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Fuzzy Relative Homotopy and Fuzzy Weak Equivalence with Some Results

Azhar Kareem Jabr[1]*, Daher W. Al Baydli[2]

[1,2]Department, mathematics, College of education for pure science, University of Wasit, IRAQ

*Corresponding Author: Azhar Kareem Jabr

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ABSTRACT: In this paper, we first introduce my concept fuzzy homotopic and fuzzy homotopic relative and we proved that the relation fuzzy homotopic relative is a fuzzy equivalence relation. Secondly, we introduce the concepts fuzzy homotopy equivalence, fuzzy fundamental group and fuzzy weak homotopy equivalence. We have proven some important theorems.

Keywords: Fuzzy homotopic, Fuzzy homotopic relative, Fuzzy weak homotopy equivalence, Fuzzy fiber-preserving, Fuzzy fiber homotopy equivalence and Fuzzy pullback fibration

1. INTRODUCTION

A fuzzy set is a set whose elements belong to different degrees (in non-fuzzy sets, an element may or may not belong to the set. There is nothing in between these two possibilities). Fuzzy set theory was presented by the Azerbaijani L.A. Zadeh [1]and the German Theter Klaus in 1965 as an expansion of the well-known traditional set. The fuzzy homotopy theory is a field of mathematics that studies the properties of fuzzy topological spaces and their fuzzy continuous transformations. It is a fundamental concept in algebraic fuzzy topology, a branch of mathematics that uses algebraic to study fuzzy topological spaces. Submitted Allen Hatcher [1]. Also B. Z. Khalaf and D. Al Baydli,[2]. A fuzzy function between two fuzzy topological spaces $p: \mu_E \to \mu_B$ called the fuzzy projection and is said μ_E total space and μ_B be bace space. the fuzzy projection is called a fuzzy (Hurewicz) fibration if it has the fuzzy homotopy lifting property with resped to for class R of fuzzy spaces. The fuzzy fibre over b is the fuzzy subset $F_b = p^{-1}(b)$ of μ_E. We will denoted the interval $I = [0,1]$.

Definition (1.1): [2][4][5] Let Y be the universal set, A fuzzy set A in Y is a function $\mu_A : Y \to [0,1]$ the ordered pairs;

$$A = \{(y, \mu_A(y)) : y \in Y\}$$

is called the membership function,
and all $y \in Y$, the value of $\mu_A(y)$ is called the grade of membership of y in A.

Definition (1.2): [2][4] Let (Y, T) be topological space and $F(Y)$ be a family of fuzzy sets in Y .A fuzzy set $\mu_A \in F(Y)$ and $T^* \in F(Y)$. Let $[T^* = \{ H: U \to I : U \in Y\} , T^*_\delta = \{ H_\delta : H \in T^* , \delta \in I\}]$. The pair (μ_E, T^*) is called fuzzy topological space if and only if achieve the following conditions:

1) $\Phi , X \in T^*$
2) If $A, B \in T^*$ then $A \cap B \in T^*$
3) If $\{U_i\}_{i\in \wedge} \subset T^*$ then $\cup_{i\in\wedge} U_i \in T^*$

Definition (1.3): [8] let (μ_X , T^*) be fuzzy topological space and let $a , b \in \mu_X$ be any two distinct fuzzy point. We called (μ_X , T^*) fuzzy Haudorff space if and only if there exist disjoint $A , B \in T^*$ such that $a \in A$ and $b \in B$

2. FUZZY RELATIVE HOMOTOPY

Definition (2.1): [2] let μ_X and μ_Y be two fuzzy topological spaces, the fuzzy continuous maps $f: \mu_X \to \mu_Y$ and $g: \mu_X \to \mu_Y$ are said to be fuzzy homotopic if there exists a fuzzy continuous map $H: \mu_X \times I \to \mu_Y$ such that

$$H(x,0) = f(x) \quad \text{and } H(x,1) = g(x) \qquad \forall x \in \mu_X$$

If the fuzzy maps f and g are fuzzy homotopic then we denote this fact by writing $f \simeq g$. The fuzzy map H with the properties stated above is referred to as a fuzzy homotopy between f and g.

