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Some Iterations of Best Proximity Point

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ABSTRACT: The aim of this paper is to investigate the convergence of two types of iterative methods, that is Mann and Ishikawa to the optimal proximity points for cyclic Kannan and Chatterjee mappings defined on Banach spaces.

Keywords: The best proximity point, Cyclic Kannan contraction mappings, uniformly convex Banach space, Iterative sequences, Cyclic Chatterjee contraction mapping.

1. INTRODUCTION

Assume that Λ, Γ are nonempty subsets of a Banach space (**B.S**) Y, and a mapping $A: \Lambda \to \Gamma$, an element $r \in A$ is called a **fixed point [1]** to A if $Ar = r$. It is clear that $A(\Lambda) \cap \Lambda \neq \emptyset$, a fixed point of A requires this condition, but alone isn't enough. If $A(\Lambda) \cap \Lambda = \emptyset$, then $\|Ar - r\| > 0$ for all $r \in \Lambda$, which means that the set of fixed points of A is empty. In this case, one often tries to search for the point r that is as close as possible to Ar such that $\|Ar - r\| = D(\Lambda, \Gamma)$, where $D(\Lambda, \Gamma) = inf\{\|r - v\|; v \in \Gamma\}$. The existence and convergence of best proximity point is an interesting topic of optimization theory, which is why many researchers became interested in it [2], [3], on the other hand, in 1953, Mann developed an iterative approach for estimating a nonexpansive mapping's fixed point in uniformly convex Banach space (**UCBS**) [4].

$$u_{c+1} = (1 - \nu_c)u_c + \nu_c A u_c, \qquad (1)$$

where $\{\nu_c\}$ is a sequence $\in (0,1)$ so that $\lim_{c\to\infty} \nu_c = 0$ and $\sum_{c=0}^{\infty} \nu_c = \infty$. Afterwards, in the year 1974, Ishikawa proposed a new iterative process for determining the fixed points, as follows [5]:

$$u_{c+1} = (1 - \nu_c)w_c + \nu_c A k_c,$$
$$(2)$$
$$w_c = (1 - \phi_c)t_c + \phi_c A u_c,$$

where $\{\nu_c\}, \{\phi_c\}$ are sequences in [0, 1) which satisfy the following conditions:

1. $0 \leq \nu_c \leq \phi_c \leq 1, \lim_{c\to\infty} \phi_c = 0$.
2. $\sum_{c=1}^{\infty} \nu_c \phi_c = \infty$.

Later, in 2011, Sadiq Basha [6] stated the best proximity point theorems for contraction mappings for other best proximity point results, see [1], [2], [3], [5], [6].

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This paper examines the convergence of the best proximity point for cyclic **Kannan** and **Chatterjee** contraction mappings within the context of **U.C.B.S**. Additionally, we evaluate and compare the results obtained from said iterative algorithms.

Throughout this paper, given that (Λ, Γ) is a pair of non-empty subsets of a B.S, the following notions are used in the sequel.

$D(\Lambda,\Gamma) = inf\{\|r - v\|;\ r \in \Lambda, and\ v \in \Gamma\}$.
$\Lambda_o = \{r \in \Lambda;\ \|r - v\| = D(\Lambda,\Gamma)\ for\ some\ v \in \Gamma\}$.
$\Gamma_o = \{v \in \Gamma;\ \|r - v\| = D(\Lambda,\Gamma)\ for\ some\ r \in \Lambda\}$.

Definition 1.1 [7] Let Λ and Γ be non-empty subsets of B.S Y, and let a mapping $A: \Lambda \cup \Gamma \to \Lambda \cup \Gamma$ be such that $A(\Lambda) \subseteq \Gamma$ and $A(\Gamma) \subseteq \Lambda$, then A is called cyclic.

Definition 1.2 [7] Let Λ and Γ be non-empty subsets of B.S Y. A map $A: \Lambda \cup \Gamma \to \Lambda \cup \Gamma$, is a cyclic Kannan contraction map if for $\delta \in (0, \frac{1}{2})$ it satisfies:

1. $A(\Lambda) \subset \Gamma$ and $A(\Gamma) \subset \Lambda$.
2. $\|Ar - Av\| \le \delta(\|r - Ar\| + \|v - Av\|) + (1 - 2\delta)D(\Lambda,\Gamma), \forall r \in \Lambda$, and $v \in \Gamma$.

