Wasit Journal for Pure Science

Journal Homepage: https://wjps.uowasit.edu.iq/index.php/wjps/index
e-ISSN: 2790-5241 p-ISSN: 2790-5233

Proximity Relative Homotopy and Proximity Weak Equivalence

Ammar Qasim Naser[1]*, Daher Waly AL – Baydli[2]

[1,2]Department of Mathematics, College of Education for Pure Science, Wasit University, IRAQ

*Corresponding Author: Ammar Qasim Naser

DOI: https://doi.org/10.31185/wjps.440
Received 02 June 2024; Accepted 18 July 2024; Available online 30 September 2024

ABSTRACT: In this paper, we introduce and study proximity path covering property (PPCP) and let $q:(E,\delta)\longrightarrow(B,\delta_1)$ be a proximity path covering property if $n_t(0)=q(e)$, for every $e\in E$ such that $n_t^*(0)=e$ and $q\circ n_t^*=n_t$, We define the concepts of naw proximity relative homotopy (PRH) and mixed proximity relative homotopy (MPRH) and q_i has Mixed proximity relative homotopy covering property (M.R.H.P) Regarding all CW-pairs (X,A_i) , and study proximity weak equivalence (PWE), some theories and properties.

Keywords: proximity path covering property (PPCP), Proximity Relative Homotopy (PRH), Mixed Proximity Relative Homotopy (MPRH), Proximity Weak Equivalence (PWE).

1. INTRODUCTION AND PRELIMINARIES

covering space theory has led to the development of several useful generalizations. Hurewicz fibrations by short (HF) with fully path-disconnected fibers Spanier's classic method Abstractly, fibrations are defined concerning arbitrary spaces in terms of homotopy covering the standard classification of covering projections cannot be extended to classify these mappings up to homeomorphism, deck transformation. True, there exist bijective functions that are not homomorphisms from metric spaces E to B [8]. Long-standing holes in broad topological group theory have been filled in using semi-coverings and topologized fundamental groups. [4] Not everything was accomplished with topological techniques alone. Similar applications will arise from stronger links between topological group techniques and covering theoretic approaches. All inverse limits of covering projections are among the numerous appealing internal characteristics of Hurewicz fibrations (HF) with entirely path-disconnected fibers [6]. A mixed covering function for $((E_i,\delta_i),f_i,(Y,\delta_3),\alpha\,)$ isproximity continuous map $\lambda_i(e,n)(0)=e$ and $f\circ\lambda_i(e,n)(t)=n(t)$ for every $(e,n)\in(\Omega_f,\delta_5)$ and $t\in I$. Furthermore, a proximity equivalence relation produced by explicitly inverting bijective proximity weak homotopy equivalences is referred to as a "proximity weak equivalence."

Definition (1.1) [11]. Assume that $q:E\rightarrow B$ is a topological space continuous map and $n:I\rightarrow B$ a path Then $I=[0,1]$. A cover of n with a starting point $e\in q^{-1}(\,n(0))$ is a continuous map $n^*:I\rightarrow E$ then $n^*(0)=e$ and $n=q\circ n^*$.

$$\begin{tikzcd} & E \arrow[d, "q"] \\ I \arrow[ur, "n^*"] \arrow[r, "n"'] & B \end{tikzcd}$$

FIGURE 1: path covering property

*Corresponding author: ammar.q@uowasit.edu.iq

1) q is path covering property if a cover n^* exists for every $n: I \rightarrow B$ and $n \in q^{-1}(n(0))$ We do not require n^* to be unique if this holds then every path $n: I \rightarrow B$ also cover
2) q is covering (or local path covering) property if the cover n^*exists over some subinterval $[0, \epsilon]$, then $\epsilon > 0$ may depend on $e \in q^{-1}(n(0))$.
3) q are two-point path covering property if given $n: I \rightarrow B$ and $e_i \in q^{-1}(n(i))$ for $i = [0,1]$, there is a covering n^*such that $n^*(0) = e_0$ and $n^*(1) = e_1$ The basic example is a fiber bundle $q: E \rightarrow B$ with path-connected fiber

Furthermore, take note of the fact that q is surjective if B is path-connected, $E \neq \emptyset$, and $q: E \rightarrow B$ satisfies the path-covering feature. Covering constant pathways demonstrate that all its fibers are path-connect if q possesses the 2-point path covering characteristic.

Definition (1.2) [3]. Let the continuous path-covering property (C.P.C.P) be present in a map $q: E \longrightarrow B$ either each $e \in E$, the function $q: E \longrightarrow B$ is a homeomorphism.

Definition (1.3): [12] Let B be the base space There is a space E in a fiberwise space over B together with a map $q: E \longrightarrow B$, referred to be a projection.

Definition (1.4): [7] Assume that A and Y are fiberwise spaces over B, respectively, with projections p and q. A fibrewise map $\phi: A \longrightarrow Y$ such that $q \circ \phi = p$, such that $\phi A_b \subseteq Y_b$ for every point b of B Where $A_b = p^{-1}b$ and $Y_B = q^{-1}b$.

