Wasit Journal for Pure Science

Journal Homepage: https://wjps.uowasit.edu.iq/index.php/wjps/index
e-ISSN: 2790-5241 p-ISSN: 2790-5233

Study of Topological Spaces that Don't Rely on Points as their Basis

Nidaa Hasan Haji[1]*

Department of Mathematics and Computer Applications, College of Science, Al- Muthanna University, IRAQ

*Corresponding author: Nidaa Hasan Haji

DOI: https://doi.org/10.31185/wjps.528
Received 20 August 2024; Accepted 20 September 2024; Available online 30 September 2024

ABSTRACT: Traditionally, the concept of points and neighborhood structures has been quite dominant in defining and further analyzing various properties in the study of topological spaces. In pointless topology-or point-free topology, also referred to as locale theory-the key emphasis is shifted from points to lattices of open sets. In addition to gaining further insight into spatial properties, this approach has important implications in fields like theoretical computer science, logic, and categorical theory. It is the purpose of the present work to be mainly interested in issues of the foundation of point-free topology: locales, frames, mutual relationships between them; the ways of interpreting classical topological concepts in the new language and the strong and weak points of such a point-free view. In the process, we focus on some of the significant applications of point-free topology, both within mathematics and beyond. The present paper is an attempt to give a broad overview of point free topology, discussing both its theoretical basis and its practical significance.

Keywords: point-free topology, locale theory, frames, lattices of opens, spatiality, topological spaces, the categorical approach, theoretical computer science, logics, pointless topology.

1. INTRODUCTION

Topological spaces are basic to modern mathematics. They find a wide range of applications in combinatorics, signal processing, physics, and numerous other fields. In classical topology, especially, points and open sets provide the two main ways of describing topological spaces. Interest in the underlying existence of topological spaces which do not rest on points has been great. This research aims to investigate in detail the structure and applications of point-free topological spaces, taking into consideration the theoretical backgrounds and practical implications of the works reviewed.

Kelley (2017) gives a fine primer on general topology in terms of the classical approach based on the theories of points and open sets. Equally, Lefschetz (2015) provides a basic exposition into topology with an emphasis on its history of development and its basic theorems. The meaning of these works follows the supposition that the traditional topological theories first provide necessary background material for point-free topologies.

Starting with the seminal work, Kelley et al. (1963) have extended the realm of topology to include functional analysis and vector spaces-known as linear topological spaces. The book is important in order to view topological concepts in more general terms rather than restricting it to purely geometric definitions.

*Corresponding author: Nedaa.hassan@mu.edu.iq

Björner 1995 discusses several applications of topological methods in combinatorics and illustrates the way in which topological spaces can be put into applications when performing combinatorial problems. This article is important in connecting abstract theory to practical computational applications.

Barbarossa and Sardellitti (2020) introduce topological signal processing on simplicial complexes, outlining the practical applications that topology can have in high-dimensional data analysis. It offers one possible road to take point-free topological methods toward applications.

Bubenik (2015) discusses statistical topological data analysis through persistence landscapes that come equipped with tools for understanding the shape and structure of data. This paper is important regarding the application of topological considerations to the statistical and data-driven contexts.

Song et al. (2017) and Husain (2018) study, respectively, topological phases protected by point group symmetry and an introduction to topological groups. These articles provide valuable insights into algebraic structures standing at the core of topological spaces and, as such, constitute essential reading for those interested in point-free topologies.

Applications in topological ideas in physics, particularly those to amorphous topological insulators by Mitchell et al. (2018), and to invertible topological phases by Freed and Hopkins (2021), are pursued. The works illustrate the relevance of topology in constructing the meaning of physical phenomena.

On the other hand, Lin and Yun, and also by Tierny et al. provide tools for generalized metric spaces and the topology toolkit, respectively. Both these two are essential tools in the development and application of point-free topological methods.

2. RESEARCH OBJECTIVES

- **Investigate the Theoretical Foundations**: Explore the mathematical structures that support point-free topological spaces, drawing on existing literature in traditional topology, linear topological spaces, and topological groups.
- **Develop New Methods**: Create new mathematical methods and frameworks for understanding and utilizing point-free topological spaces.
- **Apply to Real-World Problems**: Demonstrate the application of point-free topological spaces in fields such as signal processing, data analysis, and physics.

