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Oscillation Criteria for Solution of the Fourth Order Nonlinear Neutral Differential Equations

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ABSTRACT: In this research, we study the properties of oscillation for all neutral differential equations Its importance in solving many engineering, physical and chemical problems and giving better results in a timely manner within a specific period of time some basic necessary and sufficient condition are established for every solution of solutions of the fourth order nonlinear

$$[(y(t)+p(t)f(y(\tau(t))]^{(4)}+q(t)g\left(y\left(\sigma(t)\right)\right)=F(t)$$

To become oscillatory, Examples are given in support our results.

Keywords: Oscillation Criteria 1, Linear Ordinary Differential Equations 2, n-th order linear systems. Non-Oscillation Criteria, properties of oscillation, Neutral differential equations (NDEs.

1. INTRODUCTION

Differential equations with delays, where the delays can manifest in both the state variables and their temporal derivatives, are known as neutral differential equations (NDEs). Since delay differential equations may be used to simulate a variety of processes, there is often a great deal of interest in researching this kind of equation.

Systems of differential equations with delays were employed in [1,2,3] to investigate the stability and dynamical characteristics of electrical power systems. Studies on the stability characteristics of macroeconomic models are among the other examples [4,5, 6].

The problems presented by the new classes of functional differential/difference equations in various application fields, along with advancements in science and technology, are driving an increasing interest in these equations. In nerve conduction, equations involving progress, delay, and a mixture of both occur. The oscillation for solving nonlinear neutral differential equations is studied by many authors

We study fourth order nonlinear neutral differential equations of the following type in this article:

$$[(\psi(\varpi)+p(\varpi)f(\psi(\tau(\varpi))]^{(4)}+q(\varpi)g\left(\psi\left(\sigma(\varpi)\right)\right)=F(\varpi) \qquad \dots(1.1)$$

Under the following assumption:

$(A_1) P(\varpi)\in C([\varpi_\infty,\infty);(0,\infty)), q(\varpi)\in C([\varpi_\infty,\infty);R)$.

$(A_2)\tau(\varpi)\sigma(\varpi)\in C([\varpi_\infty,\infty);R)$, $\lim_{\varpi\to\infty}\tau(\varpi)=\infty$, $\lim_{\varpi\to\infty}\sigma(\varpi)=\infty$,

(A_3) $F\in C(R;R), \delta_2\le\frac{f(u)}{u}\le\delta_1, uf(u)>0; \delta_1,\delta_2>0$ are constants.

$(A_4) There\ exis\varpi s\ a\ func\varpi ion\ h(\varpi)\in C^3([\varpi_0,\infty);R) such\ \varpi ha\varpi$ $\lim_{\varpi\to\infty}h(\varpi)=0\ and\ h'''(\varpi)=F(\varpi)$.

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$(A_5)\ g \in C(R;R), \dfrac{g(u)}{u} \geq \beta > 0.$
and developed several sufficient and necessary conditions to guarantee the oscillation of every solution in equation (1.1). Several examples are provided to show the outcomes that were attained.

2. Lemma: [6]

The following lemma will be used to establish our main results: Assume that $\tau(\varpi) > \varpi$,and

$$\lim_{\varpi\to\infty} inf \int_{\varpi}^{\tau(\varpi)} p(s)ds > \frac{1}{e}$$

Where $p(\varpi), \tau(\varpi) \in C([\varpi_0, \infty])$ then
The differential inequality ,$\psi'(\varpi) - p(\varpi)\psi(\tau(\varpi)) \geq 0$, has no long-term, constructive solutions.