Definition (2.2):[1] let μ_X , μ_Y be two fuzzy topological spaces and let $f: \mu_X \to \mu_Y$, $g: \mu_X \to \mu_Y$ be fuzzy continuous map, let μ_A be fuzzy subset of μ_X . we say that f and g are fuzzy homotopic relative of μ_A (denoted by $f \simeq g($ rel $\mu_A))$ if and only if there exist a (fuzzy continuous) homotopy $H: \mu_X \times [0,1] \to \mu_Y$ between f and g such that

1-$H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in \mu_X$, $s \in [0,1]$

2- $H(a,s) = f(a) = g(a)$ for all $a \in \mu_A$, $s \in [0,1]$

If $\mu_A = \phi$ we will simply say that f fuzzy homotopic to g, and we will denote it by $f \simeq g$

Remark (2.3): Note that $f \simeq g$ (rel μ_A) implies that $f(x) = g(x)$, $\forall x \in \mu_A$.

Theorem (2.4): Let μ_X , μ_Y and μ_Z be fuzzy topological spaces, and $\mu_A \subseteq \mu_X$ be a fuzzy subset. Consider the relation on the fuzzy set $C(\mu_X, \mu_Y)$ of fuzzy continuous maps from μ_X to μ_Y given by $f \simeq g$ (rel μ_A) defined before for $f, g: \mu_X \to \mu_Y$. This is a fuzzy equivalence relation

Proof: 1- **the relation is reflexive**

since for any fuzzy function $f: \mu_X \to \mu_Y$,define $H: \mu_X \times [0,1] \to \mu_Y$ by $H(x,s) = f(x)$. This yields a fuzzy homotopy between f and itself. Also, trivially, if $x \in \mu_A$ and $s \in [0,1]$, then:

$f(x) = H(x,s)$. so that H is a fuzzy homotopy relative to μ_A

2- **the relation is symmetric**

for all $f \in C(\mu_X, \mu_Y)$, using the fuzzy homotopy $H:(x,s) = f(x)$, $\forall x \in \mu_X \wedge s \in [0,1]$. On the other hand, if $f \simeq g$ (rel μ_A) by means of the fuzzy homotopy $H: \mu_X \times [0,1] \to \mu_Y$, then define $G: \mu_X \times [0,1] \to \mu_Y$ by $G(x,s) = H(x, 1 - s)$, $\forall x \in \mu_X \wedge s \in [0,1]$, is a fuzzy homotopy from g to f relative to μ_A, i.e. $g \simeq f$ (rel μ_A).

3- **the relation is transitive**

if $f, g, h \in C(\mu_X, \mu_Y)$ satisfy that $f \simeq g$ (rel μ_A) and $g \simeq h$ (rel μ_A) by means of fuzzy homotopies $F: \mu_X \times [0,1] \to \mu_Y$ and $G: \mu_X \times [0,1] \to \mu_Y$, respectively, then define the fuzzy map $H: \mu_X \times [0,1] \to \mu_Y$ by

$$H(x,s) = \begin{cases} F(x, 2s) & if \quad 0 \le s \le \frac{1}{2} \\ G(x, 2s-1) & if \quad \frac{1}{2} \le s \le 1 \end{cases}$$

be a fuzzy homotopy from f to g relative to μ_A i.e. $f \simeq h$ (rel μ_A)

From 1 ,2 and 3 we have the relation $\simeq$ is therefore a fuzzy equivalence relation.

Theorem (2.5): Let μ_X , μ_Y and μ_Z be fuzzy topological spaces, and $\mu_A \subseteq \mu_X$ is a fuzzy subset. let fuzzy continuous maps $f, g : \mu_X \to \mu_Y$ and $u, v : \mu_Y \to \mu_Z$ such that $f \simeq g$ (rel μ_A) and $u \simeq v$ (rel $f(\mu_A)$), then $u \circ f \simeq v \circ g$ (rel μ_A).

Proof: Let $H: \mu_X \times [0,1] \to \mu_Y$ be afuzzy homotopy from f to g relative to μ_A and $G: \mu_Y \times [0,1] \to \mu_Z$ be a fuzzy homotopy from u to v relative to $f(\mu_A)$.

Define the fuzzy map $K: \mu_X \times [0,1] \to \mu_Z$ K by

$$K(x,s) = G(H(x,s) \quad \forall x \in \mu_X \; and \; s \in [0,1]$$

This gives a fuzzy homotopy from $u \circ f \simeq v \circ g$ (rel μ_A), as was to be shown.