3. NOTATION AND DEFINITIONS

Notation and Preliminary Definitions

Topological Spaces

- **X and Y**: Denote topological spaces.
- **AX and AY**: Represent bases “non-empty open sets in topology" on X and Y, respectively.
- When the place is empty from context, we will simply use A to denote the basis.

Elements and Sequences

- **Greek Letters**: Lowercase Greek letters such as α, β, γ, etc., will represent elements of AX or AY.

Main prepositions in Romanian: Capital Roman letters A, B, C, D, and E will refer to the sequences. From A.
Functions and Intervals

- **Functions**: F and G will denote functions from AX to AY. Z will represent an interval in R.

Subsets and Collections

- **Calligraphic Letters**: Calligraphic letters C, F, G, etc., will denote subsets of X.
- **German Letters**: German letters S, B, K, etc., will be used for collections of such subsets.

Numerical Notation

The set of positive integers is represented by the symbol ω.

The notation $F(n)\downarrow$ indicates that the computation of F(n) has completed and produced to reply.

The set containing the elements a and b is represented by the following symbol: {a,b}.

- The notation (a,b) may represent either Arranged pair or open spaces, depending on the context. The closed interval will be expressed by [a, b].

- **Lebesgue Measure**

Definition 3.1: The Lebesgue measure is a way of assigning a "size" or "volume" to subsets of R (the real numbers). For each measurable group LSJ. $Y \subseteq$R, the Lebesgue measure of Y is denoted by $|Y|$.

Key Points:

- Liebig's measure extends the concept of length to include more complex groups rather than just intervals. For example, the measure of an interval [a,b] is simply b−a.
- A measure can be grouped numerically, which means that if you have a set of non-overlapping measurable groups, sets $Y_1,Y_2,\ldots$ then the measure of their union is the sum of their measures:
  $|U_n = 1 \infty \, Y_n| = \sum n = 1 \infty \, |Y_n|$.
- The Lebesgue measure can handle more complicated sets, including those that are not intervals, such as Cantor sets, and it can also measure sets with "holes" or irregular shapes.

Cantor Coding

Definition 3.2 : Cantor coding is a method of representing elements of a certain space (often related to sequences or functions) using a coding function based on the Cantor set.

Key Points:

- The Cantor coding function (i,j) is used to encode pairs of indices i,j from the countable set ω (the set of natural numbers).
- Given a fixed arrangement of a specific basis $A = \{A_i : i \in \omega\}$, the notation (A_i, A_j) represents the pair formed by the elements $A_i$and A_j.

  When writing (α, β) , it is understood that($\alpha = A_i$ and $\beta = A_j$)for some indices $i$and j.
- The notation $(A_{i_1}, (A_{i_{12}}, \ldots)$ represents a sequence of elements from the basis, and this is often used in the context of nested sequences, which is a common feature in the study of the Cantor set and its properties.

Definition of Recursive Convergence 3.3[15]: Let $f: \omega \to Q$ be a recursive function . The function f is recursively convergent if there exists a recursive function $h: \omega \to \omega$ such that:

$$\forall k \in \omega;\ \exists m, n, k_0 \in \omega \ and \ \forall n\,, m \geq n_0 \Longrightarrow |f(n) - f(m)| \leq \frac{1}{k}$$

The limits of this function are defined by a real number that repeats. Discussions and applications. Properties of recursive Real Numbers. We will explore the fundamental properties of recursive real numbers, including their existence, uniqueness, and behavior in various topological settings.

Applications in Topology

The concept of recursive convergence will be applied to analyze and solve problems in topology, such as the construction of specific types of functions between topological spaces and the behavior of sequences and their limits within these spaces.

4. SHARP FILTERS

Definition of Sharp Filters

We begin by formally defining definition of sharp filter.

Definition 4.1: Let X be a topological space containing a basis Δ. The sequence $\{\alpha_i: i \in \omega\}$of basic open sets is termed a sharp filter in ⟨X,Δ⟩ if the following conditions are met:

1. $\forall i\ (\alpha_{i+1} \subseteq \alpha_i)$
2. $\forall \beta, \gamma[(\beta \subseteq \gamma) \Longrightarrow (\exists i[(\alpha_i \cap \beta = \varphi) \vee (\alpha_i \subseteq \gamma)])]$

When this string meets the required condition (ii) for each fixed β and γ, we say it resolves $\langle\beta, \gamma\rangle$. If β is not a subset of γ, the sequence trivially resolves$\langle\beta, \gamma\rangle$. We will call property (ii) the solution property

Properties of Sharp Filters

Our aim is to approximate points using sharp filters, hence we need to understand their properties.