3. MAIN RESULTS

we define $Z(\varpi) = \psi(\varpi) + p(\varpi)f(\psi(\tau(\varpi))) - h(\varpi)$ ... (1.2)

Then equation (1.1) reduce to

$$Z^{(4)}(\varpi) = -q(\varpi)g(\psi(\sigma(\varpi))) \qquad ...(1.3)$$

Theorem 3.1: Assume that (A_1)–(A_2) hold, $0 \leq p(\varpi) < \frac{1}{\delta_1}$, $\sigma(\varpi) > \varpi, q(\varpi) \geq 0, \tau(\varpi) > \varpi$, and ∃ a continuous functions $\alpha(\varpi) > \varpi$, Added to the conditions required $\lim_{\varpi\to\infty} \inf q(\varpi) > 0$ ... (1.4)

$$\lim_{\varpi\to\infty} \inf \int_{\varpi}^{\sigma(\varpi)} \int_{s}^{\lambda(s)} \int_{v}^{\gamma(v)} \int_{w}^{\alpha(w)} q(k)(1 - \delta_1 p(\sigma(k))dkdwdvds > \frac{1}{\beta e} \qquad ... (1.5)$$

Following that, each of equation (1.1)'s solutions oscillates.

Proof:
Let ψ(ϖ) be an ultimately positive solution of equation (1.1) and suppose for the sake of contradiction that there is no oscillatory solution.

(The case where ψ(ϖ) ultimately becomes negative is comparable and will be excluded.).
Let $\psi(\varpi) > 0, \psi(\tau(\varpi)) > 0, \psi(\sigma(\varpi)) > 0$, for $\varpi > \varpi_0$.
From (1.3) it follows that

$$Z^{(4)}(\varpi) = -q(\varpi)g(\psi(\sigma(\varpi)) \leq 0 , \varpi > \varpi_0,$$

It follows that $Z(\varpi), Z'(\varpi), Z''(\varpi), Z'''(\varpi)$ are monotone founctions.
We have two cases for $Z'''(\varpi)$:
Case 1 : $Z'''(\varpi) < 0, \varpi \geq \varpi_1 \geq \varpi_0$, hence $Z''(\varpi) < 0$, $Z'(\varpi) < 0, Z(\varpi) < 0$, and $\lim_{\varpi\to\infty} Z(\varpi) = -\infty$.
On the other hand from (1.2) , we get $Z(\varpi) > -h(\varpi)$, it follows that $\lim_{\varpi\to\infty} h(\varpi) = \infty$,
Which is a contradiction.
Case 2: $Z'''(\varpi) > 0, \varpi \geq \varpi_1 \geq \varpi_0$, We have two cases for $Z''(\varpi)$:
Case 2.1: $Z''(\varpi) > 0 , \varpi \geq \varpi_2 \geq \varpi_1, Z'(\varpi) > 0 , Z(\varpi) > 0$, and $\lim_{\varpi\to\infty} Zv(\varpi) = \infty$ we claim that $\lim_{\varpi\to\infty} \psi(\varpi) = \infty$, otherwise there exists $k > 0$ and ϖ_k such that $\psi(\varpi) \leq k$ for $\varpi \geq \varpi_k$.
From (1.2) with virtue of $\lim_{\varpi\to\infty} h(\varpi) = 0$, we get $Z(\varpi) \leq k + k - h(\varpi)$,
This results in $\lim_{\varpi\to\infty} Z(\varpi) < \infty$, **a contradiction**.
Condition (1.4) implies that ∃ $c > 0$ and $\varpi_3 \geq \varpi_2$ s.t $q(\varpi) \geq c$ for $\varpi \geq \varpi_3$.
From(1.3) and (A_5) we got

$$Z^{(4)}(\varpi) \leq -\beta q(\varpi)(\psi(\sigma(\varpi)) \qquad ... (1.6)$$

Integrating (1.6) from ϖ_3 to ϖ we get

$$Z'''(\varpi) - Z'''(\varpi_3) \leq -\beta \int_{\varpi_3}^{\varpi} q(s) (\psi(\sigma(s)) ds ,$$

$$\leq -\beta c \int_{\varpi_3}^{\varpi} (\psi(\sigma(s)) ds ,$$

$$\int_{\varpi_3}^{\varpi} (\psi(\sigma(s))\ ds \le \frac{Z''(\varpi_3) - Z''(\varpi)}{\beta\ c} < \infty\ ,\ \ \varpi \ge \varpi_3$$

The last inequality leads to a contradiction , since

$$\lim_{\varpi\to\infty} \int_{\varpi_3}^{\varpi} (\psi(\sigma(s))\ ds = \infty$$

Cas2.2 : $Z''(\varpi) < 0\ ,\varpi \ge \varpi_2 \ge \varpi_1$, we have two cases for $Z'(\varpi)$:

Cas2.2.1: If $Z'(\varpi) < 0\ ,\varpi \ge \varpi_3 \ge \varpi_2$ we can use the same treatment in case 1.

