f-open cover of $G \vee H$, then $G \vee H \leq \bigvee_{\alpha\in\Lambda} A_\alpha$. Since $G \leq G \vee H$ and $H \leq G \vee H$, then $\mathbb{A}$ is a fuzzy f- open cover of G and fuzzy f-open cover of H. But G and H are two fuzzy f-compact sets, thus there exists a finite sub cover $\{A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n}\}$ of $\mathbb{A}$ which covering G and a finite sub cover $\{A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n}\}$ of $\mathbb{A}$ which covering H such that $G \leq \bigvee_{i=1}^{m} A_{\alpha_i}$ and $H \leq \bigvee_{j=1}^{n} A_{\alpha_j}$, therefor, $G \vee H \leq \bigvee_{k=1}^{m+n} A_{\alpha_k}$.Hence $G \vee H$ is fuzzy f-compact.

Proposition 3.7. Any fuzzy f $-$ compact space is a fuzzy compact space.
Proof. Let $\mathbb{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of a fuzzy space $\mathbb{W}$ and $\mathbb{W} = \bigvee_{\alpha\in\Lambda} A_\alpha$. Since every fuzzy open set is a fuzzy f-open and $\mathbb{W}$ is a fuzzy f-compact space, then there exists $\alpha_1, \alpha_2, \dots, \alpha_n \epsilon \Lambda$ such that $\mathbb{W} = \bigvee_{i=1}^{n} A_{\alpha_i}$. Hence $\mathbb{W}$ is fuzzy compact space.

Corollary 3.8 Let G be a fuzzy f-compact set in a FTS $(\mathbb{W}, \tau)$, then G is a fuzzy compact.
Proof. It is straightforward

Proposition 3.9. Let B be a fuzzy f-open in $\mathbb{W}$. If $A \leq B$ is a fuzzy f-compact in $\mathbb{W}$ iff A is a fuzzy f-compact in B.
Proof. Let $\mathbb{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of A by f-open sets in B. By Definition 2.28, $A_\alpha = G_\alpha \wedge B$ for each $\alpha \in \Lambda$, where $G_\alpha$is a fuzzy f-open in $\mathbb{W}$. Thus $\mathcal{S} = \{G_\alpha : \alpha \in \Lambda\}$ is a fuzzy cover of A by f-open sets in $\mathbb{W}$, but A is a fuzzy f-compact in $\mathbb{W}$, so there exists $\alpha_1, \alpha_2, \dots, \alpha_n \epsilon \Lambda$ such that $A \leq \bigvee_{i=1}^{n}(G_{\alpha_i} \wedge B) = \bigvee_{i=1}^{n}(A_{\alpha_i})$. Hence, A is a fuzzy f-compact in B. Conversely, It is straightforward.

Proposition 3.10. Let $(\mathbb{W}, \tau)$ be a FTS and $A \leq B$. If B be a fuzzy f-open set in $\mathbb{W}$. Then A is a fuzzy compact in $\mathbb{W}$, if and only if A is a fuzzy f-compact in B.
Proof. Suppose that $\{A_\alpha : \alpha \in \Lambda\}$ is a fuzzy f-open cover of A in B. From Proposition 2.28, $A_\alpha = G_\alpha \wedge B$is f- open for each $\alpha \in \Lambda$, where $G_\alpha$is a fuzzy open in$\mathbb{W}$. Thus $\mathcal{S} = \{G_\alpha : \alpha \in \Lambda\}$ is a fuzzy cover of A by fuzzy open sets in $\mathbb{W}$. But A is a fuzzy compact in $\mathbb{W}$so there exists $\alpha_1, \alpha_2, \dots, \alpha_n \epsilon \Lambda$ such that $A \leq \bigvee_{i=1}^{n}(G_{\alpha_i} \wedge B) = \bigvee_{i=1}^{n}(A_{\alpha_i})$. Hence, A is a fuzzy f-compact in B. Conversely, Suppose that $\mathcal{S} = \{G_\alpha : \alpha \in \Lambda\}$ is a fuzzy open cover of A by fuzzy f-open sets in $\mathbb{W}$.Then $\mathbb{A} = G_\alpha \wedge B$ is a fuzzy cover of A. Since G_α is a fuzzy f-open in $\mathbb{W}$ for all $\alpha \in \Lambda$ and B is a fuzzy f-open in $\mathbb{W}$. Thus $G_\alpha \wedge B$ is a fuzzy f-open in $\mathbb{W}$ for all $\alpha \in \Lambda$, but A is a fuzzy f-compact in B, so there exists $\alpha_1, \alpha_2, \dots, \alpha_n \epsilon \Lambda$ such that $A \leq \bigvee_{i=1}^{n}(G_{\alpha_i} \wedge B) \leq \bigvee_{i=1}^{n}(G_{\alpha_i})$. Hence, A is a fuzzy f-compact in $\mathbb{W}$.

Proposition 3.11. Let $(\mathbb{W}, \tau)$ be a FTS. If B be afuzzy f-open set in W and $A \leq B$. Then A is a fuzzy compact in $\mathbb{W}$. if and only if A is a fuzzy compact in B.
Proof. From Proposition 3.10 and Corollary 3.8.