Definition 1.3 [7] Let Λ and Γ be non-empty subsets of B.S Y. A map $A: \Lambda \cup \Gamma \to \Lambda \cup \Gamma$ is a cyclic Chatterjee contraction map if it satisfies:

1. $A(\Lambda) \subseteq \Gamma$ and $A(\Gamma) \subseteq \Lambda$.
2. $\|Ar - Av\| \le \zeta(\|Ar - v\| + \|Av - r\|) + (1 - 4\zeta)D(\Lambda,\Gamma), \forall r \in \Lambda$ and $v \in \Gamma$, where $\zeta \in (0, \frac{1}{4})$.

Definition 1.4 [8] Let (Λ, Γ) be a pair of non-empty subsets of B.S Y, with $\Lambda_o \neq \emptyset$. The pair (Λ, Γ) is said to have **P-property** if:

$$\left.\begin{array}{r}\|m_1 - n_1\| = D(\Lambda,\Gamma)\\ \|m_2 - n_2\| = D(\Lambda,\Gamma)\end{array}\right\} \Longrightarrow \|m_1 - m_2\| = \|n_1 - n_2\|$$

where $m_1, m_2 \in \Lambda_o$ and $n_1, n_2 \in \Gamma_o$.

Lemma 1.5 [9] Every non-empty, bounded, closed, and convex pair in a **U.C.B.S** has the p-property.

Lemma 1.6 [8] A B.S is U.C if and only if for each fixed number $l > 0$, $\exists$ a continuous strictly increasing function: $\psi: [0,\infty) \to [0,\infty)$ with $\psi(b) = 0 \Leftrightarrow b = 0$, such that $\|\varepsilon r - (1-\varepsilon)v\|^2 \le \varepsilon\|r\|^2 + (1-\varepsilon)\|v\|^2 - \varepsilon(1-\varepsilon)\psi(\|r - v\|)$, for all $\varepsilon \in [0, 1]$ and $r, v \in Y$ where $\|r\| \le l$ and $\|v\| \le l$.

Lemma 1.7 [8] Consider a strictly increasing function $\psi: [0,\infty) \to [0,\infty)$, with $\psi(0) = 0$. If a sequence $\{l_c\}$ in $[0,\infty)$ satisfies $\lim_{c\to\infty}\psi(l_c) = 0$, then $\lim_{c\to\infty} l_c = 0$.

Lemma 1.8 [10] Let Λ and Γ be non-empty closed, and convex subsets of a **U.C.B.S**. Let $\{p_c\}$ and $\{g_c\}$ be sequences in Λ, and let $\{z_c\}$ be a sequence in Γ satisfying:

1. $\|p_c - z_c\| \to D(\Lambda,\Gamma)$,
2. $\|g_c - z_c\| \to D(\Lambda,\Gamma)$.

Then $\|p_c - g_c\| \to 0$.

Lemma 1.9 [8] Let Y be a B.S, and Λ, Γ be two non-empty, closed, and convex subsets of U.C.B.S Y. Let $A: \Lambda \cup \Gamma \to \Lambda \cup \Gamma$ be a cyclic operator satisfying:

$$\|Ar - A^2 r\| \le \nu \|r - Ar\| + (1-\nu)\, D(\Lambda, \Gamma) \qquad (3)$$

for all $r \in \Lambda \cup \Gamma$, where $\nu \in [0, 1)$. Then

1. $\|A^c r - A^{c+1} r\| \le \nu^c \|r - Ar\| + (1 - \nu^c) D(\Lambda, \Gamma)$, for all $r \in \Lambda \cup \Gamma$, and $c \ge 0$.
2. $\|A^c r - A^{c+1} r\| \to D(\Lambda, \Gamma)$, as $c \to \infty$ for all $r \in \Lambda \cup \Gamma$.
3. $\|A^{2c} r - Ar^{2c+2}\| \to 0$, as $c \to \infty$ for all $r \in \Lambda \cup \Gamma$.
4. h is a best proximity point if and only if h is a fixed point of A^2.