Cas2.2.2: If $Z'(\varpi) > 0\ ,\ \varpi \ge \varpi_3 \ge \varpi_2$ it remains to consider the case

$Z(\varpi) > 0\ \ Z'(\varpi) > 0, Z''(\varpi) < 0, Z'''(\varpi) > 0$, From (2.2.2) we get

$$\psi(\varpi) \ge Z(\varpi) - \delta_1 p(\varpi)\psi(\tau(\varpi)) + h(\varpi)$$
$$= Z(\varpi) - \delta_1 p(\varpi)\left[Z(\tau(\varpi)) - p(\tau(\varpi))f(\psi\ (\tau^2(\varpi)) + h(\tau(\varpi))\right] + h(\varpi)$$
$$\ge Z(\varpi) - \delta_1 p(\varpi)Z(\tau(\varpi)) - \delta_1 p(\varpi)h(\tau(\varpi)) + h(\varpi)$$
$$\ge (1 - \delta_1 p(\varpi))Z(\varpi) - \delta_1 p(\varpi)h(\tau(\varpi)) + h(\varpi)$$

Hence for sufficiently large $\varpi_3 \ge \varpi_2$ we obtain

$\psi(\varpi) > (1 - \delta_1 p(\varpi))Z(\varpi) - \varepsilon$, for $\varepsilon > 0$, $\varpi \ge \varpi_3$

The last inequality implies that

$$\psi(\varpi) \ge (1 - \delta_1 p(\varpi))Z(\varpi) \qquad \ldots (1.7)$$

By substituting (1.7) into (1.3) we find

$$Z^{(4)}(\varpi) \le -\beta q(\varpi)(1 - \delta_1 p(\sigma(\varpi)))Z(\sigma(\varpi)) \qquad \ldots (1.8)$$

Integration (1.8) from ϖ to $\alpha(\varpi)$ where $\alpha(\varpi) > \varpi$ is a continuous function , we get

$$Z'''\left(\alpha\ (\varpi)\right) - Z'''(\varpi) \le -\beta \int_{\varpi}^{\alpha\ (\varpi)} q(s)(1 - \delta_1 p(\sigma(s)))Z(\sigma(s))\ ds,$$
$$-Z'''(\varpi) \le -\beta Z(\sigma(\varpi)) \int_{\varpi}^{\alpha\ (\varpi)} q(s)(1 - \delta_1 p(\sigma(s)))\ ds,$$
$$Z'''(\varpi) \ge \beta Z(\sigma(\varpi)) \int_{\varpi}^{\alpha\ (\varpi)} q(s)(1 - \delta_1 p(\sigma(s)))\ ds\ ,$$

Integration the last inequality from ϖ to $\gamma\ (\varpi)$ where we get

$$Z''(\gamma\ (\varpi)) - Z''(\varpi) \ge \beta \int_{\varpi}^{\gamma\ (\varpi)} Z(\sigma(s)) \int_{s}^{\alpha\ (s)} q(v)(1 - \delta_1 p(\sigma(v)))dv\ \ ds$$
$$-Z''(\varpi) \ge \beta Z(\sigma(\varpi)) \int_{\varpi}^{\gamma(\varpi)} \int_{s}^{\alpha\ (s)} q(v)(1 - \delta_1 p(\sigma(v)))dv\ \ ds$$