Definition (2.6): let μ_X and μ_Y be two fuzzy topological spaces and let $f: \mu_X \to \mu_Y$ be a fuzzy continuous map. We say that f is null fuzzy homotopic if there exists $y \in \mu_Y$ such that f is fuzzy homotopic to the fuzzy constant map$C_y: \mu_X \to \mu_Y$ that sends every $x \in \mu_X$ to y.

Definition (2.7): let μ_X be fuzzy topological space, we say that μ_X is fuzzy contractible if the identity fuzzy map on μ_X is fuzzy null homotopic, i.e. iff μ_X be fuzzy homotopic to some constant fuzzy map

3. FUZZY WEAK EQUIVELANCE

Definition (3.1): let $f: \mu_X \to \mu_Y$ be fuzzy continuous map. f is called a fuzzy homotopy equivalence if and only if there exists a fuzzy continuous map $g: \mu_Y \to \mu_X$ such that

$$fog = id_{\mu_Y} \text{ and } gof = id_{\mu_X}$$

The fuzzy map g is said to be a fuzzy homotopy inverse to f

Definition (3.2): Let μ_E be fuzzy topological and let e_0 is a fuzzy point in fuzzy topological $e_0 \in \mu_E$. A fuzzy path $\beta: [0,1] \to \mu_E$ is said fuzzy loop if and only if begins and ends at e_0 .

Definition (3.2):[2] let (μ_X, T) be fuzzy topological space and $e_0 \in \mu_X$,The set of fuzzy path homotopy classes of fuzzy loops based at e_0 is called the fuzzy fundamental group of μ_X relative to the base fuzzy point e_0 if you meet the following conditions:

1- $[f] \cdot [g] = [fg]$
2- $[f] \cdot [h] = [f]$ and $[f] \cdot [f^{-1}] = [h]$ (so $[f^{-1}] = [f]^{-1}$)

It is denoted by $\pi_1(\mu_X, e_0)$.

Definition (3.3): let $f: \mu_X \to \mu_Y$ be fuzzy continuous map of fuzzy spaces is said a fuzzy weak homotopy equivalence if the caused by fuzzy isomorphism

$$f_*: \pi_k(\mu_X, x_0) \to \pi_k(\mu_Y, f(x))$$

for all $k \geq 0$ and all fuzzy basepoints $x \in \mu_X$.

Proposition (3.4): Every fuzzy homotopy equivalence is a fuzzy weak homotopy equivalence.

Proof: suppose that $f: \mu_X \to \mu_Y$ is fuzzy homotopy equivalence
Then there exist a fuzzy continuous map $g: \mu_Y \to \mu_X$ such that $fog = id_{\mu_Y}$ and $gof = id_{\mu_X}$
In this case, the existence of these based fuzzy homotopies together with the functoriality of π_m , then $f_*og_* = id$ and $g_*of_* = id$ so that f_* and g_* are inverse fuzzy isomorphisms, it is true that a fuzzy homotopy equivalence is a weak fuzzy homotopy equivalence.

Remark (3.5): The converse of the proposition is not true**.**

i.e. (if the fuzzy weak homotopy equivalence is not necessarily to be a fuzzy homotopy equivalence).

As in the following example:

Cones from the so-called "real line two origins"

The fuzzy space μ_Y of $\mathbb{R} \times \{0, 1\}$ which was obtained by definition $(x, 0) \sim (x, 1)$ if $x \neq 0$

$f: S^1 \to \mu_Y$ is fuzzy equivalence. So μ_Y has a non-trivial fuzzy fundamental group

since S^1 is fuzzy Hausdorff spaces hen any fuzzy map $g: \mu_Y \to S^1$ must agree on the two origins

then the map $\mu_Y \to \mathbb{R}$ which identifies the two origins, we see that g is fuzzy null homotopic,

hence induces the trivial fuzzy homomorphism on all fuzzy homotopy groups,

so $g: \mu_Y \to S^1$ cannot be a fuzzy weak homotopy equivalence**.**

Although, there is only one case, If the fuzzy space μ_X and μ_Y are CW complexes and $f: \mu_X \to \mu_Y$ is a fuzzy weak homotopy equivalence then $f: \mu_X \to \mu_Y$is also fuzzy homotopy equivalence. So, in this case we say that the fuzzy weak homotopy equivalence is indeed a fuzzy homotopy equivalence.