Definition (1.5):[10] Let B be a topological space, any topology on A for which the projection q is continuous is the fibrewise topology on a fibrewise set A over B.

Definition (1.6):[13] Let a map $f: Z_1 \longrightarrow Z_2$ is a called homeomorphism and Z_1, Z_2 be topological spaces if it is continuous and has an inverse $f^{-1}: Z_2 \longrightarrow Z_1$ which is

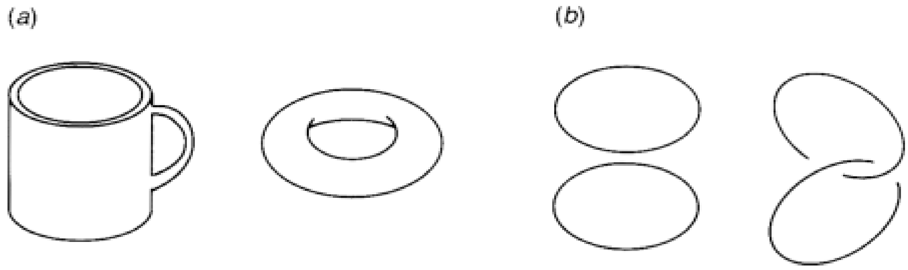


Figure 2: (a) A doughnut and a coffee cup are homeomorphic. (b) the separated rings are homeomorphic to the linked rings.

Definition (1.7): [1] A CW -space is referred to be a regular if all its attaching maps are homeomorphisms.

Definition (1.8): [2] Consider e^n to be a closed n-disk and S^{n-1} its boundary. In the given space A and a map $p: S^{n-1} \rightarrow A$, the space B obtained from A by attaching n-cell e^n via f is by the definition

$$A \cup_P e^n = A \sqcup e^n / f(m) \sim m, \quad \text{for all } m \in S^{n-1}$$

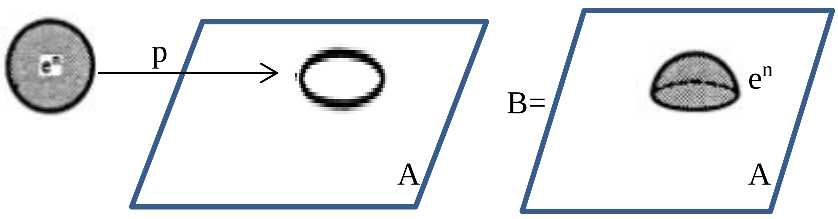


Figure 3: Attaching

Definition (1.9): [5] Let m-cell and $m \geq 0$ is a homeomorphic topological space to the open m-disk int (D^m)

Definition (1.10): [9] Assume that $q: E \longrightarrow B$ is a projection and u_t^* can be covering of u_t and $u_t: A \longrightarrow B$ be a homotopy, we say that u_t^* is regular with u_t, if and only if $u_t(a_0)$ and $u_t^*(a_0)$ are constant, as a function of t.

$$\begin{tikzcd} & E \arrow[d, "q"] \\ A \arrow[ur, "u_t^*"] \arrow[r, "u_t"'] & B \end{tikzcd}$$

Figure 4: Regular

2. PROXIMITY PATH COVERING PROPERTY

Proximity path covering property is introduced and studied

Definition (2.1): Let a map $q: (E, \delta) \longrightarrow (B, \delta_1)$ be referred to as having a proximity path covering property (PPCP by short) if, each path $n_t: (A, \delta_2) \longrightarrow (B, \delta_1)$ and there exists a path $n_t^*: (A, \delta_2) \longrightarrow (E, \delta)$

$$\begin{tikzcd} & E \arrow[d, "q"] \\ A \arrow[ur, "n_t^*"] \arrow[r, "n_t"'] & B \end{tikzcd}$$

Figure 5: Proximity path covering property

If $n_t(0) = q(e)$, for every $e \in E$ such that $n_t^*(0) = e$ and $q \circ n_t^* = n_t$ and that n_t^* is always dependent on e and n_t.
To get an exact definition, let (Ω_p, δ_3) indicate the product space's subspace
$(E, \delta) \times (B^I, \delta_1^*)$ defined by
$(\Omega_p, \delta_3) = \{(e, f) \in (E, \delta) \times (B^I, \delta_1^*) | q(e) = f(0)\}$
Define a map

$$T: (E^I, \delta^*) \longrightarrow (\Omega_p, \delta_3)$$

Through taking $T(n) = (n(0), q(n))$ for every $n: (I, \delta_2) \longrightarrow (E, \delta)$ in (E^I, δ^*) Then $q: (E, \delta) \longrightarrow (B, \delta_1)$ is said to have a proximity path covering property (PPCP) if a map is $\lambda: (\Omega_p, \delta_3) \longrightarrow (E^I, \delta^*)$ such that $T\lambda$ is mapped the identity on (Ω_p, δ_3). It is that a map $q: (E, \delta) \longrightarrow (B, \delta_1)$ has the proximity continuous path covering the property (PCPCP) if it has the absolute covering homotopy property (ACHP). The map λ is referred to as the proximity covering function in the above definition for $q: (E, \delta) \longrightarrow (B, \delta_1)$. If λ covers constant paths, then it's referred to as proximity regular covering function for $q: (E, \delta) \longrightarrow (B, \delta_1)$ and the triple $\xi = ((E, \delta), q, (B, \delta_1))$ it's referred to as proximity regular Serre fibre space.