Theorem 1.10 [11] Let Λ and Γ be non-empty subsets of uniformly, convex Suppose $A = \Lambda \cup \Gamma \to \Lambda \cup \Gamma$ is a cyclic Kannan Contraction map. Then

1. A has a unique best proximity point h in A.
2. The sequence $\{A^{2c} r\}$ converges to h for any starting point $r \in A$.
3. h is the unique fixed point of A^2.
4. Ah is a best proximity point of A in Γ.

Definition 1.11 [12] A pair (Λ, Γ) of subsets of a metric space Y satisfies property **H** provided that for every non-empty closed convex bounded pair $(\Lambda_1, \Lambda_2) \subseteq (\Lambda, \Gamma)$ we have

$$\max\{diam(\Lambda_1), diam(\Lambda_2)\} \le \omega(\Lambda_1, \Lambda_2)$$

Theorem 1.12 [12] Let (Λ, Γ) be closed convex bounded pair in strictly convex and reflexive metric space Y. Suppose that $A: \Lambda \cup \Gamma \to \Lambda \cup \Gamma$ is a cyclic contraction in the sense of Chatterjea. If (Λ, Γ) has the H-property, then A has a best proximity point in $\Lambda \cap \Gamma$.

2. MAIN RESULTS I

In this section, we provide two robust convergence theorems for determining the best proximity point. The Mann iterative process is examined in the following, in order to calculate approximations for this point.

In all theorems the pair (Λ, Γ) are nonempty bounded, closed, and convex subsets of a U.C.B.S Y.

Theorem 2.1. Suppose Λ and Γ are subsets of U.C.B.S. Y. such that $\Lambda \cap \Gamma = \emptyset$. Let $A: \Lambda \cup \Gamma \to \Lambda \cup \Gamma$ be a cyclic Kannan contraction mapping for $u_o \in \Lambda$. Define the iteration

$$u_{c+1} = (1 - \nu_c) u_c + \nu_c A^2 u_c, \text{ for all } c \in N. \qquad (4)$$

where $\varepsilon < \nu_c < 1 - \varepsilon$, $\varepsilon \in (0, \frac{1}{2}]$, then $\{u_c\}$ strongly converge to a best proximity of A.

Proof. Since A is a cyclic Kannan mapping, put $v = Ar$. In (**3**) we get $\|Ar - A^2 r\| \le \nu \|r - Ar\| + (1-\nu) D(\Lambda, \Gamma)$, where $\nu \in [0, 1)$, $\nu = \frac{\delta}{1-\delta}$. Using lemma 1.8. we get h is a best proximity point for A if and only if h is a fixed point of A^2.

$$\|u_{c+1} - h\| = \|(1-\nu_c)u_c + \nu_c A^2 u_c - (1-\nu_c)h - \nu_c A^2 u_c\|$$
$$\le (1-\nu_c)\|u_c - h\| + \nu_c \|A^2 u_c - A^2 h\|$$
$$\le \|u_c - h\|.$$

Thus $\{\|u_c - h\|\}_{c \ge 1}$ is a decreasing sequence of non-negative real numbers. Suppose that

$$\lim_{c \to \infty} \|u_c - h\| = l \ge D(\Lambda, \Gamma).$$

By using lemma 1.7 that there exists a strictly increasing and continuous function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$\|u_{c+1} - h\|^2 = \|(1-\nu_c)u_c + \nu_c A^2 u_c - (1-\nu_c)h - \nu_c A^2 h\|^2$$

$$= \left\|(1-\nu_c)(u_c-h)+\nu_c(A^2u_c-A^2h)\right\|^2$$
$$\leq (1-\nu_c)\left\|u_c-h\right\|^2+\nu_c\left\|A^2u_c-A^2h\right\|^2-\nu_c(1-\nu_c)\psi(\left\|u_c-A^2u_c\right\|)$$
$$\leq \left\|u_c-h\right\|^2-\nu_c(1-\nu_c)\psi(\left\|u_c-A^2u_c\right\|).$$