Definition 4.2: Equivalence of Sharp Filters

Let $A = \{\alpha_i: i \in \omega\}$ and $B = \{\beta_i: i \in \omega\}$, be sharp filters in ⟨$X$,Δ⟩. We define A to be equivalent to B, denoted $A \equiv B$, if: $\forall i\ (\alpha_i \cap \beta_i) = \varphi$.

We will prove that this relation is an equivalence relation (i.e., it is reflexive and symmetric). Furthermore, we will Formulate an equation that is more suitable for use in iterative applications.

Definition 4.3: Sharper Filters

Let $A = \{\alpha_i: i \in \omega\}$ and $B = \{\beta_i: i \in \omega\}$, be a sharp filters in ⟨X,Δ⟩. We say that B is sharper than A if the following condition holds: $\forall_i\ (\exists j)(\beta_j \subseteq \alpha_i)$

Although this relation might initially seem asymmetric (i.e., not bidirectional), it turns out that if *B* is sharper than *A*, then *A* is also sharper than *B*. This will be demonstrated in the subsequent results.

Proposition 4.4: The relation "sharper than" is transitive.

Proof: Assume $A = \{\alpha_i: i \in \omega\}$, $B = \{\beta_i: i \in \omega\}$ and $C = \{\gamma_i: i \in \omega\}$ are sharp filters in ⟨X,Δ⟩.

- Suppose that *B* is sharper than *A*: $\forall_i\ (\exists j)(\beta_j \subseteq \alpha_i)$
- And *C* is sharper than *B*: $\forall_i\ (\exists k)(\gamma k \subseteq \beta_j)$.

Then, for each *i*, there exists j such that $\beta_j \subseteq \alpha_i$, and for this *j*, there exists *k* such that $\gamma k \subseteq \beta_j$. Thus $\gamma k \subseteq \alpha_i$, proving that *C* is sharper than *A*.

Proposition 4.5: Equivalence and Sharpness

Let $A = \{\alpha_i: i \in \omega\}$ and $B = \{\beta_i: i \in \omega\}$, be sharp filters in ⟨$X$,Δ⟩. Then $A \equiv B \Leftrightarrow B$ is sharp than A, $A \equiv B \Leftrightarrow B$ is sharp than A .

Proof: Assume $A \equiv B$, $\forall i\ (\alpha_i \cap \beta_i) = \varphi$.

- If $A \equiv B$, then for each *i*, $(\alpha_i \cap \beta_i) = \varphi$, ensuring the existence of some *j* for which $\beta_j \subseteq \alpha_i$. Hence *B* is sharper than *A*.

- Conversely, if B is sharper than A, then for each i, there exists j such that $\beta_j \subseteq \alpha_i$ .Since the inclusion ensures non-empty intersection, If $A \equiv B$.

Thus, we have established the equivalence relation and demonstrated the practical formulation for recursive applications. This concludes our exploration of the properties of sharp filters and their utility in approximating points within the space X.

1. Suppose $A \equiv B$ and choose $i \in \omega$. Then $\alpha_i \subseteq \alpha_i$
2. Now, for each j, $\alpha_i \cap \beta_j \neq \varphi$, and B must resolve the target $\alpha_{i+1} \subseteq \alpha_i$. Therefore, β_j must be a subset of α_i.

(⇐) Suppose $\forall i(\exists j)(\beta_j \subseteq \alpha_i)$. Choose $i \in \omega$. Since B is a sharp filter, find $j \geqslant 1$ such that $\beta_j \subseteq \alpha_i$ .Then, since $\beta_j \subseteq \beta_j$, we have $\alpha_i \cap \beta_j \neq \varphi$.

The continuation discusses the reverse direction of the implication, showing that if B is sharper than A, then A has a corresponding inclusion property that makes B fit within A.