Definition (2.2): Let $((E_i, \delta_i), f_i, (Y, \delta_3), \alpha)$ Mixed proximity fibre space (by short M.P. F. S) and $i = 1, 2$ then
$(Y^I, \delta_3^*) = \{n: (I, \delta_4) \longrightarrow (Y, \delta_3)\}$,$(\Omega_{f_i}, \delta_5) \subseteq (E_i, \delta_i) \times (Y^I, \delta_3^*)$ be the subspace,
$(\Omega_f, \delta_5) = \{(e, n) \in (E_i, \delta_i) \times (Y^I, \delta_3^*) | f_i(e) = n(0)\}$. Mixed proximity covering function for $((E_i, \delta_i), f_i, (Y, \delta_3), \alpha)$ is a proximity continuous map such that $\lambda_i(e, n)(0) = e$ and $f \circ \lambda_i(e, n)(t) = n(t)$ for every $(e, n) \in (\Omega_f, \delta_5)$ and $t \in I$ thus
$\Omega_{f_i} = \{\Omega_{f_i}, \Omega_{f_2}\}$, $\lambda_i = \{\lambda_1, \lambda_2\}$ where $\lambda_1: (\Omega_{f_1}, \delta_5^*) \longrightarrow (E_1^I, \delta_1^*)$ and $\lambda_2: (\Omega_{f_2}, \delta_5^{**}) \longrightarrow (E_2^I, \delta_2^*)$ defind as
$\lambda_1(e_1, n)(0) = e_1$, $f_1 \circ \lambda_1(e_1, n)(t) = n(t)$ and
$\lambda_2(e_2, n)(0) = e_2$, $f_2 \circ \lambda_2(e_2, n)(t) = n(t)$
Thus, a mixed proximity covering function therefore associates

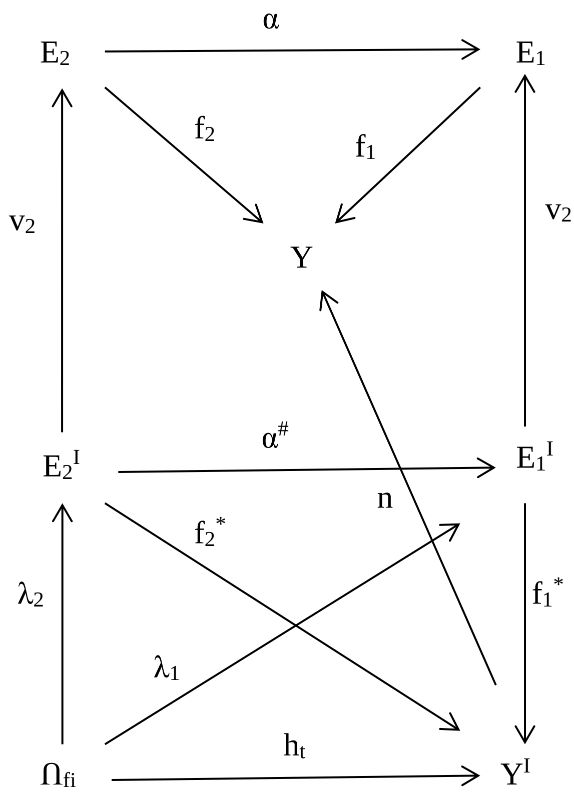


Figure 6: Mixed proximity covering function

With every $e \in E$ and every path n in Y starting at $f_i(e)$ a path $\lambda_1(e_1, n)$ in E_1 and $\lambda_2(e_2, n)$ in E_2, starting at e_1 and e_2, and is Mixed proximity cover of n since that Closed topology used for E^I the proximity continuity of λ is equivalent to that of the associated $\lambda_i: (\Omega_f, \delta_5) \times (I, \delta_4) \longrightarrow (E_i, \delta_i)$.