Therefore,

$$\mathcal{E}^2\,\psi(\left\|u_c-A^2u_c\right\|)\leq \nu_c(1-\nu_c)\psi(\left\|u_c-A^2u_c\right\|)$$
$$\leq \left\|u_{c+1}-h\right\|^2-\left\|u_c-h\right\|^2.$$

Where $c\to\infty$, we will get $\lim_{c\to\infty}\psi(\left\|u_c-A^2u_c\right\|)=0$. And ψ is strongly increasing and using lemma 1.7. we will get $\left\|u_c-A^2u_c\right\|\to 0$. Now, since every U.C.B.S is reflexive, Λ is a weakly compact set. Therefore, $\{u_c\}_{c\geq 1}$ has a weak convergence subsequence $\{u_{ci}\}_{i\geq 1}$, which is convergent to the point $r*\in\Lambda$. Since $\{u_{ci}\}_{i\geq 1}$ is convergent to $r*\in\Lambda$.

$$\left\|u_c-Au_c\right\|=\left\|u_c-A^2u_c\right\|+\left\|A^2u_c-Au_c\right\|$$
$$\leq \left\|u_c-A^2u_c\right\|+\delta(\left\|Au_c-A^2u_c\right\|+\left\|u_c-Au_c\right\|)+(1-2\delta)D(\Lambda,\Gamma)$$

$(1-\delta)\left\|u_c-Au_c\right\|\leq\ \delta\left\|Au_c-A^2u_c\right\|+(1-2\delta)D(\Lambda,\Gamma)$.

But using lemma 1.8.

$\left\|Au_c-A^2u_c\right\|\leq\nu\left\|u_c-Au_c\right\|+(1-\nu)D(\Lambda,\Gamma)$,

where $\nu=\frac{\delta}{1-\delta}$. So

$$(1-\delta)\left\|u_c-Au_c\right\|\leq\delta[\frac{\delta}{1-\delta}\left\|u_c-Au_c\right\|+(1-\frac{\delta}{1-\delta})D(\Lambda,\Gamma)]+(1-2\delta)D(\Lambda,\Gamma)$$
$$\leq\frac{\delta^2}{1-\delta}\left\|u_c-Au_c\right\|+(\delta-\frac{\delta^2}{1-\delta})D(\Lambda,\Gamma)\ +\ (1-2\delta)D(\Lambda,\Gamma)$$
$$(1-\delta-\frac{\delta^2}{1-\delta})\left\|u_c-Au_c\right\|\leq(\delta-\frac{\delta^2}{1-\delta}+1-2\delta D(\Lambda,\Gamma)$$

$(1-\delta-\frac{\delta^2}{1-\delta})\left\|u_c-Au_c\right\|\leq(1-\delta-\frac{\delta^2}{1-\delta})D(\Lambda,\Gamma)$.

So $\left\|u_c-Au_c\right\|\to D(\Lambda,\Gamma)$. Therefore, $\left\|u_c-Au_c\right\|$ convergent to $D(\Lambda,\Gamma)$.

Now, since $\{u_{ci}\}_{i\geq 1}$ converges to $r*\in\Lambda$ we have

$$\left\|r*-Ar*\right\|\leq\liminf_{i\to\infty}\left\|u_{ci}-Au_{ci}\right\|$$
$$=\lim_{i\to\infty}\left\|u_{ci}-Au_{ci}\right\|$$
$$=\lim_{c\to\infty}\left\|u_c-Au_c\right\|$$
$$=D(\Lambda,\Gamma)$$

Given that (Λ,Γ) possesses the p-property, and uniqueness of best proximity point, we must have $Ah=r*$ and h_c convergent to $r*$.

Next theorem used the Ishikawa method to give an approximate value to the best approximate point for mapping of Kannan type.