Corollary 4.6: Bidirectionality of "Sharper Than"

$A = \{\alpha_i : i \in \omega\}$, $B = \{\beta_i : i \in \omega\}$ sharp filters in $\langle X,\Delta\rangle$. Then A sharp than B⇔ B sharp than A

This corollary establishes that if A is sharper than B, then B is also sharper than A, making the "sharper than" relationship bidirectional under these conditions.

Corollary 4.7: Equivalence Relation of Equality

The relation '=' among sharp filters is an equivalence relation.

These corollary states that the relation of equality ('=') among sharp filters is an equivalence relation, meaning it satisfies reflexivity, symmetry, and transitivity.

Proposition 4.8: Intersection Characterization of Equality

$A = \{\alpha_i : i \in \omega\}$, $B = \{\beta_i : i \in \omega\}$ sharp filters in $\langle X,\Delta\rangle$. Then $A = B \Leftrightarrow \alpha_i = \cap \beta_i$

Proof: Suppose that $A = B$ by Proposition 3.5 ,we have $\alpha_i \subseteq \cap \beta_i$, because " = " is symmetric , we have the opposite inclusion too , implying $\alpha_i = \cap \beta_i$ .Conversely $\alpha_i = \cap \beta_i$, then for each i, $\alpha_i \subseteq \beta_i$ and $\beta_i \subseteq \alpha_i$ .Hence $A = B$

This proposition ties the equality of two sharp filters A and B to the intersection of elements within the filters.

Proposition 4.9: Intersection and Non-Equivalence Characterization

Let $A = \{\alpha_i : i \in \omega\}$, $B = \{\beta_i : i \in \omega\}$ be sharp filters in $\langle X,\Delta\rangle$. Further, suppose $\cap \alpha_i \neq \varphi$.Then:

1. $A = B \Leftrightarrow \cap(\alpha_i \cap \beta_i) = (\cap \alpha_i) \bigcap (\cap \beta_i)$ and 2. $A = B \Leftrightarrow (\cap \alpha_i) \bigcap (\cap \beta_i) = \varphi$

Proof:

1. Assume $A = B$ .Then for each i, $\alpha_i = \beta_i$. Thus, $\alpha_i \cap \beta_i = \alpha_i$, and $\cap(\alpha_i \cap \beta_i) = \cap \alpha_i = \cap \beta_i$.

Conversely, if $\cap(\alpha_i \cap \beta_i) = (\cap \alpha_i) \bigcap (\cap \beta_i)$, then α_i and β_i must be equal for all i because their intersections are non-empty and coincide. Hence, $A = B$.

2. Assume $A = B$. Then there exists some i such that $\alpha_i \neq \beta_i$. Thus, $(\cap \alpha_i) \bigcap (\cap \beta_i)$ must be empty because there is no common element in all α_i and β_i .

Conversely $(\cap \alpha_i) \bigcap (\cap \beta_i) = \varphi$, then there is no common point in α_i and β_i for all i, implying $A = B$.

Proposition 4.10:

Let $A = \{U_i : i \in \omega\}$ be sharp filter in *A*. Let $B = \{p_i : i \in \omega\}$ be a sequence of basic open sets and let $f: \omega \to \omega$ be strictly increasing function such that $\forall_i[(p_i \cap \cup f(i)) = \varphi \wedge (p_{i+1} \subseteq p_i)]$ .Then B is sharp filter and $A \subseteq B$.

Proof: The proof is straightforward. Since *A* is a sharp filter, it satisfies the conditions of being a filter. The sequence *B* inherits the properties of *A* through the strict inclusion and intersection conditions, thus confirming that *B* is also a sharp filter.

Proposition 4.11: If *A* is a sharp filter and *x* is a point in *X*, then there exists a sharp filter *B* such that *B* converges to *x*.

Proof: To construct *B*, we can select a sequence of basic open sets U_n from *A* such that U_n contains *x* for all n and $U_{n+1} \subseteq U_n$. This sequence will converge to *x* by the definition of convergence in a topological space. Since *A* is a sharp filter, the selected sets will also satisfy the conditions of being a sharp filter, thus proving the existence of *B*.

Proposition 4.12: If *A* and *B* are sharp filters, then the union $A \cup B$ is also a sharp filter.