Proposition (3.6): let $F_b = p^{-1}(b)$ be fuzzy fibers with $p: \mu_E \to \mu_B$ be a fuzzy fibration, each fuzzy path component of μ_B are all fuzzy homotopy equivalent.

Proof: let $\beta: [0,1] \to \mu_B$ be fuzzy path to fuzzy homotopy $h_t: F_{\beta(0)} \to \mu_B$ such that $h_t(F_{\beta(0)}) = \beta(0)$ The inclusion $F_{\beta(0)} \hookrightarrow \mu_E$ provides a fuzzy lift $h_0^*: F_{\beta(0)} \to \mu_E$

Since p is fuzzy fibration then has fuzzy homotopy lifting property for pairs $(\mu_X \times I, \mu_X \times \partial I)$

since $(I \times I, I \times \{0\} \cup \partial I \times I) \simeq (I \times I, I \times \{0\})$, then the same is true after taking products with μ_X.

we have a fuzzy homotopy $h_t^*: F_{\beta(0)} \to \mu_E$ suth that $h_t^*(F_{\beta(0)}) \subset F_{\beta(s)}$ $, \forall s \in [0,1]$. particulary, h_s^* gives a fuzzy map $L_\beta: F_{\beta(0)} \to F_{\beta(1)}$. The association $\beta \longmapsto L_\beta$ must possess the following basic properties:

(1) If $\beta \simeq \beta_1$ rel ∂I, then $L_\beta \simeq L_{\beta_1}$.

(2) A composition of fuzzy paths $\beta\beta_1$, $L_{\beta\beta_1} \simeq L_{\beta_1} L_\beta$.

To prove (1), let $\beta \simeq \beta_1$ proof required $L_\beta \simeq L_{\beta_1}$

Let $(r,s) \in I \times I$ such that $\beta(s,r)$ be a fuzzy homotopy from $\beta(r)$ to $\beta_1(s)$. This determines a fuzzy homotopy $h_{r,s}: F_{\beta(o)} \to \mu_B$ such that $h_{r,s}(F_{\beta(0)}) = \beta(r,s)$. let $h_{r,o}^*$ is the inclusion $F_{\beta(0)} \hookrightarrow \mu_E$, $\forall r \in I$ and let $h_{o,s}^*$ and $h_{1,s}^*$ be fuzzy lifts define by L_β and L_{β_1}.

By using the fuzzy homotopy lifting property for $(F_{\beta(0)} \times I, F_{\beta(0)} \times \partial I)$, we can lengthen these fuzzy lifts to fuzzy lifts $h_{r,s}^*$ $, \forall (r,s) \in I \times I$. when we take $s = 1$ we get a fuzzy homotopy $L_\beta \simeq L_{\beta_1}$

Property (2) verified since $h^*(s)$ is fuzzy lift for L_β and $h_1^*(s)$ is fuzzy lift for L_{β_1} then we get fuzzy lift defined $L_{\beta\acute{\beta}}$ by

$$L_{\beta\hat{\beta}}(s) = \begin{cases} h^*(2s) & , \quad if \quad 0 \le s \le \frac{1}{2} \\ h_1^*(2s-1) & , if \quad \frac{1}{2} \le s \le 1 \end{cases}$$

From 1 and 2 we get L_β is a fuzzy homotopy equivalence with fuzzy homotopy inverse $L_{\overline{\beta}}$, where $\overline{\beta}$ is the inverse fuzzy path of β .

Definition (3.7): let $p_1: \mu_{E_1} \to \mu_B$ and $p_2: \mu_{E_2} \to \mu_B$ be two fuzzy fibration, A fuzzy map $g: \mu_{E_1} \to \mu_{E_2}$ is called fuzzy fiber-preserving if $p_1 = p_2 g$

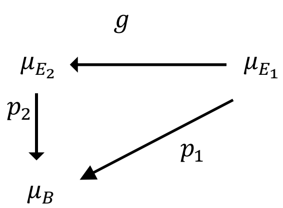

FIGURE 1. fuzzy fiber-preseving

Definition (3.8): let $g: \mu_{E_1} \to \mu_{E_2}$ be fuzzy fiber-preserving is called fuzzy fiber homotopy equivalence if and only if there exist fuzzy fiber-preserving map $h: \mu_{E_2} \to \mu_{E_1}$ such that the composition $goh \simeq id_{\mu_{E_2}}$ and $hog \simeq id_{\mu_{E_1}}$.