Example (2.3): An Example of a Mixed proximity Serre fibre space is the $((E_i, \delta_i), f_i, (Y, \delta_3), \alpha)$, $(Y^I, \delta_3^*) = \{n: (I, \delta_4) \longrightarrow (Y, \delta_3)\}$ and $f_1(n) = n(1)$, $f_2(ń) = ń(1)$, where $n, ń: (I, \delta_4) \longrightarrow (Y, \delta_3)$. Proximity covering functions $\lambda_1: (\Omega_{f_1}, \delta_5^*) \longrightarrow (E_1^I, \delta_1^*)$ for $f_1: (E_1, \delta_1) \longrightarrow (Y, \delta_3)$ and $\lambda_2: (\Omega_{f_2}, \delta_5^{**}) \longrightarrow (E_2^I, \delta_2^*)$ for $f_2: (E_2, \delta_2) \longrightarrow (Y, \delta_3)$, are defined as follows:

$$\lambda_1(\sigma, w)(t)(s) = \begin{cases} \sigma\left(\dfrac{4s}{2+t}\right) \; if \; 0 \le s \le \dfrac{2+t}{4} \\ w(4s - t - 2) \;\; if \;\; \dfrac{2+t}{4} \le s \le \dfrac{3+t}{4} \end{cases}$$

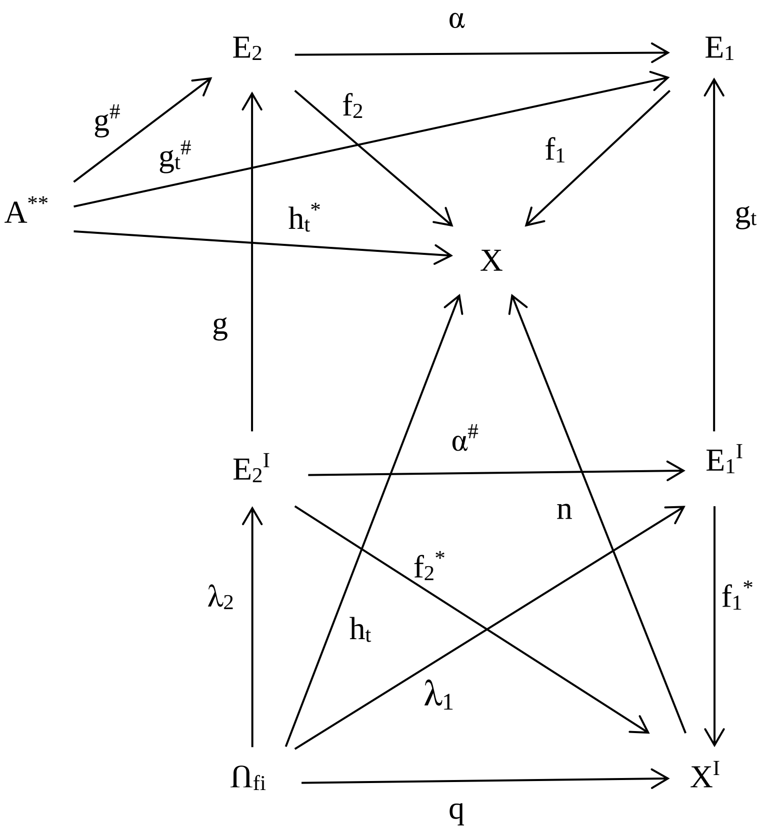
$$\lambda_1(\acute{\sigma}, \acute{w})(t)(s) = \begin{cases} \acute{\sigma}(4s - t - 3) \quad if \quad \dfrac{3+t}{4} \le s \le \dfrac{4+t}{4} \\ \acute{w}\left(\dfrac{4s - t - 4}{-t}\right) \qquad if \;\; \dfrac{4+t}{4} \le s \le 1 \end{cases}$$

Keep in mind that these λ_1, λ_2 are not regular. Because λ_1 and λ_2 are not constant.

Theorem (2.4): The Mixed proximity fibre space $((E_i, \delta_i), f_i, (Y, \delta_3), \alpha)$ is Mixed proximity Serre fibration if and only if f_i has a mixed proximity covering function.

Proof: If f_i is Mixed proximity Serre fibration, Let $(Y, \delta_3) = (\Omega_{f_i}, \delta_5)$, Let g☹$\Omega_{f_2}, \delta_5^{**}) \longrightarrow (E_2, \delta_2)$ and h_t☹$\Omega_{f_i}, \delta_5) \longrightarrow (X, \delta_6)$ be defined by $g(e_2, n)(0) = e_2$ and $h_t(e, n) = n(t)$, Then $h_0(e, n) = n(0) = f_2(e_2) = f_2 \circ g(e_2, n)$ since f_i is Mixed proximity Serre fibration. Then there exist g_t☹$\Omega_{f_1}, \delta_5^*) \longrightarrow (E_1, \delta_1)$ such that $g_0(e_1, n) = \alpha \circ g(e_2, n)(0)$,

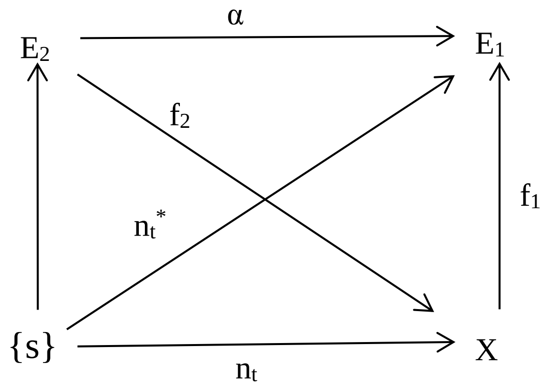
$f_1 \circ g_t = h_t$ defined as a mixed proximity covering function $\lambda_i(e,n)(t) = g_t(e,n)$
i.e:
$\lambda_1(e_1,n)(t) = g_t(e_1,n) \Longrightarrow \lambda_1(e_1,n)(0) = g_0(e_1,n) = e_1$
$f_1 \circ \lambda_1(e_1,n)(t) = f_1 \circ g_t(e_1,n) \Longrightarrow f_1 \circ \lambda_1(e_1,n) = h_t(e_1,n) = n(t)$