Theorem 2.2. Suppose Λ and Γ are subsets of U.C.B.S Y, such that $\Lambda\cap\Gamma=\emptyset$. Let $A:\Lambda\cup\Gamma\to\Lambda\cup\Gamma$ be a cyclic Kannan contraction mapping for $u_o\in\Lambda$, and define the iteration:

$$u_{c+1}=(1-\nu_c)w_c+\nu_cA^2w_c, \qquad (5)$$
$$w_c=(1-\phi_c)u_c+\phi_cA^2u_c, \qquad \forall c\in N$$

where

$$0<\varepsilon<\nu_c\leq 1,$$
$$0<\varepsilon\leq\phi_c(1-\phi_c). \quad \forall c\in N$$

Then, $\left\|u_c-A^2u_c\right\|$ is convergent to zero, and $\{u_c\}$ strongly convergent to best proximity point of A.

Proof. Using a comparable justification as presented in the proof of the theorem (2.1) we can show that every best proximity of A is fixed point for A^2. Now using Ishikawa iteration to $A^2:\Gamma\to\Gamma$, one has

$$\left\|u_{c+1}-h\right\|=\left\|(1-\nu_c)w_c+\nu_cA^2w_c-(1-\nu_c)h-\nu_cA^2h\right\|$$
$$\leq(1-\nu_c)\left\|w_c-h\right\|+\nu_c\left\|A^2w_c-A^2h\right\|$$

$$\leq (1-\nu_c)\|w_c-h\| + \nu_c\|w_c-h\| = \|w_c-h\|.$$

Also,

$$\|w_c-h\| = \|(1-\phi_c)u_c+\phi_c A^2u_c-(1-\phi_c)h-\phi_c A^2h\|$$
$$\leq (1-\phi_c)\|u_c-h\| + \phi_c\|A^2u_c - A^2h\|$$
$$\leq (1-\phi_c)\|u_c-h\| + \phi_c\|u_c-h\| = \|u_c-h\|.$$

Therefore, $\|u_{c+1}-h\| \leq \|u_c-h\|$.

So, the sequence $\{\|u_c-h\|\}$ is descending, and therefore $\lim_{c\to\infty}\|u_c-h\|$ exists for any $h\in Fix(A^2|\Gamma_o)$. Since Y is a U.C, there are strictly increasing and continuous function $\psi:[0,\infty)\to[0,\infty)$ with $\psi(0)=0$ such that

$$\|u_{c+1}-h\|^2 = \|(1-\nu_c)k_c+\nu_c A^2w_c-(1-\nu_c)h-\nu_c A^2h\|^2$$
$$\leq (1-\nu_c)\|w_c-h\|^2+\nu_c\|A^2w_c-h\|^2-\nu_c(1-\nu_c)\psi(\|w_c-A^2w_c\|)$$
$$\leq (1-\nu_c)\|k_c-h\|^2+\nu_c\|(1-\phi_c)u_c+\phi_c A^2w_c-h\|^2$$
$$-\nu_c(1-\nu_c)\psi(\|w_c-A^2w_c\|)$$
$$\leq (1-\nu_c)\|w_c-h\|^2+\nu_c\|(1-\phi_c)u_c-h\|^2+\nu_c\phi_c\|A^2u_c-A^2h\|^2$$
$$-\nu_c\phi_c(1-\phi_c)\psi(\|u_c-A^2u_c\|)-\nu_c(1-\nu_c)\psi(\|w_c-A^2w_c\|)$$
$$=\|u_c-h\|^2-\nu_c\phi_c(1-\phi_c)\psi(\|u_c-A^2u_c\|)-\nu_c(1-\nu_c)\psi(\|w_c-A^2w_c\|)$$

Therefore,

$$\nu_c\phi_c(1-\phi_c)\psi(\|u_c-A^2u_c\|)\leq\nu_c\phi_c(1-\phi_c)\psi(\|u_c-A^2u_c\|)+\nu_c(1-\nu_c)\psi(\|w_c-A^2w_c\|)$$
$$\leq (1-\nu_c)\|u_c-h\|^2+\nu_c(1-\phi_c)\|u_c-h\|^2$$
$$\leq \|u_c-h\|^2-\|u_{c+1}-h\|^2.$$

Hence,

$\varepsilon^2\psi(\|u_c-A^2u_c\|)\leq\|u_c-h\|^2-\|u_{c+1}-h\|^2$. Which ensures that $\lim_{c\to\infty}\|A^2u_c-u_c\| = 0$. Using the same argument as the proof of Theorem 2.1. $\{u_c\}$ converges to best proximity point of A in Λ.

