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Applications of $\delta -$ Continuous Function

Zahraa Mohsen dawod[1]*, Ali Khalaf Hussain[2]

[1,2]Department of Mathematics, College of Education for Pure Science, Wassit university, IRAQ

*Corresponding Author: Zahraa Mohsen Dawod

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ABSTRACT: This paper introduces a new class of functions, termed **δ-continuous functions**, and investigates their fundamental characterizations and properties. Particular attention is given to their connections with other well-known classes of functions in topology and functional analysis. δ-1-continuity, viewed as a natural extension of classical continuity, provides a broader framework that captures settings where standard continuity may be insufficient. This generalization proves especially relevant in the study of spaces with distinctive structural features or in abstract mathematical contexts. The findings not only enrich the theory of generalized continuity but also highlight the potential applications of δ-1-continuous functions in advancing mathematical analysis.

Keywords: δ-Continuous functions, $\delta - open\ set$, $\delta b - open$, $\delta b -$ closed, $\delta -$ semiopen, semi open set

1. INTRODUCTION

The study of generalized forms of continuity has attracted considerable attention in topology due to its ability to extend and refine classical concepts. Traditional continuity, while powerful, is often insufficient to capture the subtleties of more complex or abstract spaces. To address these limitations, mathematicians have introduced various generalizations such as semi-continuity, δ-continuity, and related notions. These approaches provide a broader framework for analyzing topological structures and have significant implications in both pure and applied mathematics.

In this context, the concept of ideals on topological spaces plays an essential role. An ideal provides a systematic way of handling subsets and allows the construction of ideal topological spaces, enriching the study of closure, interior, and local functions. Such spaces often reveal new properties and lead to the development of novel function classes that extend beyond the classical framework. For instance, δ-open, δ-closed, and regular open sets provide important tools in understanding generalized continuity.

The motivation behind this work is to investigate new forms of continuity that arise from these generalizations. By introducing and analyzing new classes of functions, we aim to establish their fundamental properties, explore their relationships with existing types of functions, and demonstrate their significance within topology. These developments not only contribute to the theoretical foundations of generalized continuity but also open avenues for further research in functional analysis and related areas.

In what follows, we recall essential definitions and results concerning semi-open sets, δ-open sets, and related function types. Building on this groundwork, we introduce almost δ-continuous and δb-continuous functions, present their characterizations, and study their relationships with existing classes. This systematic approach provides a deeper understanding of the structural behavior of generalized continuity in topological spaces.

Throughout this paper $Cl(A)$ and $Int\ (A)$ denote the closure and the interior of A, respectively. Let (X, τ) be a topological space and let I an ideal of subsets of X. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: (1) If A ∈ I and $B \subseteq I$, then B ∈ I; (2) If A ∈ I and B ∈ I, then $A \cup B \in I$. An ideal topological space is a topological space (X,τ) with an ideal I on X and is denoted by (X, τ, I). For a subset A⊂X, $A^*(I)$ = {x ∈ X | U ∩ A∉I for each neighborhood U of x} is called the local function of A with respect to I and τ [4]. We simply write A^* instead of $A^*(I)$ o be brief. X* is often a proper subset of X. The hypothesis $X^*(I)$ is equivalent to the hypothesis [5]. For every ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ, generated by β(I,τ) = {U\I: U ∈ τ and I ∈ I}, but in general $\beta(I, \tau)$ is not always a topology [2]. Additionally, $Cl^*(A) = A \cup (A)^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

Definition 1.1. A subset A of a topological space (X, τ) is said to be
(1) semi open [10] if $A \subset Cl(Int(A))$,
(2) δ-semi open [11] if $A \subset Cl(\delta - Int(A))$,
(3) b-open [6] (γ-open [7]) if $A \subset Int(Cl(A)) \cup Cl(Int(A))$,
(4) δb–open [9] (z-open [8]) if $A \subset Int(Cl(A)) \cup Cl(\delta - Int(A))$.
(5) regular-open [12] (briefly, r-open) if $A = \text{int}(\text{cl}(A))$
(6) regular-closed [12] (briefly, r-closed) if $A = \text{cl}(\text{int}(A))$
[7]A function $f: (X, \tau) \to (Y, \tau)$ is said to be δ-continuous [10] if for each $x \in X$ and each open set $V\ of\ f(x)$ there exists an open set $U\ of\ x$, such that: $f(int(cl(U))) \subseteq int(cl(V))$
Lemma 1. [5] A function f: $(X, \tau) \to (Y, \tau)$ is almost continuous
(i) If the inverse image of every regularly open subset of Y is open in X. or equivalently
(ii) The inverse image of every regularly closed subset of Y is closed in X.