Integration the last inequality from ϖ to $\lambda(\varpi)$ where we get

$$-Z'(\lambda(\varpi)) + Z'(\varpi) \ge \beta \int_{\varpi}^{\lambda(\varpi)} Z(\sigma(\varpi)) \int_{s}^{\gamma(s)} \int_{v}^{\alpha\ (v)} q(w)(1 - \delta_1 p(\sigma(w)))dw\ dv\ \ ds$$
$$Z'(\varpi) \ge \beta Z(\sigma(\varpi)) \int_{\varpi}^{\lambda(\varpi)} \int_{s}^{\gamma(s)} \int_{v}^{\alpha\ (v)} q(w)(1 - \delta_1 p(\sigma(w)))dw\ dv\ \ ds \quad \ldots (1.9)$$
$$Z'(\varpi) - \beta Z(\sigma(\varpi)) \int_{\varpi}^{\lambda(\varpi)} \int_{s}^{\gamma(s)} \int_{v}^{\alpha\ (v)} q(w)(1 - \delta_1 p(\sigma(w)))dw\ dv\ \ ds\ \ \ge 0$$

Then by lemma (1.5.3), The last inequality cannot have a positive solution in the end due to condition (2.2.5), which is contradictory.

Corollary3.1 : The conclusion of theorem(3 .1), remains true if we replace the condition (1.9) by the condition

$$\limsup_{t\to\infty} \int_{\sigma^{-1}(\varpi)}^{\varpi} \int_{s}^{\lambda(s)} \int_{v}^{\gamma(v)} \int_{w}^{\alpha\ (w)} q(k)(1 - \delta_1 p(\sigma(k)))dkdw\ dvds\ > \frac{1}{\beta}\ (1.10)$$

Proof According to the inequality (1.9), the evidence is comparable to the theorem 2.2.1 proof.

To do this, integrating (1.9) by $\sigma^{-1}(\varpi)$ to ϖ we get

$$Z(\varpi) - Z(\sigma^{-1}(\varpi)) \ge \beta \int_{\sigma^{-1}(\varpi)}^{\varpi} Z(\sigma(s)) \int_{s}^{\gamma(s)} \int_{v}^{\alpha(v)} |q(w)|\ (1 - \delta_1 p(\sigma(w)))dw\ dv\ ds,$$

$$Z(\varpi) \ge \beta\ Z(\varpi) \int_{\sigma^{-1}(\varpi)}^{\varpi} \int_{s}^{\alpha(s)} \int_{v}^{\alpha(v)} |q(w)|\ (1 - \delta_1 p(\sigma(w)))dwdvds,$$

$$\int_{\sigma^{-1}(\varpi)}^{\varpi}\int_{s}^{\alpha(s)}\int_{v}^{\alpha(v)}|q(w)|\,(1-\delta_1 p(\sigma(w)))dw\ dv\ ds \le \frac{1}{\beta},$$

This works contradictory to (1.10).

***Example 3.1*: Consider about the differential equation that is neutral.**

$[\psi(\varpi)+e^{-\varpi}\, f\big(\psi(\varpi-2\pi)\big)]^4 + e^{\frac{-\pi}{2}}\, g(\psi(\varpi+\frac{3\pi}{2}) =$

$e^{-2\pi}\sin\varpi - 4e^{\varpi}\sin\varpi - e^{\varpi+\frac{3\pi}{4}}\cos\varpi\,;\ \ \varpi \ge 1 \quad (E1)$

where

$\tau(\varpi)=\varpi-2\pi\,,\sigma(\varpi)=\varpi+\dfrac{3\pi}{2}\,,p(\varpi)=e^{-\varpi},f(\psi)=\ \psi,q(\varpi)=e^{-\frac{\pi}{2}},$

$F(\varpi)=e^{-2\pi}\sin\varpi-4e^{\varpi}\sin\varpi-e^{\varpi+\frac{3\pi}{4}}\cos\varpi,\ g(\psi)=e^{\frac{\pi}{2}}\,\psi$

for $\varpi \ge 1$,which implies that $\beta = e^{\frac{\pi}{2}}$.