Conversely: If f_i has a mixed proximity covering function
Let $g^{\#}: (Y, \delta_3) \longrightarrow (E_2, \delta_2)$ and $h_t^*: (Y, \delta_3) \longrightarrow (X, \delta_6)$ such that $f_2 \circ g^{\#} = h_0^*$ consider $u_y: (I, \delta_4) \longrightarrow (X, \delta_6)$ such that $n_y(t) = h_t^*(y)$ and define $g_t^{\#}: (Y, \delta_3) \longrightarrow (E_1, \delta_1)$ as $g_t^{\#}(e_1) = \lambda_1[\alpha \circ g^{\#}(e_1), n_y](t)$ Thus $g_0^{\#} = \alpha \circ g^{\#}$ and $f_1 \circ g_t^{\#} = h_t^*$
Hence g_t is the Mixed proximity covering the homotopy property (MPCHP) of h_t concerning A^{**} since A^{**} is CW-complex
Therefore f_i is a Mixed proximity Serre fibration.

Corollary (2.5): Let $((E_i, \delta_i), f_i, (Y, \delta_3), \alpha)$ be Mixed proximity Serre fibration, then any u path in X such that $n_0 = f_2(e_2) = f_1 \circ \alpha(e_2)$ can be Mixed proximity covering the path in (E_1, δ_1).

Proof:



A path $u:(I,\delta_4)\longrightarrow(X,\delta_6)$ can be regarded as a homotopy $n_t:(\{s\},\delta)\longrightarrow(X,\delta_6)$ where $(\{s\},\delta)$ is a proximity point space, and a point $e_2\in(E_2,\delta_2)$ such that $f_2(e_2)=n_0$, corresponds to map $g:(\{s\},\delta)\longrightarrow(E_2,\delta_2)$ such that $(f_2\circ g)\{s\}=n_0(\{s\})$, and since f_i is a Mixed proximity Serre fibration there exists a path n_t^* in (E_1,δ_1) such that $n_0^*=\alpha(e_2)$ and $f_1\circ n_t^*=n_t$ where $n_t^*:(\{s\},\delta)\longrightarrow(E_1,\delta_1)$ is a Mixed proximity covering of $n_t:(\{s\},\delta)\longrightarrow(X,\delta_6)$.

3. PROXIMITY RELATIVE HOMOTOPY

In this section, we introduce and examine the new concept of proximity relative homotopy by short (P. R. H) and some applications.

Definition (3.1): *Let $f:(X,\delta_1)\longrightarrow(Y,\delta_3)$ and $g:(X,\delta_1)\longrightarrow(Y,\delta_3)$ be a proximity continuous function and $(A,\delta)\subset(X,\delta_1)$. Then f,g is said to be proximity relative homotopy to (A,δ), If there is proximity homotopy G between f,g such that $G(x,t)$ is independent of t for $x\in(A,\delta)$ or $F(X,t)=f(x)$ for everyone $x\in(A,\delta)$ and for everyone $t\in(I,\delta_2)$*

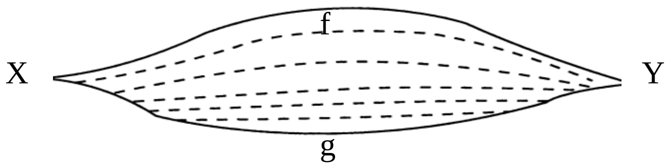


Figure 7: Proximity relative homotopy

In this case, it is clear that $f(x)=g(x)$ for everyone $x\in(A,\delta)$ The homotopy G is called proximity homotopy relative to A and is written as $f\simeq_\delta g$ (relative A) and when G is to be referred, $G:f\simeq_\delta g$ *(relative A)* Then $G(x,0)=f(x)$ and $G(x,1)=g(x)$ for each $x\in(X,\delta_1)$ and $F(y,t)=f(y)=g(y)$ for every $y\in(A,\delta)$ and for every $t\in(I,\delta_2)$, If $(A,\delta)=\varphi$, then proximity homotopy relative to A becomes only proximity homotopic. Thus ordinary proximity homotopy is a particular case of proximity-relative homotopy.

Theorem (3.2): The $\simeq_\delta$ is a proximity equivalence relation.

