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Some Results Concerning Directed Graphs Over Commutative Rings

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ABSTRACT: This paper delves into the idea of a directed graph that presented and illustrated, A directed graph for every finite, commutative ring is presented in this paper by utilizing the additive and multiplicative properties. This is achieved based on the mapping (a, b) → (a + b, a • b). The essential objective of this research work is to examine the characteristics of these directed graphs, specifically the valuable information they offer about the ring itself. To this end, we have adopted and presented several theorems and Corollaries with their proofs to support the obtained results.

Keywords: Associative ring, graph, semiprime ideal, nil-clean element, idempotent element.

1. INTRODUCTION

Graph theory and algebra are two branches of mathematics that focus on the construction and analysis of structures. While algebra is a fundamental field dating back to the 16th century, graph theory is a field of study that has been developing since the 18th century. A Swiss mathematician's solution to the Königsberg bridge problem by graphically being it led to the emergence of graph theory as a tool for visually modeling real-world problems. As sub-theoretical applications developed, graph theory developed in its own way and set up itself as an evolving discipline. While algebraic structures have sets with defined operations, graphs represent relations over sets of vertices. Synergies between algebraic and graph structures are explored in algebraic graph theory and combinatorial strategies from graph theory are used to prove important results in group theory. Graph theory has used powerful combinatorial strategies to demonstrate important and widely recognized outcomes in group theory. For instance, it has been shown that any finite group can serve as the automorphism collection of a connected graph (see [1]). The concept of connecting a commutative ring to a chart was first proposed by Istvan Beck [3]. He introduced the concept of a zero-divisor graph for a commutative ring with unity. V. Ramanathan [4] The connection between commutative ring theory and graph theory was further investigated by examining the topological characteristics of the intersection graph of ideals. Additionally, David F. Anderson et al. [2] focused on the graphs associated with commutative rings. For many years, researchers have been interested in exploring the connections between algebraic structures and specific graphs. One example is the study of finite groups and their Cayley graphs, which has been a widely researched topic. In more recent times, there has been a shift in focus towards graphs derived from commutative rings. This shift has resulted in a significant amount of research in this field over the last three decades.

Lastly, Muhammad, et. al. [5] The researchers explored fundamental characteristics of the graph Γ(R) and examined properties of Γ(M2(Zn)), which represents matrices over Zn. Additionally, they analyzed the diameter of the bipartite graph linked to the quaternion ring.. This paper will look at some properties concerning the directed graphs of commutative rings and focus on the facts of the graph which deliver gives about the ring.

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2. RELATED WORKS

During this work, this article relies on concepts from both ring theory and graph theory. Therefore, we will first discuss the basic concepts required for this article. We will then elaborate on these concepts as they become necessary. For a more detailed clarification of ring concept, please refer to other resources, see [7], [10], [11], [9] and of graph theory, see [8].

In the following, we list out some fundamental definitions in graph theory from [4], [6] and [13].

Definition 2.1: Other findings related to the overall layout of the digraph of commutative rings include:

- By examining the digraph of a commutative ring, we can ascertain if the ring is an integral domain.
- We can determine if a given ideal is prime.

Definition 2.2: A directed graph of G contain cycles if are path that start and end at the same vertex with one edge. contain the edges are two pairs of vertices an order pair (a, b).

Definition 2.3: An induced subgraph of G, denoted as G subgraph of G, is a subgraph where the set of vertices, denoted as V(G), and the set of edges, denoted as E(G), satisfy $V(\bar{G}) \leq V(G)$ and $E(\bar{G}) \leq E(G)$ if G contains all edges of G.

Definition 2.4: Let (a, b) (z, y) $\in \phi$ (R). Suppose (a, b) $\rightarrow$ (z, y) so denoted that a path and walk and (z, y) denoted a downstream from (a, b) and (a,b) denoted upstream from (z,y). In addition, the subgraph of an ideal in the digraph of a ring can be seen as a combination of connected components within the digraph of the ring. This situation occurs at times, as well as in cases where the subgraph of the ideal includes vertices connected to it from outside the ideal.

In addition to that, we say a ring R is a leading ring if we take the relation x Ry= (0) for all x, y $\in$ R. then either x=0 or y=0. The set of units of a ring R is denoted via U(R).

The following lemma will use in the next results.

Lemma 2.5 [12] If R is a ring, then the intersection of prime ideals is always a semiprime ideal. On the other hand, every semiprime ideal can be expressed as the intersection of prime ideals.

3. THE MAIN RESULTS

Theorem 3.1

Suppose $(\sigma_1,\tau_1) \rightarrow (\sigma_2,\tau_2) \rightarrow \ldots\ldots (\sigma_n,\tau_n)$ from an absorbed path of length n in ψ(R), then $\sigma_k\tau_k = \tau_k\,\sigma_k$, where R is a ring with unit.