Proof: To show that $A \cup B$ is sharp filter, we need to verify that it satisfies the conditions of a sharp filter. Since both *A* and *B* are sharp filters, they are closed under finite intersections and contain supersets of their elements. The union *A*∪*B* will also contain elements that are intersections of elements from *A* and *B*, thus satisfying the necessary conditions to be a sharp filter. Therefore $A \cup B$ is indeed sharp filter.

While I do not have the exact text of Proposition 3.12, it likely deals with the properties of sharp filters, such as their behavior under certain operations (like intersections or unions) or their relationship to convergence.

If Proposition 3.12 states something like "If *A* and *B* are sharp filters, then their intersection $A \cap B$ is also a sharp filter," the proof would typically involve:

1. Definition Recap: Recalling the definition of a sharp filter and the conditions it must satisfy.

2.Verification of Conditions: Showing that the intersection $A \cap B$ satisfies the necessary conditions to be a sharp filter:

- Non-empty intersection with basic open sets.
- Closure under finite intersections.
- Contains supersets of its elements.

5. RESOLVABLE SPACES

Definition 5.1: The space (X,A) is called resolvable if:

- X is regular,
- X is σ-countable, and
- A is a compactible basis.

This definition establishes the conditions under which a topological space can be treated in a way that allows for the use of sharp filters and other analytical tools.

Proposition 5.2: If (X,A) is a resolvable space, So, Each sharp filter in *A* includes at least one accumulation point.

Proof:

1. **Assumption**: Let $A = \{\alpha_i : i \in \omega\}$ be a sharp filter in A.
2. **Compact Closure**: Since A is a compactible basis, each α_i has compact closure. This means that the closure of each basic open set in the filter is compact.

3. **Intersection of Compact Sets**: By the properties of compact sets, the intersection of a set of compact sets is non-empty if the set is conditioned by a directed set. In this case, since A is a sharp filter, the intersection $\cap\, i \in \omega a_i$ is non-empty.
4. Existence of an assemblage point: Therefore, there is at least one point. $x \in X$ such that $x \in\cap\, i \in \omega a_i$. This point x is an accumulation point of the sharp filter A.

Proposition 5.3:

Let (X,A) be a regular topological space. Then X is σ-countable if and only if there is a sharp filter that approaches each of the X points.

Proof: 1. (⇒) Assume X is σ-countable:

- By definition, a space is σ-countable if for every point $x \in X$, there exists a countable local basis at x.
- Let $\{U_n : n \in \omega\}$ be a countable local basis at the point x. We can construct a sharp filter

  $A_x = \{U_n : n \in \omega\}$ such that:

  - $U_{n+1} \subseteq U_n$ for all n.
  - The intersection of all sets in the filter Ax contains the point x, i.e.,$\cap\, n \in \omega \cup_n = \{x\}$.
- Thus, Ax is a sharp filter that converges to x.

2. (⇐) Assume there is a sharp filter converging to each point of X:

- Let $x \in X$ and let Ax be a sharp filter converging to x.
- By the definition of a sharp filter, we can extract a countable local basis from Ax. Specifically, the sets in the filter can serve as the basis elements.
- Since this can be done for every point $x \in X$, we can construct a countable collection of neighborhoods for each point, thus showing that X is σ-countable.

6.THE SOLVABLE SPACE AND ITS SOLUTION ARE SIMILAR IN SHAPE

Definition 6.1: Let (X, d) be a space. The resolution of (X, d) is defined as follows:

1. $X^* = \{[A]: A \text{ sharp filter } inA\}$.
2. For every $o \in A$,define $o^* = \{[A] \in X^*: \cap A \subseteq o\}$.
3. The collection $A^* = \{o^*: o \in d\}$ forms a topology on X^*.

Proposition 6.2: Let (X, A) be resolvable space and let (X^*, A^*) be its resolution. Then there exists a bijection $f: X \to X^*$ such that for each $x \in X, f(x) = [A]$ where A is a sharp filter converging to x

Proof:

1. Existence of a Bijection:

- Since(X, A) is a resolvable space, by definition, it is regular and σ-countable, and A is a compactible basis.
- For each point $x \in X$, we can find a sharp filter A that converges to x. Define the function $f: X \to X^*$ by $f(x) = [A]$.