The complement of a δb-open set is said to be δb-closed ([8]). If A is a subset of a space (X, τ), then the δb-closure of A, denoted by $\delta b - Cl(A)$, is the smallest δb-closed set containing A ([8]). The family of all $\delta b - open$, $\delta b -$ closed, $\delta -$ semiopen, semi open and $b -$ open, regular open and regular closed sets of a space (X, τ)will be denoted by $\delta BO(X)$, $\delta BC(X)$, $\delta SO(X)$, $SO(X) and\ BO(X)$, RO(X), RL(X) respectively.

2. ALMOST δb-CONTINUOUS FUNCTIONS

In this section, we introduce almost δb-continuous functions, characterizations and properties of these functions.

Definition 2.1.[3] A function $f : (X, \tau) \to (Y, \phi)$ is said to be almost δb-continuous if for each $x \in X$ and each $V \in RO(Y)$ containing $f(x)$, there exists $U \in \delta BO(X)$ containing x such that $f(U) \subset V$.

Theorem 2.2.[3] For a function $f : (X, \tau) \longrightarrow (Y, \phi)$, the following properties are equivalent:
(1) f is almost δb-continuous;
(2) For each $x \in X$ and $V \in \phi$ containing $f(x)$, there exists a subset $U \in \delta BO(X)$ containing x such that $f(U) \subset Int(Cl(V))$;
(3) $f^{-1}(V) \in \delta BO(X)$ for every $V \in RO(Y)$;
(4) $f^{-1}(F) \in \delta BC(X)\ for\ every\ F \in RC(Y)$.

Theorem 2.3.[3] For a function $f : (X, \tau) \longrightarrow (Y, \phi)$, the following properties are equivalent:
(1) f is almost δb-continuous;
(2) $f(bCl\delta(A)) \subset \delta - Cl((f(A)))$ for every subset A of X;
(3) $\delta b - Cl(f^{-1}(B)) \subset f^{-1}(\delta - Cl(B))$ for every subset $B\ of\ Y$;
(4) $f^{-1}(F) \in \delta BC(X)$ for every δ-closed set F of Y;
(5) $f^{-1}(V) \in \delta BO(X)$ for every δ-open set V of Y.

Proof.
(1) → (2) Let A be a subset of X. Since $\delta - Cl((f(A)))$ is a δ-closed set in Y, it is denoted by $\cap \{F\alpha : F\alpha\ (Y, \phi), \alpha\ \Delta\}, where\ \Delta\ is\ an\ index\ set. Then\ we\ have\ A \subset f^{-1}(\delta - Cl((f(A)))) = \cap \{f^{-1}(F\alpha) : \alpha \in \Delta\} \in \delta BC(X)$ by Theorem 3.2. So, we obtain $\delta b - Cl(A) \subset f^{-1}(\delta - Cl((f(A))))\ and\ hence\ f(bCl\delta(A)) \subset \delta - Cl((f(A)))$.

(2) → (3) Let B be a subset of Y. We $have\ f\ )\delta b - Cl(f\ -1\ (B))(\ \subset\ \delta - Cl(\ f(f^{-1}\ (B)\ )\ )\ \subset\ \delta - Cl\ (B)\ and\ hence\ \delta b - Cl(f^{-1}\ (B))\ \subset\ f^{-1}\ (\delta - Cl\ (B))$.

(3) → (4) Let F be any δ-closed set of Y. We have $\delta b - Cl(f^{-1}(F))\ \subset\ f^{-1}\ (\delta - Cl\ (F))\ = f^{-1}\ (F)\ and\ f^{-1}\ (F)\ is\ \delta b - closed\ in\ X$.
(4) → (5) Let V be any δ-open set of Y. Using (4), we have that $f^{-1}\ (Y - V) = X - f^{-1}\ (V)\ \in \delta BC(X)\ and\ so\ f^{-1}\ (V)\ \in\ \delta BO(X)$.
(5) →(1) Let V be any regular open set of Y. Since V is δ −open set in Y, $f^{-1}\ (V)\ \in\ \delta BO(X)$ and hence by Theorem 3.2, f is almost δb-continuous.

Lemma 2.4. ([9]) Let A and B be subsets of a space (X, τ). If $A\ \in\ \delta O(X)\ and\ B\ \in\ \delta BO(X), then$

$$A\ \cap\ B\ \in\ \delta BO(A).$$

Theorem 2.5. If $f\ :\ X\ \to\ Y$ is almost δb-continuous and A is a δ-open subspace of X, then the restriction $f\ |A$ is almost $\delta - b$ −continuous.