Proof:

In this proof, we apply the induction mathematical strategy. Based on the definition, we write $\sigma_2 = \sigma_1 + \tau_1$ and $\tau_2 = \sigma_1\tau_1$. Obviously, this proposition is true when n=2.

In the next step, we suppose the proposal is true for n=k. Here k is a fixed but arbitrary positive integer. Consequently, $\sigma_k = \sigma_1 + \sum_{i=1}^{k-1}\tau_i$ and $\tau_k = \tau_1 \prod_{i=1}^{k-1}\sigma_i$ after that, we exam the intention to be true when n=k+1. According to the hypothesis i.e. via the definition:

$\sigma_k + \tau_k = \sigma_{k+1}$ and $\sigma_k\tau_k = \tau_{k+1}$. Hence, the inductive hypothesis supplies the following relations:

$\sigma_{k+1} = \sigma_1 + \tau_k + \sum_{i=1}^{k-1}\tau_i$ and $\tau_{k+1}. = \tau_1\,\sigma_k\,\pi_{i=1}^{k-1}\,\sigma_i$.

Then, we substitute the relation $\sigma_k\tau_k = \tau_{k+1}$ in the left side of the identity $\tau_{k+1} = \tau_1\sigma_k \prod_{i=1}^{k-1}\sigma_i$, we observe that $\sigma_k\,\tau_k = \tau_1\sigma_k \prod_{i=1}^{k-1}\sigma_i$, where $\tau_k = \tau_1 \prod_{i=1}^{k-1}\sigma_i$, then $\tau_1 = \frac{\tau_k}{\prod_{i=1}^{k-1}\sigma_i}$, we see that $\sigma_k\,\tau_k = \frac{\tau_k}{\prod_{i=1}^{k-1}\sigma_i} \cdot \sigma_k \cdot \prod_{i=1}^{k-1}\sigma_i$. Consequently, $\sigma_k\,\tau_k = \tau_k\,\sigma_k$ which concludes the proof.

An element δ of a ring R is named nil-clean if $\delta = \sigma + \tau$, $\sigma^2 = \sigma \in$ R . i.e. σ is an idempotent element and τ is a nilpotent element. When $\sigma\,\tau = \tau\,\sigma$ the element δ is called a strong nil-clean of R.

In the next corollary, we employ the previous definition and applying the same style of the proof of Theorem 3.1, one can prove the following.

Corollary 3.2

Let σ and τ be idempotent and nilpotent elements respectively of a ring R such that $(\sigma, \tau) \rightarrow \ldots \rightarrow (\sigma_n, \tau_n) \rightarrow (\delta, 0)$ for some $\delta \in$R , then δ acts as a strong nil-clean element, Where R is a ring with unit

Proof:

Based on the results of Theorem 3.1, we see that $\sigma_n\,\tau_n = \tau_n\,\sigma_n$,besides that, via the definition $\delta = \sigma_n + \tau_n$. Obviously, δ is a strong nil-clean, as desired.

Theorem 3.3

Let $(\sigma_1, \tau_1) \rightarrow (\sigma_2, \tau_2) \rightarrow \ldots\ldots \rightarrow (\sigma_n, \tau_n) \rightarrow (\sigma_1, \tau_1)$ be a cycle of length n in ψ(v) , where v is a vector space, then

(i) τ_n forms a linear combination of $\tau_1, \tau_2,\ldots, \tau_{n-1}$ with scalar one.
(ii) The vectors τ_1 , τ_2 ,, τ_n are linearly dependent.

Proof:(i)

From the reference [13: Corollary 3.5], we note that $\sum_{i=1}^{n} \tau_i$ .Consequently,
$\tau_1 + \tau_2 +\ldots+ \tau_n$ =0. This relation yields.
τ_n = - τ_1 - τ_2 -...τ_3- τ_{n-1}. Hence, we have τ_n as a linear combination.

(ii) Immediately from Branch (I) we deduce that $\sum_{i=1}^{n} \tau_i$ =0, Moreover $\tau_1 + \tau_2 +\ldots+ \tau_n$ =0. Here, the scalars k_1=k_2 =…. =k_n=1 That mean the scalars are non- zero. Hence, the previous equation forms linearly independent.

Proposition 3.4

Suppose c=(σ_1, τ_1)→ (σ_2, τ_2)→…..→(σ_n, τ_n)→ (σ_1, τ_1) and c ⊂ $\psi(R)$ for same ring R such that $\tau_n \in \cup (R)$ *then* $\prod_{i=j}^{n-1} \sigma_i \in \cup (R)$.