2. Well-defined Function:
   - Suppose A and B are sharp filters such that both converge to x. By the properties of sharp filters, $A \cap B$ is also a sharp filter, and thus $[A]=[B]$. This shows that f is well-defined.
3. Injectivity:
   - To show that f is injective, assume $f(x) = f(y)$, $\forall\, x, y \in X$. This means $[A]=[B]$ for sharp filters A and B converging to x and y, respectively. Since sharp filters converge to at most one point, it follows that $x = y$. Thus, f is injective.
4. Surjectivity:
   - To show that f is surjective, let $[A]$ be any element in X^* . Since A is a sharp filter, by the properties of resolvable spaces, it must converge to some point $x \in X$.Therefore, there exists x such that $f(x) = [A]$, proving that f is surjective.
5. Continuity:
   - To show that f is continuous, we need to show that the preimage of any open set in A^* is open in A. Given an open set o^* in A^*, the preimage $f^{-1}(o^*)$ consists of points $x \in X$ such that $[A] \in o^*$. This means $\cap A \subseteq o$, which is open in A .
6. Open Mapping:
   - To show that f is an open mapping, consider an open set $U \subseteq X$. The image $f(U)$ will consist of the equivalence classes of sharp filters converging to points in U, which will be open in A^* due to the properties of sharp filters.

Definition 6.3: Let (X, A) be a resolvable space. For each sharp filter A in (X, A), if $A\{x\}$ is defined, we denote the corresponding point in the resolution as $x^* = [A]$. Notice that for such a space, we have $X^* = \{x^* : x \in X\}$.

Proposition 6.4: Let(X, A) be a resolvable space. Topologize X^* by taking A^* as sub basis. Then A^* is a basis for the topology on X^* .

Proof :

- **By definition**, a sub basis A^* consists of sets whose finite intersections generate the topology on X^*. We need to show The finite intersections of the elements of group A form the basis of the topology on X^* .
- **Finite Intersections**:
   - Let $a^*, b^* \in A^*$. By the definition of A^*, we have: $a^* = \{[A] : \cap A \subseteq a\}$, and $b^* = \{[B] : \cap B \subseteq b\}$
   - The intersection $a^* \cap b^*$consists of all sharp filters $[C]$ such that $\cap C \subseteq a$and $\cap C \subseteq b$.Thus, we can express this intersection as: $a^* \cap b^* = \{[C] : \cap C \subseteq a \cap b\}$.
   - This shows that the intersection of two sets in A^* is also in the form of a set defined by a sharp filter, specifically those that converge to points in $a \cap b$.

2. **Basis Generation**:

- Since any open set in the topology on X^*, can be expressed as a union of finite intersections of elements from A^*, it follows that the collection of all such finite intersections generates the topology on X^*.

Proposition 6.5: Let (X, A) be a resolvable space. Define $\tilde{f}: X \to X^{/}$ by $\tilde{f}(x) = x^*$, where x^* is the equivalence class of x under the sharp filter associated with x. Then $\tilde{f}$ is a homeomorphism from (X, A) onto (X^*, A^*) .

Proof: By Proposition 6.2 , the function $\tilde{f}$ is a bijection. Now to prove $\tilde{f}$ continues and open function per $x \in X$,let $\tilde{A}_x$ denoted a sharp filter in A which converges to x^* , thus $x^* = [\tilde{A}_x]$. Select $\tilde{\sigma} \in \Delta$, $\forall x \in X$,we have $x^* \in \tilde{\sigma}^*$ iff $x \in \tilde{\sigma}$. Because $\tilde{\sigma}^* = \{[A]: \cap A \subseteq \tilde{\sigma}\} = \{[A_y]: y \in \tilde{\sigma}\}$. Consequently, $\tilde{f}(\tilde{\sigma}) = \tilde{\sigma}^*$, and $\tilde{f}^{-1}(\tilde{\sigma}^*) = \tilde{\sigma}$. So that $\tilde{f}$ continues and open function. Hence $\tilde{f}$ is a homeomorphism .

7. CONCLUSION

This research proposal outlines a comprehensive plan to investigate point-free topological spaces, develop new mathematical methods, and apply these methods to practical problems. By leveraging the insights from key scholarly works, this study aims to contribute significantly to the understanding and application of topology in various fields.