Proof.
Let V be any regular open set of Y. Then we have $f^{-1}(V)\ \in\ \delta BO(X)$ by Theorem 3.2. Therefore, we have $f\ |_A$ is almost δb-continuous.

Definition (2.6):[4]
A function $f: X \to\ Y$ is said to be δ-continuous if for each $x\ \in\ X$ and each open set $V\ of\ f(x)$ there exists an open set $U\ of\ x$, such that:

$$f(int(cl(U))) \subseteq\ int(cl(V))$$

Recall that, the family of all regular open subsets of (X, τ) forms a base for a smaller topology τ_s on X, called the semi regularization of τ. Sometimes, we write X for (X, τ) and $X\_s\ \ for\ (X, \tau\_s\ )$.
Define a function $f_{s\,:}\,X_s \longrightarrow Y_s$ associated with a function $f: X \longrightarrow\ Y$ as follows; $f(s\ )\ (x)\ =\ f(x)$, for each $x \in X_S$
The following theorem appears Without the details of the proof, we give its proof for completeness.

Theorem (3.2):
For a function $f: X \to\ Y$. the following are equivalent:
(1) f is δ-continuous
(2) For each $x\ \epsilon\ X$ and each regular open set V containing $f(x)$, there exists a regular open set U containing x, such that $f(U)\ \subseteq\ V$
$f\ ([A]_\delta)$ for every $A\ \subseteq\ X$

$[\ f^{-1}(B)]_\delta\ \subseteq f^{-1}([B]_\delta)$ for every $B \subseteq\ Y$
For every regular closed set of Y , $f^{-1}\ (F)$ is $\delta - closed$ in X
(6) For every δ-closed set F of Y, $f^{-1}(F)$ is δ-closed in X.
(7) For every δ-open set V of Y, $f^{-1}\ (V)$ is δ-open in X.
(8) For every regular open set V of Y , $f^{-1}(V)$ is δ-open in X.

Proof:
(1)→ (2) Let $x \in\ X$. let V be a regular open set containing $f(x)$ sine V is open and f is δ-continuous, then there exist an open set $U^{-1} of\ x$, such that:

$$f(int(cl(U_1))) \subseteq\ int(cl(V))$$

Let $U = int(cl(U_1))$ then U is a regular open set
Also, since V is regular open, then $\mathrm{int}(cl(V))\ =\ V$, so $f(U) \subseteq\ V$
(2)→ (3). We have to prove that $f([A]_\delta\ ) \subseteq\ [f(A)]_\delta$
for every $A \subseteq X$
Let $y \in f([A]_\delta)$, then there exists $x \in A_\delta$ such that $y = f(x)$, we have to show that $y\ \in\ [f(A)]_\delta$
Let V b a regular open set in Y containing y. Now, by (2), there exists a regular open set U in X containing x, such that $f(U)\ \subseteq V$, but $x\ \in\ [A]_\delta$ then $U\ \cap A \neq \emptyset$. Now:
$\emptyset \neq U \cap A\ \to \emptyset \neq f(U \cap A\ )$
So $\emptyset \neq f(U \cap A) \subseteq f(U\ (\cap\ f(A)\ \subseteq V \cap f(A)$
Hence $V\ \cap f(A)\ \neq \emptyset$, which means that $y\ \in\ [(A)]_{\delta.}$
Hence $f([A]_\delta)$
(3)→ (4) Let $B \subseteq Y, let\ A = f^{-1}(B)$
By (3), we have $f([A])_\delta\ \subseteq\ [(A)]_\delta$

Hence $f([f^{-1}(B)]_{\delta} \subseteq f(f^{-1}(B))]_{\delta}$
Therefore $f(f^{-1}(B)]_{\delta}) \subseteq [B]_{\delta}$
Hence $[f^{-1}(B)]_{\delta} \subseteq f^{-1}([B]_{\delta})$.
(4) → (5) Let F be a regular closed set F in Y.
Now F is δ-closed in Y. Then by (4), we have:

where L_{α} is regular open. So:

$$F = \bigcap_{\alpha\in\Omega} L_{\alpha}^{C}$$

Where L_{α}^{C} is regular closed.