Proof:

Where the reference [13: Theorem 3.4], provide the following relations: τ_n=$\tau_j \prod_{i=j}^{n-1} \sigma_i$ for j ∈z such that 1≤ $j \leq n-1$. Moreover, τ_n=$\tau_n \prod_{i=1}^{n} \sigma_i$. By reason of $\tau_n \in \cup (R)$ the previous relation becomes $\prod_{i=1}^{n} \sigma_i$ =1.
Consequently, $\sigma_i \in \cup (R)$ for all i. Employing this result with $\tau_n \in \cup (R)$, we deduce that $\tau_n . \tau_j^{-1} = \prod_{i=j}^{n-1} \sigma_i$. Then the least expression modifies to $\tau_n . \tau_j^{-1}$ ∈ ∪(R) yields $\prod_{i=j}^{n-1} \sigma_i$ ∈ ∪(R).

Theorem 3.5

Let Us be a subring of a ring R. Then no vertex with one entry not from S and one entry from S connects to ψ(S).

Proof:

At the beginning, we assume there exists some σ, τ ∈ R such that σ ∈ S and $\tau \notin S$ and (σ, τ)→(g, h) where g, h ∈ S. The definition of ψ refers to g. H= τ . By reason of g, h ∈ S via closure property of S yields τ ∈ S. this expression makes a contradiction Consequently, no vertex with one entry not from S and one entry from S connects to ψ(s).

Theorem 3.6

If U is the primary ideal of the ring R, then ψ(U) is the sum of the connected components of ψ(R).

Proof:

The information of the hypothesis delivers U acts as a primary ideal of a commutative ring R. suppose there exists some elements of a ring R say σ and τ such that $\sigma, \tau \notin$ U and (σ, τ) →(g,h) where g,h ∈ U . Thus, $\sigma . \tau$ =h. multiplying by τ^{m-1} , m > 0 ,we observe that $\sigma . \tau . \tau^{m-1}$ = h . τ^{m-1} yields $\sigma . \tau^{m}$= h. τ^{m-1} .Due to the relations that h ∈ U , τ ∈ R and U plays the role of a primary ideal of R . we can write h. τ^{m-1} ∈ U. This result leads to the left side of the previous expression $\sigma . \tau^{m}$ ∈ U. Basically, U is a primary ideal.

Which mean having two options: Either σ ∈ U or τ^{m} ∈ U. In fact, both options make a contradiction. Therefore, no vertex with both entries not in U connects to ψ(U).

Corollary 3.7

Let Z(R) be an ideal of a ring R. Then:
(i) Z(R) is primary ideal of R.
(ii) ψ (Z(R)) is an assembly of related parts of.ψ(R).

Proof:(i)

Based on the hypothesis. We have Z(R) is an ideal of a ring R. Let $\sigma . \tau$ ∈ Z under the condition $\sigma, \tau \notin$ Z(R). Moreover, ($\sigma . \tau$).x =0 for same x ∈ R\ {0}, where Z(R) is the set of zero-divisors of R. Using an associative property the previous expression modifies to σ.(τ.x) =0. Here, we employ the Z(R) action which is set of zero-divisors of R. We observe that, τ.x≠ 0, where $\tau \notin$ Z(R). Thus, we harvest σ.(τ.x) =0 left multiplying by σ^{m-1} , m> 0. We see that σ^{m}.(τ.x) =0, say to the relation τ.x≠ 0 ,we arrive to σ^{m} ∈ Z(R). This result yields a contradiction. Consequently, Z(R) is a prime ideal of R.

(ii) Depend on the same protocol of the proof of Branch(I) and Theorem 3.5, we find that ψ(Z(R)) is the union of all of ψ(R)'s connected parts.

According to the reference [6], An M-set is defined as the set of non-zero elements that are found in the intersection of a prime ideal ∩ $p_\propto$ of a semiprime ring R, provided that the following property is satisfied: δ ∈ R, then δ^2∈ M-set while: δ ∉ M-set that is, δ^2 ∈ M-set ⊆∩ $p_\propto$ whilst a ∉ M-set ⊆∩ $p_\propto$.

Theorem 3.8

Since U is an M-set of a semiprime ring R., then ψ(U) is the union of all of V(R)'s connected components.

Proof:

Assume there is a vertex (σ, τ) ∉ V (ψ (U)) connect to ψ(U). Basically, U⊆ ∩ $p_\propto$, where U is M-set and any digraph of $p_\propto$ is a union of connected components have (σ, τ) guarantee a vertex in all $\psi(p_\propto)$. Consequently, σ, τ ∈ ∩ $p_\propto$ which is

U. This result provides that $\sigma, \tau \in$ U. Here $\sigma, \tau \in$ U leads to a contradiction. Obviously, ψ(U) forms a union of connected components of V(R).