Proposition 3.12. Let $(\mathbb{W}, \tau)$ be a FTS. If B be a Fuzzy set in W and $A \leq B$. Then A is a fuzzy compact in $\mathbb{W}$. if A is a fuzzy compact in B .
Proof. be $K = \{G_\alpha : \alpha \in \Lambda\}$a fuzzy open cover of A in $\mathbb{W}$. Since $A \leq B$ and $A \leq G_\alpha$. then $H = \{G_\alpha \wedge B : \alpha \in \Lambda\}$is a fuzzy cover of A. But G_α is a fuzzy open in $\mathbb{W}$ for all $\alpha \in \Lambda$ then$G_\alpha \wedge B$ is a fuzzy open in B for all $\alpha \in \Lambda$, by Assumption, A is a fuzzy compact in B, so there exists$\alpha_1, \alpha_2, \dots, \alpha_n \epsilon \Lambda$ such that $A \leq \bigvee_{i=1}^{n}(G_{\alpha_i} \wedge B) \leq \bigvee_{i=1}^{n}(G_{\alpha_i})$. Hence, A is a fuzzy compact in $\mathbb{W}$.

Proposition 3.13. A fuzzy f-closed subset of a fuzzy f- compact space $(\mathbb{W}, \tau)$ is fuzzy compact.
Proof. Let G be a fuzzy f-closed subset of a fuzzy f- compact space W and $\{A_\alpha : \alpha \in \Lambda\}$ is a fuzzy open cover of G in $\mathbb{W}$. which implies that $G \leq \bigvee_{\alpha\in\Lambda} A_\alpha$. Thus, G has a fuzzy f-open cover$\{A_\alpha : \alpha \in \Lambda\}$. Since $G^c$is f-open, then the family $\{A_\alpha : \alpha \in \Lambda\} \vee G^c$ is a fuzzy f-open cover of $\mathbb{W}$, which is a fuzzy f-compact space. Thus there exists $\alpha_1, \alpha_2, \dots, \alpha_n \epsilon \Lambda$ such thst $\bigvee_{i=1}^{n} A_{\alpha_i} \vee \{G^c\} = 1$. Since $\{A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n}, G^c\}$ is finite subcover of X and $G \leq 1 = \bigvee_{i=1}^{n} A_{\alpha_i} \vee \{G^c\}$, but $G \nleq G^c$, therefor $G \leq \bigvee_{i=1}^{n} A_{\alpha_i}$. Hence, G is a fuzzy f-compact.

Theorem 3.14. Each fuzzy f-compact subset of a fuzzy feebly Hausdroff topological space is fuzzy f-closed.

Proof. Let G be a fuzzy f-compact subset of a f-Hausdroff space 𝕎 and $x_\alpha \in \overline{G}^f$, from Theorem 2.36 , so there exists fuzzy net $x^n_{\alpha_n}$ such that $x^n_{\alpha_n} \xrightarrow{f} x_\alpha$. Since G is fuzzy f-compact and 𝕎 is fuzzy f- T_2-space, then due to Corollary 3.5 and Proposition 2.35, we have $x_\alpha \in G$. Thus $\overline{G}^f \le G$ and so G is fuzzy f-closed set.

Theorem 3.15. In any fuzzy space, the intersection of a fuzzy f-compact set with a fuzzy f-closed set is fuzzy f-compact.

Proof. Assume that A is a fuzzy f-compact set and B is a fuzzy f-closed set, let $x^n_{\alpha_n}$ be a fuzzy net in A. Since A is fuzzy f-compact, then by Corollary 3.5, $x^n_{\alpha_n} \xrightarrow{f} x_\alpha$ for some $x_\alpha \in FP(\mathbb{W})$ and by Proposition 2.33, $x_\alpha \in \overline{A}^f$. We have B is fuzzy f-closed, then $x_\alpha \in B$. Therefore $x_\alpha \in A \wedge B$ and. Hence $A \wedge B$ is fuzzy f-compact.

Definition 3.16. In a FTS 𝕎, a fuzzy set G is said to be f-compactly fuzzy f-closed set if $G \wedge K$ is fuzzy f-compact, for every fuzzy f-compact set K in 𝕎.

Proposition 3.17. Every fuzzy f-closed subset of a FTS 𝕎 is f-compactly fuzzy f-closed.

Proof. Assume that G be a fuzzy f-closed subset of a fuzzy space 𝕎 and K is a fuzzy f-compact set. Then from Theorem 3.15, $G \wedge K$ is a fuzzy f-compact. Hence G is a f-compactly fuzzy f-closed set .

Theorem 3.18. In a fuzzy feebly Hausdorff space 𝕎, a fuzzy set G of 𝕎 is f-compactly fuzzy f-closed if and only if G is fuzzy f-closed.