Now, by (5) $f^{-1}(L_{\alpha}^{C})$ is δ -closed in X. Therefore:

$$f^{-1}(F) = f^{-1}(\bigcap_{\alpha\in\Omega} L_{\alpha}^{C}) = \bigcap_{\alpha\in\Omega} f^{-1}(L_{\alpha}^{C})$$

So $f^{-1}(F)$ is δ -closed set in

(6) → (7). Let V be δ -open set in Y, then V^{C} is $\delta - closed$ set in Y, So by (6), $f^{-1}(V^{C})$ is $\delta - closed$ set in X
But $f^{-1}(V^{C}) = (f^{-1}(V))^{C}$
So $f^{-1}(V)$ is δ -open set in X

(7) → (8) Let V be a regular open set in Y, so V is δ -open set in Y and hence by (7), we have $f^{-1}(V)$ is δ -open in X.
(8) ⟶ (1) Let $x \in X$ let V be regular open set in Y containing $F(x)$.
Now, by (8), $f^{-1}(V)$ is δ -open set in X.
Now, $x \in f^{-1}(V)$ But every δ- open set is a union of regular open sets
Hence there exists a regular open set U,
Such that $x \in U \subseteq f^{-1}(V)$
Now, $f(U) \subseteq V$
So $f(int(cl(U)) \subseteq int(cl(V))$
So f is δ-continuous.

Theorem (3.3): A function $f: X \to Y$ is δ-continuous if and only if $f_s : X_s \to Ys$ is continuous

Proof:
(→)Suppose that $f: X \to Y$ is δ- continuous, we have to show that $f_s : X_s \to Ys$ is continuous.
Let W be an open set in $Y_S = (Y, \tau_S)$
So, $W = \bigcup_{\alpha\in\Omega} V_{\alpha}$, where V_{α} is regular open set in Y.

Now,

$$f^{-1}(W) = f^{-1}\left(\bigcup_{\alpha\in\Omega} V_{\alpha}\right) = \bigcup_{\alpha\in\Omega} f^{-1}(V_{\alpha})$$

But f is δ-continuous, so $f^{-1}(V_{\alpha})$ is δ-open set in X, and $\bigcup_{\alpha\in\Omega} f^{-1}(V_{\alpha})$ is δ-open in X, so

$$\bigcup_{\alpha\in\Omega} f^{-1}(V_{\alpha}) \in \tau_{\alpha}$$

But, $\tau_{\delta} = \tau_s$. So,

$$\bigcup_{\alpha\in\Omega} f^{-1}(V_{\alpha}) \in \tau_s$$

Hence $f_s : X_s \to Y_s$ is continuous.

(←) Consider $f_s : X_s \to Y_s$ is continuous.

We will show that $f : X \to Y$ is δ-continuous.

Let V be regular open set in Y

Hence V is an open set in Y_s and therefore f^{-1}(V) is an open set in X_s

Thus $f^{-1}(V) \in \tau_s = \tau_\delta$

Hence f^{-1} (V) is δ-open set in X
Therefore, $f : X \to Y$ is δ-continuous.

Definition (3.4). [1]: A function $f: X \to Y$ is called $\delta^* -$continuous $f^{-1}(V)$ is δ-open in X for each open subset V of Y.

Definition (3.5):[1] A function $f: X \to Y$ is called $\delta^{**} -$continuous if $f^{-1}(V)$ is open in X, for each δ-open subset $V\ of\ Y$.

Remarks (3.6):[1]
1. If f a function $f: X \to Y$ is $\delta^* -$continuous, then it is δ-continuous.

proof :
Let W be δ-open in Y
Hence W is open in Y
But f is $\delta^* -$continuous in X
Hence $f^{-1}(W)$ is δ-open in X
Hence f is δ-continuous.

2. If a function $f: X \to Y$ is δ-continuous, then it is
$\delta^{**} -$continuous

Proof:
Let W be δ-open in Y
But f is δ-continuous.
Hence $f^{-1}(W)$ is δ-open in X
Therefore, $f^{-1}(W)$ is open in X.

Lemma (3.7), [2]:
Let $\{ X_\alpha : \alpha \in \Omega \}$ be a family of spaces, and $X = \prod_{\alpha\in\Omega} X_\alpha$,then $X_s = \prod_{\alpha\in\Omega}(X_\alpha)_s$
Theorem (3.8):
Let $f_\alpha: X_\alpha \to Y_\alpha$ $(\alpha \in \Omega)$ be a family of functions. Then the product function $f : \prod_{\alpha\in\Omega} X_\alpha \to \prod_{\alpha\in\Omega} Y_\alpha$ is δ^*-continuous if and only if $f_\alpha: X_\alpha \to Y_\alpha$ is δ^*-continuous, for each $\alpha \in \Omega$