Theorem 3.9
Suppose U is a semiprime ideal of a ring R, then ψ(U) forms a union of connected components of V (ψ (U)).
Proof:
At the beginning, we employ the information which mention in Lemma 2.5. Writing U=$\cap_{\propto=1}^{n}\ p_{\propto}$, where $p_{\propto}$ is a prime ideal such that $\propto$ =1,2,…..,n. Assume there exists a vertex $(\sigma, \tau) \notin$ v(ψ (U)) connected to ψ(U). Basically, the digraph of any $p_{\propto}$ is union of connected components. Hence, (σ, τ) must be vertex in all $\psi(p_{\propto})$. Consequently, σ and τ are in $\cap_{\propto=1}^{n}\ p_{\propto}$ yields $\sigma, \tau \in$ U. This result leads to a contradiction.
Theorem 3.10
Let f: V(ψ(R)) →V(ψ(S)) be an additive abjection and anti-zoomorphism then ψ(R) $\gtrsim$ ψ(S), where ψ(R) and ψ(S) are the digraphs of two commutative rings R and S.
Proof:
Basically, we have F: R→ S are abjection and anthropomorphisms ring. Moreover, for all $\sigma, \tau \in$ R, F (σ, τ)= F (τ). F (σ), where F acts as anthropomorphisms. Also, F $(\sigma + \tau)$= F (σ)+F (τ) = F (τ)+F (σ). this implies that if (σ, τ)→(a,b), then(F(σ),F(τ))→(F(τ)+F(σ), F(τ).F(σ))= (F$(\tau + \sigma)$, F$(\tau . \sigma)$) . Due to $\tau, \sigma \in$ R is commutative ring, then = (F $(\sigma + \tau)$, F (($\sigma . \tau$) = (F(a), F(b)). Finally, we arrive to ψ(R) $\gtrsim$ ψ(S).

Proposition 3.11
Let σ be a nilpotent element and τ be an idempotent element of a loop R. Then $(\sigma . \tau)$→…. → $(\delta$,0) for $\delta \in$ R.
Proof:
Basically, the nil (R) is closed under adding and development (σ, τ)→…..→(δ,x) where $\delta \in$ R. Due to σ acts as a nilpotent element, then $\sigma^n = 0$ and τ acts as idempotent then $\tau^2 = \tau$ .Moreover, taking (σ, τ) dwonstream n+2 steps (σ, τ)→…..→($\delta, \sigma^{x1}\ \tau + \sigma^{x2}\ \tau$ +…..) where x_i+2 = n+2+1. We now move to having the result that apiece summand is equivalent to zero. If $x_i \geq n$ yields σ^{xi}= 0. Consequently $\sigma^{xi}b = 0$. Thus, we harvest all the summands $\sigma^{x1}\tau + \sigma^{x2}\tau$+$\sigma^{x3}\tau + \cdots$. are zero. More precisely, $\sigma^{x1}\tau$ +$\sigma^{x2}\tau$+$\sigma^{x3}\tau$+…. =0.
This result means (σ, τ)→…. →($\delta, 0$) . This step completes the proof of the proposition.
Proposition 3.12
Let σ *and* τ be anon-zero vector of a vector space (V, +,). Then (σ, τ)→($\sigma + \tau$,0) if and only if σ *and* τ are orthogonal of V.
Proof:
Suppose that σ *and* τ are non-zero vectors and (σ, τ)→($\sigma + \tau$,0). Employing the definition, we arrive to $\sigma . \tau = 0$. This outcome refers to that σ *and* τ are orthogonal of V. Conversely, assume that σ *and* τ are orthogonal of V. This case yields that (σ, τ)→($\sigma + \tau$,0). Hence, the definition delivers that $\sigma \neq 0$ and $\tau \neq 0$.
Proposition 3.13
Suppose σ *and* τ belong to a ring R such that (σ, τ)→($\sigma + \tau$,0), where R has no zero-divisors then either $\sigma = 0$ *or* $\tau = 0$, if and only if R is a prime ring.
Proof:
From the hypothesis, we have (σ, τ)→($\sigma + \tau$,0). The definition provides $\sigma . \tau$ =0. Basically, R has a no non-zero-divisors. Thus either $\sigma = 0$ *or* $\tau = 0$ which mean R acts as a main ring. Conversely, R is a main ring supply two options: Either $\sigma = 0$ *or* $\tau = 0$, where $\sigma . \tau$ =0 this completes the proof.

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