Definition (3.9): Let $p: \mu_E \to \mu_B$ be a fuzzy fibration and let $g: \mu_A \to \mu_B$ be fuzzy continuous map. The fuzzy projection $q: \mu_g \to \mu_A$ such that $\mu_g \subset \mu_A \times \mu_E$ defined by $\mu_g = \{(a,e) \in \mu_A \times \mu_E : g(a) = p(e)\}$, then q is called fuzzy pullback fibration and the diagram is commutative such that $po\acute{g} = goq$.

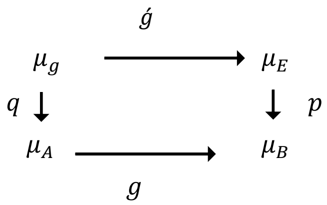

FIGURE 2. fuzzy pullback fibration

The fuzzy homotopy lifting property verified for $q: \mu_g \to \mu_A$ since a fuzzy homotopy $h_t: \mu_X \to \mu_A$ we get both condition

1- A fuzzy lift $h_t^*: \mu_X \to \mu_g$
2- a fuzzy lift to μ_E of the composed fuzzy homotopy $g \circ h_t$

Proposition (3.10): let $p: \mu_E \to \mu_B$ be a fuzzy fibration and let $g_t: \mu_A \to \mu_B$ be a fuzzy homotopy , then fuzzy pullback fibrations $q_1: \mu_{g_1} \to \mu_A$ and $q_2: \mu_{g_2} \to \mu_A$ are fuzzy fiber homotopy equivalent.

Proof: suppose that $G: \mu_A \times I \to \mu_B$ be the fuzzy homotopy g_t and $Q: \mu_g \to \mu_A \times I$ be fuzzy fibration such that Q contains q_1 and q_2 over $\mu_A \times \{0\}$ and $\mu_A \times \{1\}$. Therefore, it is sufficient for us to prove the following: if $p: \mu_E \to \mu_B \times I$ is a fuzzy fibration then the restricted fuzzy fibrations $F_r = p^{-1}(\mu_B \times \{r\}) \to \mu_B$ are all fuzzy fiber homotopy equivalent, $\forall r \in [0\,,1]$.

To prove this, the idea is to create a constructive imitation of fuzzy homotopy equivalences L_β in proof **Proposition (2.6)**.

Let $\beta: [0\,,1] \to I$ be fuzzy path gives to a fuzzy fiber-preserving map $L_\beta: F_{\beta(0)} \to F_{\beta(1)}$ by fuzzy lifting the fuzzy homotopy $h_t: F_{\beta(0)} \to \mu_B \times I$, $h_t(x) = (p(x)\,, \beta(t))$, starting with the inclusion $E_{\beta(0)} \hookrightarrow \mu_E$. As before, one shows the two basic properties (1) and (2), noting that in (1) he fuzzy homotopy $L_\gamma \simeq L_{\acute{\gamma}}$ is fuzzy fiber-preserving since it is obtained by lifting a fuzzy homotopy $g_t: E_{\beta(0)} \times [0\,,1] \to \mu_B \times I$ of the form $g_t(x\,,u) = (p(x)\,, -)$. From (1) and (2) it follows that L_β is a fuzzy fiber homotopy equivalence with inverse $L_{\bar{\beta}}$.

Corollary (3.11): If $p: \mu_E \to \mu_B$ be fuzzy fibration such that μ_B be a fuzzy contractible base then p is fuzzy homotopy equivalent to a product fuzzy fibration $\mu_B \times \mu_F \to \mu_B$.

Proof: let $p: \mu_E \to \mu_B$ be a fuzzy fibration and let the identity fuzzy map $id: \mu_B \to \mu_B$ is μ_E itself. The fuzzy pullback of μ_E and since μ_B is fuzzy contractible then a fuzzy constant map $\mu_B \to \mu_B$ is a product $\mu_B \times \mu_F$.

Proposition (3.12): let $p: \mu_E \to \mu_B$ be a fuzzy fibration, the inclusion $\mu_E \hookrightarrow \mu_{E_P}$ be a fuzzy homotopy equivalence.