If one searches for the condition (1.5), they will discover that all of the requirements of theorem 3.1 are met:

Let $\alpha(\varpi)=\gamma(\varpi)=\lambda(\varpi)=\varpi+\frac{\pi}{2},\ \sigma(\varpi)=\varpi-\pi<\varpi,\ \delta_1=1.$

$$\liminf_{\varpi\to\infty}\int_{\varpi}^{\sigma(\varpi)}\int_{s}^{\lambda(s)}\int_{v}^{\gamma(v)}\int_{w}^{\alpha(w)} q(k)(1-\delta_1 p\big(\sigma(\,k)\big)dkdwdvds =$$

$$e^{-\frac{\pi}{2}}\lim_{\varpi\to\infty}\int_{\varpi}^{\varpi+\frac{3\pi}{2}}\int_{s}^{s+\frac{\pi}{2}}\int_{v}^{v+\frac{\pi}{2}}\int_{w}^{w+\frac{\pi}{2}}\left(1-e^{-k-\frac{3\pi}{2}}\right)dk\ dw\ dv\ ds = 3.796 > \frac{1}{e^{1+(\pi)/2}}$$

So, every solution of equation$(E1)$ oscillates for instance $\psi(\varpi)=e^{\varpi}\sin\varpi$ is such a solution.

Theorem 3.2: Assume that $(A_1)-(A_2)$ hold,$0 \le p(\varpi) < \frac{1}{\delta_1}$, , $q(\varpi)\le 0, \tau(\varpi)>\varpi, \sigma(\varpi)<\varpi$ and there exist a continuous functions $\alpha(\varpi)\ge\varpi, \gamma(\varpi)>\varpi, \lambda(\varpi)>\varpi$ such that $\big(\sigma(\alpha(\gamma(\lambda(\varpi))))\big)<\varpi$ in addition to the conditions

$$\liminf_{\varpi\to\infty}\int_{\sigma(\alpha(\gamma(\lambda(\varpi))))}^{\varpi}\int_{s}^{\lambda(s)}\int_{v}^{\gamma(v)}\int_{w}^{\alpha(w)}|q(k)|\left(1-\delta_1 p\big(\sigma(k)\big)\right)dk\ dwdv\ ds\ > \tfrac{1}{\beta e}\ (1.11)$$

Following that, every bounded solution to equation (1.1) suffers oscillations.

Proof. Suppose that equation (1.1) has a non-scillatory solution for the sake of contradiction. Let y(t) be the ultimately positive solution of (1.1); the scenario where y(t) is finally negative is identical and will be left altogether. Let $\psi(\varpi) > 0, \psi(\tau(\varpi)) > 0, \psi(\sigma(\varpi)) > 0$ for $\varpi \ge \varpi_0$.

From (1.3) it follows that $Z^{(4)}(\varpi) = -q(\varpi)g(\psi(\sigma(\varpi))) \ge 0,\ \ \varpi \ge \varpi_0.$

It follows that $Z(\varpi),\ Z'(\varpi),\ Z''(\varpi), Z'''(\varpi)$ are monotone functions.

We have two cases for $z'''(\varpi)$:

<u>Case 1.</u>$Z'''(\varpi) > 0,\ \varpi \ge \varpi_1 \ge \varpi_0$, hence $Z''(\varpi) > 0, Z'(\varpi) > 0,\ Z(\varpi) > 0$ and $\lim_{\varpi\to\infty} Z(\varpi) = \infty$,On the other side since $\psi(\varpi)$ is bounded, then there exists $k > 0$, such that $\psi(\varpi) \le k$, it follows that from (1.2)

$Z(\varpi) \le \psi(\varpi) + \delta_1 p(\varpi)\psi(\tau(\varpi))) - h(\varpi)$. Hence $Z(\varpi) < k + k - h(\varpi)$, which implies that $\lim_{\varpi\to\infty} Z(\varpi) < \infty$ a contradiction

<u>Case 2.</u> $Z'''(\varpi) < 0$,*we have two cases for* $Z''(\varpi)$:

<u>Case 2.1</u> $Z''(\varpi) < 0, \varpi \ge \varpi_2 \ge \varpi_1$ hence

$Z(\varpi) < 0, Z'(\varpi) < 0$, and it follows that $\lim_{\varpi\to\infty} Z(\varpi) = -\infty$.

From (1.2) we get $Z(\varpi) \geq -h(\varpi)$, which implies that

$\lim_{\varpi\to\infty} Z(\varpi) = 0$, which is a contradiction.