Proof

- **Reflexivity**. The proximity homotopy $F(x,t)=f(x)$ gives that $f\simeq f$
- **Symmetry.** If $F:f\simeq_\delta g$ then $G:g\simeq_\delta f$ because $G(x,t)=F(x,1-t)$ is proximity homotopy between g and f.
- **Transitivity**. If $F:f\simeq_\delta g$ and $G:g\simeq_\delta h$, Let H be given by

$$H(x,t)=\begin{cases}F(x,2t) & , \quad 0\le t\le\frac{1}{2}\\ G(x,2t-1) & , \quad \frac{1}{2}\le t\le 1\end{cases}$$

Then H is proximity continuous and also $H:f\simeq_\delta h$.

Theorem (3.3): The relation proximity homotopic relative to (A,δ) is a proximity equivalence relation.

Proof

- **Reflexivity**. Let $f:(X,\delta_1)\longrightarrow(Y,\delta_3)$ and $F(x,t)=f(x)$. Then, F is a proximity continuous $F(x,0)=F(x,1)=f(x)$ and $F(x,t)=f(x)$ for every $x\in(X,\delta_1)$, So $f\simeq_\delta f$ where (A,δ) may be any subset of X.

- **Symmetry**. Suppose $f\simeq_\delta g$ and $f:(X,\delta_1)\longrightarrow(Y,\delta_3)$, $g:(X,\delta_1)\longrightarrow(Y,\delta_3)$ Then there exists a proximity continuous mapping $F:(X,\delta_1)\times(I,\delta_2)\longrightarrow(Y,\delta_3)$ such that $F(x,0)=f(x)$ and $F(x,1)=g(x)$ for all $x\in(X,\delta_1)$
Further $F(y,t)=f(y)=g(y)$ if $y\in(A,\delta)$. Let $G:(X,\delta_1)\times(I,\delta_2)\longrightarrow(Y,\delta_3)$ be defined by $G(x,t)=F(x,1-t)$. Then G is proximity continuous and $G(x,0)=g(x)$ and $G(x,1)=f(x)$ Also $G(y,t)=f(y)=g(y)$ for all $y\in(A,\delta)$ so $g\simeq_\delta f$.

- **Transitivity**. Let $f\simeq_\delta g$ and $g\simeq_\delta h$ and $f:(X,\delta_1)\longrightarrow(Y,\delta_3)$, $g:(X,\delta_1)\longrightarrow(Y,\delta_3)$, $h:(Y,\delta_3)\longrightarrow(Z,\delta_4)$ there exists a proximity continuous mapping $F,G:(X,\delta_1)\times(I,\delta_2)\longrightarrow(Y,\delta_3)$ then $F(x,0)=f(x)$ and $F(x,1)=g(x)$ for all $x\in(X,\delta_1)$ Further $F(y,t)=f(y)=g(y)$ if $y\in(A,\delta)$. And $G(x,0)=g(x)$ and $G(x,1)=h(x)$, $G(y,t)=g(y)=h(y)$ for $y\in(A,\delta)$
Let a transformation $H:(X,\delta_1)\times(I,\delta_2)\longrightarrow(Y,\delta_3)$ be defined by

$$H(x,t) = F(x,2t), 0 \le t \le \frac{1}{2}$$
$$H(x,t) = G(x,2t-1), \frac{1}{2} \le t \le 1$$

Then H is a proximity continuous such that $H(x,0) = f(x)$ and $H(x,1) = h(x)$ such that $H(y,t) = f(y) = h(y)$ for all $y \in (A,\delta)$ This shows that $f \simeq_\delta h$.

4. MIXED PROXIMITY RELATIVE HOMOTOPY

Definition (4.1): Let $E_1 \subset (X_1,\delta_1)$, $E_2 \subset (X_2,\delta_2)$ and $f_1, g_1: (X_1,\delta_1) \longrightarrow (Y,\delta_3)$, $f_2, g_2: (X_2,\delta_2) \longrightarrow (Y,\delta_3)$ be a proximity continuous function. If $H: (X_1,\delta_1) \times (I,\delta) \longrightarrow (Y,\delta_3)$ and $G: (X_2,\delta_2) \times (I,\delta) \longrightarrow (Y,\delta_3)$ are proximity continuous functions such that $H(x_1,0) = f_1(x_1)$, $H(x_1,1) = g_1(x_1)$ and $G(x_2,0) = f_2(x_2)$, $G(x_2,1) = g_2(x_2)$ for everyone $x_1 \in X_1$, $x_2 \in X_2$ and $H(e_1,t) = f_1(e_1) = g_1(e_1)$, $G(e_2,t) = f_2(e_2) = g_2(e_2)$ for everyone $e_1 \in E_1$, $e_2 \in E_2$ and for everyone $t \in I$, then H, G is called Mixed proximity relative homotopy, and f_i is said to be a homotopy to g_i relative to E_i, where $i = 1,2$