3. MAIN RESULTS II

In this section we calculate the approximate value for the best approximate point for Chatterjee mapping using two types of iterations. The first one is the Mann iteration.

Theorem 3.1. Suppose Λ and Γ are subsets of U.C.B.S Y, such that $\Lambda\cap\Gamma=\emptyset$. Let $A:\Lambda\cup\Gamma\to\Lambda\cup\Gamma$ be a cyclic Chatterjee contraction mapping that satisfies the H-property for $t_o\in\Lambda$. Define the iteration

$$u_{c+1}=(1-\nu_c)u_c+\nu_c A^2u_c,\quad \forall c\in N.$$

Where $\varepsilon<\nu_c<1-\varepsilon$, $\varepsilon\in(0,\frac{1}{2}]$. Then $\{u_c\}$ is strongly convergent to best proximity point of A.

Proof. Using theorem (1.10), there exists $h\in\Gamma_o$ such that $\|h-Ah\| = D(\Lambda,\Gamma)$, to calculate h, we shall first show that h is a fixed point for the self-mapping $A^2:\Gamma\to\Gamma$.

$$D(\Lambda,\Gamma)\leq\|A^2h-Ah\|$$
$$\leq\zeta(\|A^2h-h\|+\|Ah-Ah\|)+(1-4\zeta)D(\Lambda,\Gamma)$$
$$\leq\zeta\|A^2h-h\|+(1-4\zeta)D(\Lambda,\Gamma)$$
$$\leq\zeta\|A^2h-h\|+(1-4\zeta)D(\Lambda,\Gamma).$$

Using lemma 1.7, $\|A^2h-h\|\to 0$. We get $D(\Lambda,\Gamma)\leq\|A^2h-Ah\|<(1-4\zeta)D(\Lambda,\Gamma)<D(\Lambda,\Gamma)$.

Since

$\|A^2h-Ah\|=D(\Lambda,\Gamma)$.

$\|A^2h-h\|=\|Ah-Ah\|=0$.

By lemma 1.7, therefore, $A^2h=h$. Now,

$$\|u_{c+1}-h\| = \|(1-\nu_c)u_c+A^2u_c-(1-\nu_c)h-\nu_c A^2h\|$$
$$\leq (1-\nu_c)\|u_c-h\|+\nu_c\|A^2u_c-A^2h\|$$

$\leq \|u_c - h\|$.

So, $\{\|u_c - h\|\}_{c\geq 1}$ is a sequence of non-negative real numbers which is decreasing. Assume that $\lim_{c\to\infty}\|u_c - h\| = l \geq D(\Lambda, \Gamma)$. Now, by using lemma (1.5), that there is an increasing and continuous function $\psi: [0,\infty) \to [0,\infty)$, with $\psi(0) = 0$ such that

$$\|u_{c+1} - h\|^2 = \|(1-\nu_c)u_c - \nu_c A^2 u_c - (1-\nu_c)h - \nu_c A^2 h\|^2$$
$$= \|(1-\nu_c)(u_c - h) + \nu_c(A^2 u_c - A^2 h)\|^2$$
$$\leq (1-\nu_c)\|u_c - h\|^2 + \nu_c\|A^2 u_c - A^2 h\|^2 - \alpha_c(1-\nu_c)\psi(\|u_c - A^2 u_c\|)$$
$$\leq \|u_c - h\|^2 - \nu_c(1-\nu_c)\psi(\|u_c - A^2 u_c\|).$$

Therefore,

$$\varepsilon^2\psi(\|u_c - A^2 u_c\|) < \nu_c(1-\nu_c)\psi(\|u_c - A^2 u_c\|)$$
$$\leq \|u_{c+1} - h\|^2 - \|u_c - h\|^2.$$