Proof:
$\Longrightarrow$ Let $f: \prod_{\alpha\in\Omega} X_\alpha \to \prod_{\alpha\in\Omega} Y_\alpha$ is δ^*-continuous
Then $f : (\prod_{\alpha\in\Omega} X_\alpha)_s \to \prod_{\alpha\in\Omega} Y_\alpha$ is continuous
Then $f : \prod_{\alpha\in\Omega}(X_\alpha)_s \to \prod_{\alpha\in\Omega} Y_\alpha$ is continuous
Therefore, $f_\alpha : (X_\alpha)_s \to Y_\alpha$ is continuous, for each $\alpha \in \Omega$
Hence $f_\alpha : X_\alpha \to Y_\alpha$ is δ^*-continuous, for each $\alpha \in \Omega$
$\Longleftarrow Let\ f_\alpha : X_\alpha \to Y_\alpha$ is $\delta^* -$ continuous , for each $\alpha \in \Omega$
Then $f_\alpha : (X_\alpha)_s \to Y_\alpha$ is continuous, for each $\alpha \in \Omega$
Then f: $\prod_{\alpha\in\Omega}(X_\alpha)_s \to \prod_{\alpha\in\Omega} Y_\alpha$ is continuous
So: f : $(\prod_{\alpha\in\Omega} X_\alpha)_s \to \prod_{\alpha\in\Omega} Y_\alpha$ is continuous
Hence $f : \prod_{\alpha\in\Omega} X_\alpha \to \prod_{\alpha\in\Omega} Y_\alpha$ is δ^*-continuous
Similar theorem can be stated for δ^{**}continuous functions.
Now, we study a new type of δ-continuous functions.

Definition (3.9), [3]:
A function $f: X \to Y$ is called contra-δ-continuous if $f^{-1}(V)$ is
$\delta -$ closed in X for each open set V in Y.

Example (3.10):
Let $X = \{a, b, c\}$
Let τ_d be the discrete topology on X
Let $Y = \{1, 2, 3\}$, $\tau_y = \{\emptyset, Y, \{1\}, \{2\}, \{1,2\}\}$
Let $f : X \to Y$ be a function defined as follows:
$F(a) = 1, f(b) = f(c) = 2$
Then f is contra-δ-continuous

Remark (3.11):
If $f: X \to Y$ is contra-δ-continuous thent $f^{-1}(V)$ is δ-open set in X, for each V closed in Y.

Proof:
Let V be closed set in Y
Hence V^{C} is open in Y
Therefore, $f^{-1}(V^{C})$ is δ-closed (since f is contra-δ-continuous function)
But $f^{-1}(V^{C}) = (f^{-1}(V))^{C}$
Hence $(f^{-1}(V))^{C}$ is δ-closed
Therefore, $f^{-1}(V)$ is δ-open.

Theorem (3.12):[3]
If $f : X \to Y$ is δ-continuous and $g : Y \to Z$ is contra$-\delta-$ continuous, then gof is contra$-\delta-$continuous.

Proof:
Let V be an open set in Z, and since g is contra-δ-continuous, then $g^{-1}(V)$ is δ-closed set in Y
But f is δ-continuous, then $f^{-1}\big(g^{-1}(V)\big)$ is δ-closed in X. Now:

$$(gof)^{-1}(V) = f^{-1}(g^{-1}(V))$$

which is δ-closed

Hence gof is contra-δ-continuous.
Recall that, $f : X \to Y$ is contra-continuous if $f^{-1}(V)$ is closed for each open set V in Y.

Proposition (3.13): [3(i)]
Every contra-δ-continuous function is contra-continuous.

Proof:
Let $f : X \to Y$ be contra-δ-continuous function
Let V be an open set in Y
Then $f^{-1}(V)$ is δ-closed in X
But every δ-closed set is closed
So $f^{-1}(V)$ is closed in X, and hence f is contra-continuous

Lemma (3.14)
If $f: X \to Y$ is a δ-continuous function and K is N-closed relative to X, then $f(K)$ is N-closed relative to Y.

Proof:
Let K be N-closed relative to X, so by proposition (3.2.5) K is compact in X_s
But $f_s : X_s \to Y_s$ is continuous, then $f(K)$ is compact in Y_s , and hence by proposition (3.2.5), $f(K)$ is N-closed relative to Y.
Theorem (3.15): Near-compactness is preserved under $\delta-$continuous surjections.
Corollary (3.16): Near-compactness is preserved under almost-continuous and almost-open surjections.

Proof:
Every almost-continuous function is θ-continuous but every θ-continuous and almost-open function is δ-continuous

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