Proof. Let G be a f-compactly fuzzy f-closed set in. If $x_\alpha \in \overline{G}^f$, then by Proposition 2.36, there exists a fuzzy net $x^n_{\alpha_n}$ in G, such that $x^n_{\alpha_n} \xrightarrow{f} x_\alpha$. Thus by Corollary 3.4, $F = \{x^n_{\alpha_n}, x_\alpha\}$ is a fuzzy f-compact set. Since G is f-compactly fuzzy f-closed, then $G \wedge F$ is a fuzzy f-compact set in 𝕎 . But 𝕎 is a fuzzy feebly Hausdorff space, then from Theorem 3.14, $G \wedge F$ is fuzzy f-closed. Since $x^n_{\alpha_n} \xrightarrow{f} x_\alpha$ and $x^n_{\alpha_n} \in G \wedge F$, then from Theorem 2.36 we have $x_\alpha \in G \wedge F$, and $x_\alpha \in G$. Hence $\overline{G}^f \le G$ and G is a fuzzy f-closed set. Conversely, By Proposition 3.18.

Theorem 3.19. A fuzzy f-regular space 𝕎 is fuzzy f-compact if and only if there exist a fuzzy dense D of 𝕎 such that every fuzzy filterbase in D have a fuzzy f-cluster point in 𝕎, with $D^\circ \neq 0$

Proof. By Theorem 3.2.

Conversely, we prove if there exist a fuzzy dense D of 𝕎 such that any fuzzy filter base in D have a fuzzy f-cluster point in 𝕎, then 𝕎 is a fuzzy f-compact. Let D be a fuzzy dense set and 𝕎 is not fuzzy f-compact, then there exist a cover $\{U_j : j \in J\}$of fuzzy f-open set in 𝕎 with no finite fuzzy subcover. Since 𝕎 is a fuzzy f-regular, then there exists fuzzy f-open cover $\{V_i : i \in I\}$ of 𝕎 such that for each j there exist i such that $\overline{V_i}^f \le U_j$. By Theorem above, since $\mathbb{W} = \overline{D}^f$, $\{V_i : i \in I\}$ is a fuzzy f-open cover of $\overline{D}^f$with no finite subcover. Therefore, the collection $\mathcal{B} = \{D \wedge (1 - \vee V_{i_k}), k = 1,2, \dots, n\}$ is a fuzzy filterbase in D. By assumption, $\mathcal{B}$ has a fuzzy f-cluster point x_α. Then $x_\alpha \in \overline{D}^f$ implies $x_\alpha \in V_i$ for some i and so V_i is a fuzzy f-open set containing x. Then$(D \wedge (1 - V_i) \wedge V_i = 0$ contradicts the fact that x is a fuzzy f-cluster point of $\mathcal{B}$. Hence $\overline{D}^f = \mathbb{W}$ is a fuzzy f-compact.

Corollary 3.20. A fuzzy f-regular space 𝕎 is fuzzy compact if and only if there exist a fuzzy dense D of 𝕎 such that every fuzzy filterbase in D have a fuzzy f- cluster point in 𝕎, with $D^\circ \neq 0$

Proof. Clear.

REFERENCES

[1] L. A. Zadeh, “fuzzy sets,” Information and control, 8(1965), 338-353.
[2] C. L. Chang, “Fuzzy topological spaces,”J. Math. Anal. Appl, 24(1968), 182-190.
[3] Maheshwari S.N & Tapi U. Feebly open sets . Ann ,Univ . Timisoaras. (1978).
[4] Lee , Je –Yoon C. and Gyu – Ihn , “A Fuzzy Feebly open set in fuzzy Topological spaces” uou Report , vol. 17 , No. 1 , pp. 139-142, 1986.
[5] M. H. Rashid and D. M. Ali, “Separation axioms in maxed fuzzy topological spaces,” Bangladesh, J. Acad. Sci, 32(2), 211-220, (2008).
[6] S. M. Jaber and M. Y. Muhsin "On Fuzzy (P,α,Pα)-compact subspace."J. Phys.: Conf. ser. (2021) doi: 10.1088/1742-6596/1897/1/012048.

[7] Kareem, H, Reyadh, N "Fuzzy Compact and Coercive Mappings ", Journal of Karbala University, 10(3),2012, 111-121.

[8] Rubasri.M and Palanisamy.M, "On fuzzy pre- α -open sets and fuzzy contrapre- α - continuous functions in fuzzy topological space", IJARIIE-ISSN (O)-2395-4396,vol-3 Issue-4 ,PP.1601-1608, 2017.

[9] Lee, Je-yoon,Gyu-Ihn "A Fuzzy feebly open set in fuzzy topological spaces",uou Report Vol. 17.No.1.pp.139-142.1986.

[10] S. M. Jaber, “Fuzzy pre compact space,” J.Phys.:Conf . Ser. 1591012073. DoT 10.1088/1742-6596/1591/1/012073. (2020).

[11] Anjana Bhattacharyya, “p^*-Closure Operator and p^*-Regularity in Fuzzy Setting,” Mathematica Moravica.Vol. 19-1 (2015), 131–139.

[12] Khalaf, Bushra Zeidan. "On Structure Fuzzy Fibration." Wasit Journal for Pure sciences 3.1 (2024): 33-37.