We will get $\lim_{c\to\infty}\psi(\|u_c - A^2 u_c\|) = 0$, where c is convergent to ∞. in view of the truth that ψ is strictly increasing. We will get $\|u_c - A^2 u_c\|$ is convergent to zero. Now, since every U.C.B.S is reflexive, Λ is a weakly compact set, therefore $\{u_c\}$ has a weak convergent subsequence $\{u_{ci}\}_{i\geq 1}$which is convergent to some point $r* \in \Lambda$

$$\|u_c - Au_c\| \leq \|u_c - A^2 u_c\| + \|A^2 u_c - Au_c\|$$
$$\leq \|u_c - A^2 u_c\| + \zeta(\|A^2 u_c - u_c\| + \|Au_c - Au_c\|) + (1-4\zeta)D(\Lambda, \Gamma).$$

But $\|A^2 u_c - u_c\| \to 0$, so $\|u_c - Au_c\| \leq (1-4\zeta)D(\Lambda,\Gamma) \leq D(\Lambda,\Gamma)$.

Therefore, $\|u_c - Au_c\|$ is convergent to $D(\Lambda,\Gamma)$ Because $\{u_{ci}\}_{i\geq 1} \to r* \in \Lambda$. We have $\|r* - Ar*\| \leq \liminf_{i\to\infty}\|u_{ci} - Au_{ci}\|$

$$\leq \lim_{i\to\infty}\|u_{ci} - Au_{ci}\|$$
$$\leq \lim_{c\to\infty}\|u_c - Au_c\|$$
$$= D(\Lambda,\Gamma)$$

Given that (Λ,Γ) possesses the p-property, and uniqueness of best proximity point, we must have $Ah = r*$ and h_c convergent to $r*$.

Secondly, we use the Ishikawa iteration in calculation.

Theorem 3.2. Suppose that Λ and Γ are subsets of U.C.B.S Y such that $\Lambda \cap \Gamma = \emptyset$. Let $A: \Lambda \cup \Gamma \to \Lambda \cup \Gamma$ be a cyclic Chatterjee contraction mapping that satisfies H-property for $t_o \in \Lambda$. Define the iteration:

$$u_{c+1} = (1-\nu_c)w_c + \nu_c A^2 w_c,$$
$$w_c = (1-\phi_c)t_c + \phi_c A^2 u_c, \forall c \in N.$$

Where $0 < \varepsilon < \nu_c \leq 1$, $0 < \varepsilon \leq \phi_c(1-\phi_c), \forall c \in N$. Then $\|u_c - A^2 u_c\|$ convergent to zero and $\{u_c\}$ strongly convergent to best proximity point of A.

Proof. Using the same reasoning as the one in the proof of theorem (3.1). Now using Ishikawa iteration to $A^2: \Gamma \to \Gamma$ one has,

$$\|u_{c+1} - h\| = \|(1-\nu_c)w_c + \nu_c A^2 w_c - (1-\nu_c)h - \nu_c A^2 h\|$$
$$\leq (1-\nu_c)\|w_c - h\| + \nu_c\|A^2 w_c - A^2 h\|$$
$$\leq (1-\nu_c)\|w_c - h\| + \nu_c\|w_c - h\| = \|w_c - h\|.$$

Also,

$$\|w_c - h\| = \|(1-\phi_c)u_c + \phi_c A^2 u_c - (1-\phi_c)h - \phi_c A^2 h\|$$
$$\leq (1-\phi_c)\|u_c - h\| + \phi_c\|A^2 u_c - A^2 h\|$$
$$\leq (1-\phi_c)\|u_c - h\| + \phi_c\|u_c - h\| = \|u_c - h\|.$$

Therefore, $\|u_{c+1} - h\| \leq \|u_c - h\|$.