Case 2.2 $Z''(\varpi) > 0$ *, we have two cases for* $Z'(\varpi)$:

Case 2.2.1 $Z'(\varpi) > 0$, the proof is similar to case1.

Case 2.2.1. $Z'(\varpi) < 0,\ \varpi \geq \varpi_2 \geq \varpi_1,$ it remains to consider the case

$$Z(\varpi) > 0, Z'(\varpi) < 0, Z''(\varpi) > 0, Z'''(\varpi) < 0, Z^{(4)} \geq 0, \qquad \varpi \geq \varpi_3 \geq \varpi_2,$$

From (1.3), (A_5) and substituting (2.2.7) we get

$$Z^{(4)}(\varpi) \geq \beta\, |q(\varpi)|\, (1 - \delta_1 p(s))\, Z(\sigma(\varpi)) \qquad (1.12)$$

Integrating (1.12) from ϖ to $\alpha(\varpi)$ we get

$$Z'''(\alpha(\varpi)) - Z'''(\varpi) \geq \beta \int_{\varpi}^{\alpha(\varpi)} |q(s)|(1 - \delta_1 p(s)) Z(\sigma(s))\, ds,$$

$$-Z'''(\varpi) \geq \beta\, Z(\sigma(\alpha(\varpi))) \int_{\varpi}^{\alpha(\varpi)} |q(s)|(1 - \delta_1 p(\sigma(s)))\, ds,$$

Integrating from ϖ to $\gamma(\varpi)$ we get

$$-Z''(\gamma(\varpi)) + Z''(\varpi) \geq \beta \int_{\varpi}^{\gamma(\varpi)} Z(\sigma(\alpha(s))) \int_{s}^{\alpha(s)} |q(v)| \left(1 - \delta_1 p(\sigma(v))\right) dv\, ds,$$

$$Z''(\varpi) \geq \beta Z(\sigma(\alpha(\gamma(\varpi)))) \int_{\varpi}^{\gamma(\varpi)} \int_{s}^{\alpha(s)} |q(v)| \left(1 - \delta_1 p(\sigma(v))\right) dv\, ds$$

Integrating from ϖ to $\lambda(\varpi)$ we get

$$Z'(\lambda(\varpi)) - Z'(\varpi) \geq \beta \int_{\varpi}^{\lambda(\varpi)} Z(\sigma(\alpha(\gamma(\varpi)))) \int_{s}^{\gamma(s)} \int_{v}^{\alpha(v)} |q(w)| \left(1 - \delta_1 p(\sigma(w))\right) dw dv\, ds$$

$$-Z'(\varpi) \geq \beta Z(\sigma(\alpha(\gamma(\lambda(\varpi))))) \int_{\varpi}^{\lambda(\varpi)} \int_{s}^{\gamma(s)} \int_{v}^{\alpha(v)} |q(w)| \left(1 - \delta_1 p(\sigma(w))\right) dw dv\, ds \quad (2.2.13)$$

$$Z'(\varpi) + \beta Z(\sigma(\alpha(\gamma(\lambda(\varpi))))) \int_{\varpi}^{\lambda(\varpi)} \int_{s}^{\gamma(s)} \int_{v}^{\alpha(v)} |q(w)| \left(1 - \delta_1 p(\sigma(w))\right) dw dv\, ds \ \leq 0$$

Then by lemma (2) , with virtue of condition (1.11) the last inequality can not have eventually positive solution which is a contradiction .

Corollary *3.2*: The conclusion of theorem (3.2), remain true if we replace the condition (1.11) by the condition

$$\lim_{\varpi\to\infty} \sup \int_{\varpi}^{\mu(\varpi)} \int_{s}^{\lambda(s)} \int_{v}^{\gamma(v)} \int_{w}^{\alpha(w)} q(k)(1 - \delta_1 p(\sigma(k)) dk dw dv ds > \frac{1}{\beta} \qquad \ldots (1.14)$$

