On Composition of Continuity in ideal Topological spaces

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Abstract
We define new classes of sets called pre-$\hat{\delta}$-closed set and pre-$\hat{\delta}$-open set. Also we define new ideal topological spaces is called $\mathbb{T}_{p\hat{\delta}}$-spaces, the notion of pre-$\hat{\delta}$-closed set in is applied to study a new class of functions, continuous function and composition of continuity in ideal topological spaces, so that we investigate some properties, characterizations of these functions.

Keywords: Ideal topological spaces, continuous functions, Pre-$\hat{\delta}$-closed, Pre-$\hat{\delta}$-open,$\mathbb{T}_{p\hat{\delta}}$.-space.

1 Introduction

A nonempty collection $\mathbb{I}$ of subsets in a topological space $(\mathbb{X},\mathbb{T})$ is said to be an ideal if it satisfies

- $\mathcal{A} \in \mathbb{I}$ and $\text{ß} \subseteq \mathcal{A}$ implies $\text{ß} \in \mathbb{I}$.
- $\mathcal{A} \in \mathbb{I}$ and $\text{ß} \in \mathbb{I}$ implies $\mathcal{A} \cup \text{ß} \in \mathbb{I}$.

A topological space $(\mathbb{X},\mathbb{T})$ with an ideal $\mathbb{I}$ is called an ideal topological space or simply ideal space, if $P(\mathbb{X})$ is the set of all subsets of $\mathbb{X}$. A set operator $(\cdot)^*: P(\mathbb{X}) \to P(\mathbb{X})$ is called a local function [1] of a subset $\mathcal{A}$ with respect to the topology $\mathbb{T}$ and ideal $\mathbb{I}$ is defined as $A^*(\mathbb{X},\mathbb{T}) = \{x \in \mathbb{X} : \mathcal{W} \cap \mathcal{A} \notin \mathbb{I}, \forall \mathcal{W} \in \mathbb{T}(\mathbb{X})\}$ where $\mathbb{T}(\mathbb{X}) = \{\mathcal{W} \in \mathbb{T}: x \in \mathcal{W}\}$. A kuratowski closure operator $cl^*(\cdot)$ for a topology $\mathbb{T}^*(\mathbb{I},\mathbb{T})$, called the $*$-topology; finar than $\mathbb{T}$ is defined by $cl^*(\mathcal{A}) = \mathcal{A}^*(\mathbb{I},\mathbb{T}) \cup \mathcal{A}$ [2]. Levine [3]; velicko [4] introduced the notions of generalized closed (briefly g-closed) and δ-closed sets respectively and studied their basic properties. The notion of $\mathbb{I}g$-closed sets first introduced by Dontchev [5] in 1999; Navaneetha Krishanan and Joseph [6] further investigated and characterized $\mathbb{I}g$-closed sets. Julian Dontchev and

Maximilian Ganster [7]; Yuksel; Acikgoz and Noiri [8] introduced and studied the notions of δ-generalized closed (briefly δg-closed) and δ-$\mathbb{I}$-closed sets respectively.

Pre-$\hat{\delta}$-closed sets a novel type of sets that will be defined in this paper with fundamental characteristics.

2 Fundamental Concepts

Definition 2.1. Let $\mathcal{A}$ subset of a topological space ($\mathbb{X}$,$\mathbb{T}$,$\mathbb{I}$) is said a:

- Semi-open set [9] if $\mathcal{A} \subseteq cl\ (int(\mathcal{A}))$.
- Semi-closed set [10] if $int(cl(\mathcal{A})) \subseteq \mathcal{A}$.
- Pre-open set [11] if $\mathcal{A} \subseteq int(cl(\mathcal{A}))$.
- Pre-closed set [12] if $cl(int(\mathcal{A})) \subseteq \mathcal{A}$.
- Regular open set [13] if $\mathcal{A} = int(cl(\mathcal{A}))$.
- Regular closed set [14] if $\mathcal{A} = cl(int(\mathcal{A}))$.

The semi- closure (respectively, pre-closure) of a subset $\mathcal{A}$ of $(\mathbb{X}, \mathbb{T})$ is the intersection of all semi-closed (respectively, pre-closed) sets containing $\mathcal{A}$ and is denoted by $scl(\mathcal{A})$ (respectively, $pcl(\mathcal{A})$).

Definition 2.2. [8] Let $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ be an ideal topological-space, let $\mathcal{A}$ a subset of $\mathbb{X}$ and x is a point of $\mathbb{X}$, then: x is called a δ-$\mathbb{I}$-cluster points of $\mathcal{A}$ if $\mathcal{A} \cap (int\ cl^*(\mathcal{W})) \neq \phi$ for all open neighborhood $\mathcal{W}$ of x.
The family of each δ-$\mathbb{I}$-cluster points of $\mathcal{A}$ is said the δ-$\mathbb{I}$-closure of $\mathcal{A}$ and is denoted by $[\mathcal{A}]_{\delta-\mathbb{I}}$.
A subset $\mathcal{A}$ is called to be δ-$\mathbb{I}$-closed if $[\mathcal{A}]_{\delta-\mathbb{I}} = \mathcal{A}$.
The complement of a δ-$\mathbb{I}$-closed set of $\mathbb{X}$ is called to be δ-$\mathbb{I}$-open.