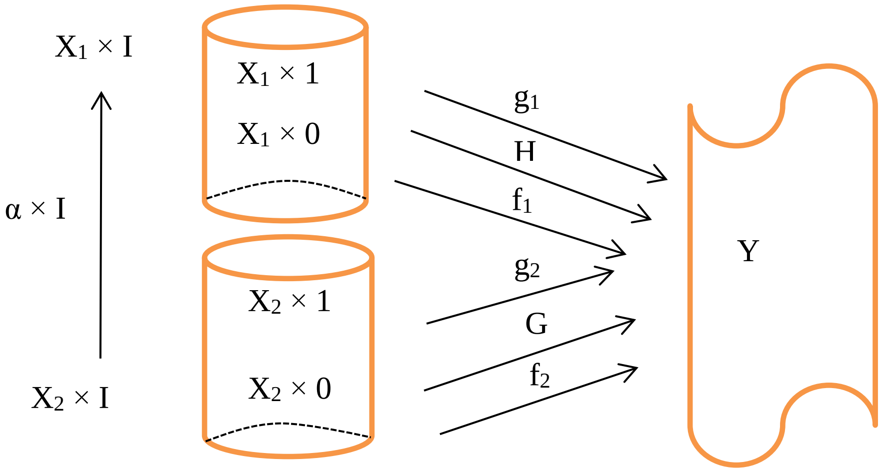


Figure 8: Mixed proximity relative homotopy

Theorem (4.2): Let $q_i: (E_i,\delta_i) \longrightarrow (B,\delta_3)$ be proximity function where $i = 1, 2$, as a result, these

1. q_i Mixed proximity Serre fibration.
2. Regarding all m- discs D^m, q_i has the proximity homotopy covering property
3. Regarding all m- discs (D^m, S^{m-1}), q_i has Mixed proximity relative homotopy covering property
4. q_i has Mixed proximity relative homotopy covering property Regarding all CW-pairs (X, A_i)

Proof: 1) $\Longrightarrow$ 2): The define lead this direct

2) $\Longrightarrow$ 3): that the pair $(D_i^m \times I, \delta_4), (D_i^m \times 0 \cup S_i^{m-1} \times I, \delta_5))$ is not hard to prove is homeomorphic Regarding the pair $((D_i^m, \delta_4), D_i^m \times 0)$ This simply leads to the desired inference.

3) $\Longrightarrow$ 4): Assume a proximity cover H_i is already provided on $A_i \times I$ We expand H_i over $X_i^m \times I \cup A_i \times I$ through induction on m in the inductive stage, we reduce to creating a proximity homotopy in a diagram of the following form:

$$\begin{tikzcd} D_i^m \times 0 \cup S_i^{m-1} \times I \arrow[r] \arrow[d] & E_i \arrow[d, "q_i"] \\ {} \arrow[ur, dashed, "H_i"] & {} \end{tikzcd}$$

$$D_i^m \times I \xrightarrow{\quad h_t \quad} B$$

It is assumed that such a proximity homotopy exists (3).

4) $\Longrightarrow$ 1): This is immediate and requires taking $A_i = \emptyset$ Considering point B to be in (4) we have proved that

Remark 1: Assume $q_i: E_i \longrightarrow B$ be a map. If q_i satisfies the proximity homotopy covering property concerning all CW-complex Having maximum dimension n, they consider it to be a Mixed proximity Serre fibration.

5. PROXIMITY WEAK EQUIVALENCE

Definition (5.1):[1] If a map $f: X \longrightarrow Y$ induces isomorphisms, it is referred to as a weak homotopy equivalence. $\pi_m(X, x_0) \longrightarrow \pi_m(Y, f(x_0))$ for everyone $m \geq 0$ and every basepoint option x_0

Definition (5.2):[1] Let (E, q, B) be a fiber structure Let X be any space and let $g: X \longrightarrow B$ be any continuous map into base B. Let $Eg \subset E \times X$ be the subspace $Eg = \{(e, x) \in E \times X / p(e) = g(x)$ cartesian produet, Let $q^*: Eg \longrightarrow X$ be the projection $q^*(e, x) = x$. Then p^* is called the pullback of q by g

Letting $\pi: Eg \longrightarrow E$ be the projection $\pi(e, x) = e$, it is immediate from the definitions that the diagram is commutative such that $q \circ \pi = g \circ q^*$

$$\begin{tikzcd}
Eg \arrow[r, "\pi"] \arrow[d, "q^*"'] & E \arrow[d, "q"] \\
X \arrow[r, "g"'] & B
\end{tikzcd}$$

Figure 9: Pullback

Definition (5.3): Let $\mathrm{q}_1: (\mathrm{E}_1, \delta_1) \longrightarrow (\mathrm{B}, \delta)$, $\mathrm{q}_2: (\mathrm{E}_2, \delta_2) \longrightarrow (\mathrm{B}, \delta)$ be two maps of the proximity continuous path covering property