So, the sequence $\{\|u_c - h\|\}$ is descending, and therefore $\lim_{c\to\infty}\|u_c - h\|$ exists for any $h \in Fix(A^2|\Gamma_o)$. We notice that there is a significant increase and continuous function ψ. Since Y is a U.C, there are strictly increasing and continuous function ψ: $[0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$\|u_{c+1} - h\|^2 = \|(1-\nu_c)k_c + \nu_c A^2 k_c - (1-\nu_c)h - \nu_c A^2 h\|^2$$
$$= \|(1-\nu_c)(w_c - h) + \nu_c(A^2 k_c - A^2 h)\|^2$$
$$\leq (1-\nu_c)\|w_c - h\|^2 + \nu_c\|A^2 w_c - A^2 h\|^2$$
$$-\nu_c(1-\nu_c)\psi(\|w_c - A^2 w_c\|$$
$$\leq (1-\nu_c)\|w_c - h\|^2 + \nu_c(1-\phi_c)\|u_c - h\|^2$$
$$+\nu_c\phi_c(1-\phi_c)\|A^2 u_c - A^2 h\|^2 - \nu_c\phi_c(1-\phi_c)\psi(\|u_c - A^2 u_c\|)$$
$$-\nu_c(1-\nu_c)\psi(\|w_c - A^2 w_c\|)$$

$$= \|u_c - h\|^2 - \nu_c\phi_c(1-\phi_c)\psi(\|u_c - A^2 u_c\|) - \nu_c(1-\nu_c)\psi(\|w_c - A^2 w_c\|)$$

Therefore,

$$\nu_c\phi_c(1-\phi_c)\psi(\|u_c - A^2 u_c\|) \leq \nu_c\phi_c(1-\phi_c)\psi(\|u_c - A^2 u_c\|) + \nu_c(1-\nu_c)\psi(\|w_c - A^2 w_c\|)$$
$$\leq (1-\nu_c)\|u_c - h\|^2 + \nu_c(1-\nu_c)\|u_c - h\|^2$$
$$\leq \|u_c - h\|^2 - \|u_{c+1} - h\|^2.$$

Hence,

$\varepsilon^2\psi(\|u_c - A^2 u_c\|) \leq \|u_c - h\|^2 - \|u_{c+1} - h\|^2$, where c is convergent to ∞, we will get $\lim_{c\to\infty}\psi(\|u_c - A^2 u_c\|)$ equal to zero.

Using the same argument as the proof of Theorem 2.1. $\{t_c\}$ converges to best proximity point of A in Λ.

REFERENCES

[1] L. J. Khaleel and B. A. A. Ahmed, "Fixed point theorem for set valued mapping with rational condition," *Iraqi J. Sci.*, vol. 61, no. 4, pp. 805-810, 2020.

[2] K. A. Jassim and S. S. Abed, "Some results on best proximity in geodesic spaces," *J. Phys. Conf. Ser.*, vol. 1804, no. 1, p. 012030, 2021.

[3] Z. H. Maibed and S. S. Hussein, "Some theorems of fixed-point approximations by iteration processes," *J. Phys. Conf. Ser.*, vol. 1818, no. 1, p. 012153, 2021.

[4] W. R. Mann, "Mean value methods in iteration," *Proc. Am. Math. Soc.*, vol. 4, pp. 506-510, 1953.

[5] E. Karapinar, "Best proximity points of cyclic mappings," *Appl. Math. Lett.*, vol. 25, pp. 1761-1766, 2012.

[6] S. Basha, "Best proximity point theorems," *J. Approx. Theory*, vol. 163, pp. 1772-1781, 2011.

[7] E. Karapinar and I. M. Erhan, "Best proximity point on different type contractions," *Appl. Math. Inf. Sci.*, vol. 5, no. 3, pp. 558-569, 2011.

[8] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *J. Math. Appl.*, vol. 323, pp. 1001-1006, 2006.

[9] A. Abkar and M. Gabeleh, "Global optimal solutions of non-cyclic mappings in metric spaces," *J. Optim. Theory Appl.*, vol. 153, pp. 298-305, 2012.

[10] S. Barootkoob, H. Lakzian, and Z. D. Mitrovic, "The best proximity points for weak MT-cyclic Reich type contractions," *J. Math. Ext.*, vol. 16, no. 5, pp. 1-21, 2016.

[11] M. A. Petric, "Best proximity point theorems for weak cyclic Kannan contractions," *Filomat*, vol. 25, no. 1, pp. 145-154, 2011.

[12] A. Fernandez-Leon and M. Gabeleh, "Best proximity pair theorems for non-cyclic mappings in Banach and metric spaces," *Fixed Point Theory*, vol. 17, no. 1, pp. 63-84, 2016.