Remark 2.1. We can write $[\mathcal{A}]_{\delta-\mathbb{I}} = \{x \in \mathbb{X}: int(cl*(\mathcal{W}) \cap \mathcal{A} \neq \phi$, for all $\mathcal{W} \in \mathbb{T}(\mathbb{X})\}$. We use the notation σcl($\mathcal{A}$) =$[\mathcal{A}]_{\delta-\mathbb{I}}$.

Lemma 2.1. [8] Let $\mathcal{A}$ and ß be subset of an ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$. Then the following properties satisfy:

- $\mathcal{A} \subseteq \sigma cl(\mathcal{A})$.
- If $\mathcal{A} \subset$ ß, then $\sigma cl(\mathcal{A}) \subset \sigma cl(ß)$.
- $\sigma cl(\mathcal{A}) = \bigcap\{ \mathcal{G} \subset \mathbb{X} : \mathcal{A} \subset \mathcal{G}$ and $\mathcal{G}$ is δ-$\mathbb{I}$-closed$\}$.
- If $\mathcal{A}$ is δ-$\mathbb{I}$-closed set of $\mathbb{X}$ for all $\propto \epsilon \Delta$; then $\bigcap\{\mathcal{A}_{\propto}: \propto \epsilon \Delta\}$ is δ-$\mathbb{I}$-closed.
- $\sigma cl(\mathcal{A})$ is δ-$\mathbb{I}$-closed.

Remark 2.2. It is well-known that the family of regular open sets of $(\mathbb{X}, \mathbb{T})$ is a basis for a topology which is weaker than $\mathbb{T}$.
This topology is called the semi-regularization of $\mathbb{T}$ and is denoted by $\mathbb{T}_s$.
Actually, $\mathbb{T}_s$ is the same as the family of δ-open sets of $(\mathbb{X}, \mathbb{T})$.

Lemma 2.2. [8] Let $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ be an ideal topological-space, and $\mathbb{T}_{\delta-\mathbb{I}}$ =$\{\mathcal{A} \subset \mathbb{X} : \mathcal{A}$ is δ-$\mathbb{I}$-open set of $(\mathbb{X}, \mathbb{T}, \mathbb{I})\}$. Then $\mathrm{T}_{\delta-\mathbb{I}}$ is a topology such that $\mathbb{T}_s \subset \mathbb{T}_{\delta-\mathbb{I}} \subset \mathbb{T}$.

Remark 2.3. [8] $\mathbb{T}_s$ (respectively, $\mathbb{T}_{\delta-\mathbb{I}}$) is the topology created by the family of δ-open sets (respectively, δ-$\mathbb{I}$-open sets).

Lemma 2.3. Let ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$) be an ideal topology space, and $\mathcal{A}$ a subset of $\mathbb{X}$.
$\sigma cl(\mathcal{A})$ = {x ∊ $\mathbb{X}$: $\mathcal{A}$ ⋂ int(cl*($\mathcal{W}$)) ≠ ϕ; for each $\mathcal{W}$ ∊ $\mathbb{T}$($\mathbb{X}$)} is closed.
Proof. If x ∊ cl($\sigma cl(\mathcal{A})$) , and $\mathcal{W}$ ∊ $\mathbb{T}$($\mathbb{X}$), then $\mathcal{W}$ ⋂ $\sigma cl(\mathcal{A})$ ≠ ϕ.
Then y ∊ $\mathcal{W}$ ⋂ σcl(Ą) for some y ∊ $\mathbb{X}$.
Since $\mathcal{W}$ ∊ $\mathbb{T}$(y) and y ∊ $\sigma cl(\mathcal{A})$, from the definition of $\sigma cl(\mathcal{A})$ we have $\mathcal{A}$ ⋂ int(cl*($\mathcal{W}$)) ≠ ϕ. Therefore, x ∊ $\sigma cl(\mathcal{A})$. So cl($\sigma cl(\mathcal{A})$) ⊂ $\sigma cl(\mathcal{A})$ and hence $\sigma cl(\mathcal{A})$ is closed.

Definition 2.3. Let ($\mathbb{X}$,$\mathbb{T}$) be a topological-space; a subset $\mathcal{A}$ of $\mathbb{X}$ is said to be :

- g-closed set [3] if cl($\mathcal{A}$) ⊆ $\mathcal{W}$, whenever $\mathcal{A}$⊂ $\mathcal{W}$ and $\mathcal{W}$ is open in ($\mathbb{X}$, $\mathbb{T}$)
- δ-closed set [4] if $\mathcal{A}$ = $cl_\delta(\mathcal{A})$; where $\delta cl(\mathcal{A})$ =$cl_\delta(\mathcal{A})$= {x ϵ $\mathbb{X}$: (int(cl($\mathcal{W}$)) ⋂ $\mathcal{A}$ ≠ ϕ, $\mathcal{W}$ ϵ $\mathbb{T}$ and x ϵ $\mathcal{W}$}.
- δ-generalized closed set (short, δg-closed) set [7] if $cl_\delta(\mathcal{A})$ ⊆ $\mathcal{W}$, whenever $\mathcal{A}$ ⊆ $\mathcal{W}$ and $\mathcal{W}$ is open.
- δĝ-closed set [15] if $cl_\delta(\mathcal{A})$ ⊆ $\mathcal{W}$, whenever $\mathcal{A}$ ⊆ $\mathcal{W}$ and $\mathcal{W}$ is ĝ-0pen set in ($\mathbb{X}$, $\mathbb{T}$).