1) If present a homeomorphism $Z: (E_1, \delta_1) \longrightarrow (E_2, \delta_2)$ we say q_1, q_2 are proximity equivalent such that $q_2 \circ Z = q_1$ Therefore, h is referred to as a proximity equivalency.
2) Proximity weak equivalence between q_1, q_2 is a triple (q_3, f_1, f_2) where $q_3: (E_3, \delta_4) \longrightarrow (B, \delta)$ has the continuous proximity path-covering property and $f_1: (E_3, \delta_4) \longrightarrow (E_1, \delta_1)$, $f_2: (E_3, \delta_4) \longrightarrow (E_2, \delta_2)$ by proximity weak homotopy equivalences then $q_i \circ f_i = q_3$ where $i = 1,2$. If proximity weak equivalence exists between $q_1$and q_2 by short $q_1 \sim_s q_2$

$$\begin{tikzcd}
E_1 \arrow[rr, bend left, "Z"] \arrow[dr, "q_1"'] & E_3 \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "q_3"] & E_2 \arrow[dl, "q_2"] \\
 & {} &
\end{tikzcd}$$

B

Figure 10: Proximity weak equivalencies

3) We state q_1, q_2 are if a finite chain of proximity weak equivalencies exists, then the pair is proximity-weak equivalence $q_1 = c_1 \sim_s c_2 \sim_s \cdots \sim_s c_m = q_2$. If a chain exists, we say q_1 and q_2 are a proximity-weak equivalence.

Remark 2: Be aware that if we possess $q_2 \circ Z = q_1$ then maps $\mathrm{q_i}: (\mathrm{E_i}, \delta_i) \longrightarrow (\mathrm{B}, \delta)$ and $i = 1{,}2$ with the (PCPCP), such that f has the (PCPCP) and f also has to be surjective.

Theorem (5.4): Let $\mathrm{q_i}: (\mathrm{E_i}, \delta_\mathrm{i}) \longrightarrow (\mathrm{A}, \delta)$ be a proximity function where $\mathrm{i} = 1{,}2$ and suppose a proximity path connected. Then if $\mathrm{q_i}$ is a proximity Serre fibration, any two proximate fibers of $\mathrm{q_i}$ are proximity weakly equivalent.

Proof: Let the pulling back over a proximity path $\alpha: (\mathrm{I}, \delta_3) \longrightarrow (\mathrm{A}, \delta)$ where $\mathrm{i} = 1{,}2$ we reduce to case (A, δ) contractible Then the extensive sequence of exact a proximity homotopy demonstrates that for each proximity fibre $\mathrm{q_i^{-1} b_i}$, the inclusion $\mathrm{q_1^{-1} b_1} \subset (\mathrm{E_1}, \delta_1)$ and $\mathrm{qb_2} \subset (\mathrm{E_2}, \delta_2)$ are proximity weakly equivalent. Hence any two proximity fibres are proximity weakly equivalent.

CONFLICTS OF INTEREST

In this Paper we have some results as shown below:

1. Mixed proximity Serre fibre space is not regular because not constant.
2. The Mixed proximity fibre space is Mixed proximity Serre fibration if and only if f_i has a Mixed proximity covering function.
3. If q_i Mixed proximity Serre fibration thenp_i has Mixed proximity relative homotopy covering property Regarding all CW-pairs (X, A_i)
4. If $\mathrm{q_i}$ is a proximity Serre fibration, any two proximity fibers of $\mathrm{q_i}$ are proximity weakly equivalent.

REFERENCES

[1] Allen Hatcher. Algebraic topology, Cambridge University, Press, New York,NY,USA. 2002.

[2] Bott, Raoul, and Loring W. Tu. *Differential forms in algebraic topology*. Vol. 82. New York: Springer, 1982.

[3] Brazas, Jeremy, and Atish Mitra. "On maps with continuous path lifting." *arXiv preprint arXiv:2006.03667* (2020).

[4] Brazas, Jeremy. "Open subgroups of free topological groups." *arXiv preprint arXiv:1209.5486* (2012).

[5] Bui Anh Tuan, Discrete Vector Fields and the Cohomology of Certain Arithmetic and Crystallographic Groups, PhD thesis, National University of Ireland, Galway, 2015.

[6] Conner, Gregory R., Wolfgang Herfort, and Petar Pavešić. "Geometry of compact lifting spaces. " *Monatshefte für Mathematik* 193 (2020): 591-603.

[7] Crabb, M. C., & James, I. M. *Fibrewise homotopy theory*. Springer Science & Business Media. (2012).

[8] Greenberg, Marvin J. Algebraic topology: a first course. CRC Press, 2018.

[9] Habeeb, Zahraa Yassin, and Daher Al Baydli. "Mixed Serre Fibration." *Wasit Journal for Pure Sciences* vo1.No2 (2022): 50-60.

[10] James, Ioan Mackenzie. "Fibrewise topology." *(No Title)* (1989).

[11] Kollár, János. "Fundamental groups and path lifting for algebraic varieties." Singularities and Their Interaction with Geometry and Low Dimensional Topology: In Honor of András Némethi. Cham: Springer International Publishing. 97-116. 2021.

[12] Michael Charles Crabb, Loan Mackenzie James Fibrewise homotopy theory Spring,1998.

[13] Nakahara,Mikio. Geometry, topology and physics. CRC Press, 2018.