Where$\sigma(\alpha(\gamma(\lambda(\mu(\varpi))))) > \varpi$

Proof: The proof is similar to the proof of theorem 3.2 up to the inequality (1.13). To complete the proof integrating (1.13) from ϖ ϖo $\mu(\varpi)$, we get

$$-Z(\mu(\varpi)) + Z(\varpi) \geq \beta \int_{\varpi}^{\mu(\varpi)} Z(\sigma(\alpha(\gamma(\lambda(\varpi))))) \int_{\varpi}^{\lambda(\varpi)} \int_{s}^{\gamma(s)} \int_{v}^{\alpha\,(v)} q(k)(1 - \delta_1 p(\sigma(k))) dk\, dw\, dv\, ds$$

$$Z(\varpi) \geq \beta\, Z\left(\sigma\left(\alpha\left(\gamma(\lambda(\mu(\varpi)))\right)\right)\right) \int_{\varpi}^{\mu(\varpi)} \int_{\varpi}^{\theta(\varpi)} \int_{s}^{\gamma(s)} \int_{v}^{\alpha\,(v)} q(k)(1 - \delta_1 p(\sigma(k))) dk\, dw\, dv\, ds,$$

$$\frac{Z(\varpi)}{Z\left(\sigma\left(\alpha\left(\gamma(\lambda(\mu(\varpi)))\right)\right)\right)} \geq \beta \int_{\varpi}^{\mu(\varpi)} \int_{\varpi}^{\theta(\varpi)} \int_{s}^{\gamma(s)} \int_{v}^{\alpha\,(v)} q(w)(1 - \delta_1 p(\sigma(w))) dw\, dv\, ds,$$

$$\int_{\varpi}^{\mu(\varpi)}\int_{\varpi}^{\lambda(\varpi)}\int_{s}^{\gamma(s)}\int_{v}^{\alpha(v)} q(w)(1-\delta_1 p(\sigma(w)))dw\,dv\,ds \le \frac{1}{\beta},$$

Which is contradiction with (1.14).

***Example 3.2*:** Consider the neutral differential equation

$$[\psi(\varpi)+e^{-\varpi} f(\psi(\varpi+\pi))]^4 - e^{\frac{-\pi}{2}} g(\psi(\varpi-4\pi) = -e^{\pi}\cos\varpi - 4e^{\varpi}\cos\varpi - e^{\frac{-17\pi}{4}}\cos\varpi;\ \ \varpi \ge 1 \quad (E2)$$

where

$\tau(\varpi) = \varpi+\pi,\ \sigma(\varpi) = \varpi - 4\pi,\ p(\varpi) = e^{-\varpi}, f(\psi) = \psi, q(\varpi) = -e^{-\frac{\pi}{2}},$

$F(\varpi) = -e^{\pi}\cos\varpi - 4e^{\varpi}\cos\varpi - e^{\frac{-17\pi}{4}}\cos\varpi,\ g(\psi) = e^{\frac{\pi}{4}}\psi$

for $\varpi \ge 1$,which implies that $\beta = e^{\frac{\pi}{4}}$.

One can find that all conditions of theorem 3.2 are hold, to see the condition (1.11):

Let $\alpha(\varpi) = \gamma(\varpi) = \lambda(\varpi) = \varpi + \pi,\ \sigma(\alpha(\gamma(\lambda(\varpi)))) = \varpi - \pi < \varpi,\ \delta_1 = 1.$

$$\liminf_{\varpi\to\infty}\int_{\sigma(\alpha(\gamma(\lambda(\varpi))))}^{\varpi}\int_{s}^{\lambda(s)}\int_{v}^{\gamma(v)}\int_{w}^{\alpha(w)} |q(k)|\left(1-\delta_1 p(\sigma(k))\right)dk\,dwdv\,ds =$$

$$e^{-\frac{\pi}{2}}\lim_{\varpi\to\infty}\int_{\varpi-\pi}^{\varpi}\int_{s}^{s+\pi}\int_{v}^{v+\pi}\int_{w}^{w+\pi}(1-e^{-k+4\pi})\,dk\,dw\,dv\,ds = \frac{\pi^4}{e^{\frac{\pi}{2}}} > \frac{1}{e^{\frac{\pi}{4}}}$$

So, every solution of equation$(E2)$ oscillates for instance $\psi(\varpi) = e^{\varpi}\cos\varpi$ is such a solution.

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