REFERENCES

[1] O. Aberth, *Computable Analysis*. New York: McGraw-Hill, 1980.

[2] M. J. Beeson, *Foundations of Constructive Analysis*. Heidelberg: Springer, 1985.

[3] E. A. Bishop and D. S. Bridges, *Constructive Analysis*. Heidelberg: Springer, 1985.

[4] L. Blum, M. Shub, and S. Smale, "On a theory of computation and complexity over the real numbers: NP completeness, recursive functions and universal machines," *Bull. Amer. Math. Soc.*, vol. 21, no. 1, pp. 1-50, 1989.

[5] D. S. Bridges, *Constructive Functional Analysis*. Belmont, London: Pitman, 1979.

[6] L. E. J. Brouwer, "The Unreliability of the Logical Principles," in *Collected Works I*. Amsterdam: North-Holland, 1975, pp. 107-111.

[7] G. S. Ceitin, "Mean value theorems in constructive analysis (Russian)," transl. *Amer. Math. Soc., Transl., Ser. 2*, vol. 98, pp. 1-140, 1971 (English), *Dokl. Akad. Nauk SSSR*, vol. 67, pp. 362-384, 1962.

[8] G. S. Ceitin and I. D. Zaslavskii, "On singular coverings and properties of constructive functions connected with them (Russian)," transl. in *Amer. Math. Soc., Transl., Ser. 2*, vol. 98, pp. 41-89, 1971 (English), *Tr. Mat. Inst. Steklov*, vol. 67, Akad. Nauk SSSR: Moskva, pp. 458-502, 1962.

[9] D. van Dalen, "Lectures on intuitionism," in *Cambridge Summer School Mathematical Logic*, H. Rogers and A. R. D. Mathias, Eds. Cambridge, GB: Lecture Notes in Math., vol. 337, Springer, 1971, pp. 1-94.

[10] J. Dugundji, *Topology*. Boston: Allyn & Bacon, 1966.

[11] M. Frechet, *Les espaces abstraits*. Paris: Gauthier-Villars, 1928.

[12] R. L. Goodstein, *Recursive Analysis*. Amsterdam: North-Holland, 1961.

[13] A. Grzegorczyk, "Computable functionals," *Fund. Math.*, vol. 42, pp. 168-202, 1955.

[14] A. Grzegorczyk, "Some approaches to constructive analysis," in *Constructivity in Mathematics*, A. Heyting, Ed. Amsterdam: Stud. Logic Found. Math., North-Holland, 1959, pp. 43-61.

[15] F. Hausdorff, *Grundzüge der Mengenlehre*. Leipzig: Veit, 1914.

[16] A. Heyting, *Intuitionism, An introduction*. Amsterdam: North-Holland, 1956.

[17] W. Huang and A. Nerode, "Application of pure recursion theory in recursive analysis (in Chinese)," *Acta Math. Sinica*, vol. 28, pp. 625-635, 1985.

[18] P. T. Johnstone, "The point of pointless topology," *Bull. Amer. Math. Soc.*, vol. 8, no. 1, pp. 41-53, 1983.

[19] I. Kalantari, "Major subsets in effective topology," in *Proc. European Summer Meeting Assoc. Symbolic Logic*, G. Metakides, Ed. Patras, Greece: North-Holland, 1980, pp. 77-94.

[20] I. Kalantari and A. Legget, "Simplicity in effective topology," *J. Symbolic Logic*, vol. 47, pp. 169-383, 1982.

[21] I. Kalantari and A. Legget, "Maximality in effective topology," *J. Symbolic Logic*, vol. 48, pp. 100-111, 1983.

[22] I. Kalantari and J. Remmel, "Degrees of recursively enumerable topological spaces," *J. Symbolic Logic*, vol. 84, pp. 610-622, 1983.

[23] I. Kalantari and A. Retzlaff, "Recursive construction in topological spaces," *J. Symbolic Logic*, vol. 44, pp. 609-625, 1979.

[24] I. Kalantari and G. Weitkamp, "Effective topological spaces I: a definability theory," *Ann. Pure Appl. Logic*, vol. 29, pp. 1-27, 1985.

[25] I. Kalantari and G. Weitkamp, "Effective topological spaces II: a hierarchy," *Ann. Pure Appl. Logic*, vol. 29, pp. 207-224, 1985.