Proof: Assume that i is denote the inclusion $i: \mu_E \hookrightarrow \mu_{E_p}$. As $p: \mu_E \to \mu_B$ be fuzzy fibration then p has fuzzy homotopy lifting property to the fuzzy homotopy $h_t: \mu_{E_p} \to \mu_B$ define by $h_t(e\,,\beta) = \beta(t)$, the fuzzy lifting $h_t^*: \mu_{E_p} \to \mu_E$ define by $h_t^*(e\,,\beta) = e$ such that the first coordinate of a fuzzy homotopy $g_t: \mu_{E_p} \to \mu_{E_p}$ while it is the second coordinate is the restricted by the fuzzy paths β to the interval [t ,1]. g_t is fuzzy fiber-preserving since in the fuzzy paths the end point has not changed. Saves us $g_0 = id\,, g_1(\mu_{E_p}) \subset \mu_E$, and $g_t(\mu_E) \subset \mu_E$, $\forall\, t \in [0{,}1]$

Then from $ig_1 \simeq id$ via g_t and $g_1 i \simeq id$ via $g_t|\mu_E$ we get i is a fuzzy homotopy equivalence.

Remark (3.13): [1] **(The Five-Lemma)** let $\mu_A\,, \mu_B\,, \mu_C\,, \mu_D\,, \mu_E\,, \mu_{\acute{A}}\,, \mu_{\acute{B}}\,, \mu_{\acute{C}}\,, \mu_{\acute{D}}$ and $\mu_{\acute{E}}$ are fuzzy topological spaces and $f_1\,, f_2\,, f_3\,, f_4$ and f_5 are fuzzy maps between to fuzzy topological spaces such that all two rows are exact. Diagram is commutative abelian groups

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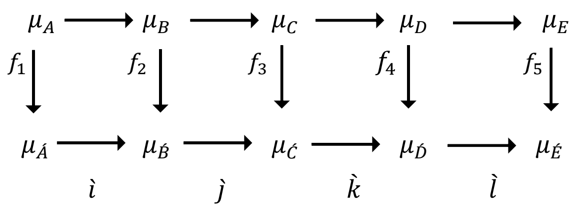
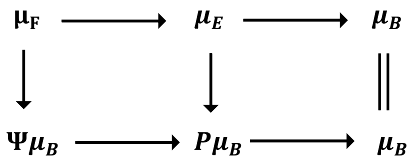


FIGURE 3. Five-lemma

If f_1 , f_2 , f_4 and f_5 are isomorphisms, then f_3 is an isomorphism also.

Proposition (3.14): If $\mu_F \to \mu_E \to \mu_B$ is a fuzzy fibration such that $\mu_E$is fuzzy contractible, then there exit $\mu_F \to \Psi\mu_B$ is a fuzzy weak homotopy equivalenc.

Proof : let $p: \mu_E \to \mu_B$ be fuzzy projection, If we configure the contraction of μ_E then we have for each fuzzy point $e \in \mu_E$ a fuzzy path α_e in μ_B from $p(e)$ to a basepoint $b_o = p(e_0)$, when e_0 is the fuzzy point to which μ_E contracts



This yields a fuzzy map $\mu_E \to P\mu_B$, $e \mapsto \overline{\alpha_e}$, which is composition with the fuzzy fibration $p\mu_B \to \mu_B$ is P.
By compliance with this we get a fuzzy map $\mu_F \to \Psi\mu_B$ such that $F = p^{-1}(b_0)$ be fuzzy fiber,
Since the fuzzy maps $\mu_F \to \mu_E \to \mu_B$ is the long exact sequence for $\Psi\mu_B \to P\mu_B \to \mu_B$ is also long exact sequence and since μ_E and $P\mu_B$ are fuzzy contractible then using **The Five-Lemma** we get the fuzzy map $\mu_F \to P\mu_B$ is a fuzzy weak homotopy equivalence.

4. CONCLUSION

In this paper we first conducted a study homotopic relative and we provided a definition fuzzy homotopic relative and we proved that the relation fuzzy homotopic relative is a fuzzy equivalence relation. Secondly, we did a study fuzzy weak homotopy equivalence and we provided a new definition it, and we concluded Every fuzzy homotopy equivalence is a fuzzy weak homotopy equivalence but the opposite is not true, every fibers over each fuzzy path component are all fuzzy homotopy equivalent, every fuzzy pullback fibrations are fuzzy fiber homotopy equivalent and every fuzzy homotopy fibers are fuzzy homotopy equivalent to the actual fuzzy fibers.

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