Definition 2.4. Let ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$) be an ideal space. A subset $\mathcal{A}$ of $\mathbb{X}$ is said to be:
$\mathbb{I}$g-closed set [5] if $\mathcal{A}$ * ⊆ $\mathcal{W}$, whenever $\mathcal{A}$ ⊆ $\mathcal{W}$ and $\mathcal{W}$ is open in $\mathbb{X}$.

Definition 2.5. [16] Let ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$) be an ideal space, a subset $\mathcal{A}$ of $\mathbb{X}$ is called $\hat{\delta}$-closed if $\sigma cl(\mathcal{A})$ ⊂ $\mathcal{W}$, whenever $\mathcal{A}$ ⊂ $\mathcal{W}$, and $\mathcal{W}$ is open in ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$).
The complement of $\hat{\delta}$-closed set in ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$) is called $\hat{\delta}$-open set in ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$).

Definition 2.6. In ideal topological-space ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$); let $\mathcal{A}$ ⊂ $\mathbb{X}$, $\mathcal{A}$ is called pre-$\hat{\delta}$-closed if $\sigma cl(\mathcal{A})$ ⊂ $\mathcal{W}$ whenever $\mathcal{A}$ ⊂ $\mathcal{W}$ and $\mathcal{W}$ is pre-open in ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$).
The complement of pre-$\hat{\delta}$-closed in ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$) is called pre-$\hat{\delta}$-open set in ($\mathbb{X}$, $\mathbb{T}$, $\mathbb{I}$).

Example 2.1. Let $\mathbb{X} = \{e_1, e_2, e_3\}$, $\mathbb{T} = \{\mathbb{X}, \phi, \{e_1\}, \{e_2\}, \{e_1, e_2\}\}$, $\mathbb{I} = \{\phi, \{e_3\}\}$.
Let $\mathcal{A} = \{e_1, e_3\}$ then $\mathcal{A}$ is pre-$\hat{\delta}$-closed.

Remark 2.4. Each Pre-$\hat{\delta}$-closed is $\hat{\delta}$-closed, but the opposite of is not true. It is clear from the following example.

Example 2.2. Let $\mathbb{X} = \{e_1, e_2, e_3, e_4\}$; $\mathbb{T} = \{\mathbb{X}, \phi, \{e_1\}, \{e_2\}, \{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\}$, and $\mathbb{I} = \{\phi, \{e_1\}\}$. Let $\mathcal{A} = \{e_1, e_4\}$, then $\mathcal{A}$ is $\hat{\delta}$-closed but not pre-$\hat{\delta}$-closed.

Remark 2.5. The collection of all pre-$\hat{\delta}$-closed set in ($\mathbb{X}, \mathbb{T}, \mathbb{I}$) denoted by $P\hat{\delta}C(\mathbb{X})$; the family of all pre-$\hat{\delta}$-open sets denoted by P$\hat{\delta}$O($\mathbb{X}$).

Definition 2.7. [17]
A function $\Gamma: (\mathbb{X}, \mathbb{T}) \to (\mathbb{Y}, \mathbb{T})$, continuous function if $\Gamma^{-1}(\mathbb{Y})$ is an open set in $\mathbb{X}$, for all open set in $\mathbb{Y}$.

3 Some types of Continuous Function

In this section, a new classes of continuous functions are defined and studied the relations between these concepts.

Definition 3.1.
A function $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ is called:

- Pre-$\hat{\delta}$-continuous function denoted by p$\hat{\delta}$-continuous function, if $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set in $\mathbb{X}$, whenever $\mathcal{V}$ is a pre-$\hat{\delta}$-open set in $\mathbb{Y}$.
- $\hat{\delta}$-irresolute function if $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$, whenever $\mathcal{V}$ is a $\hat{\delta}$-open set in $\mathbb{Y}$.
- $\hat{\delta}$-continuous function if $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$, whenever $\mathcal{V}$ is a pre-$\hat{\delta}$-open set in $\mathbb{Y}$.
- Strongly pre-$\hat{\delta}$-continuous function denoted by strongly p$\hat{\delta}$-continuous function if $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set in $\mathbb{X}$, whenever $\mathcal{V}$ is a $\hat{\delta}$-open set in $\mathbb{Y}$.

The following diagram shows the relations among the variant concepts were introduced in Definition (3.1).

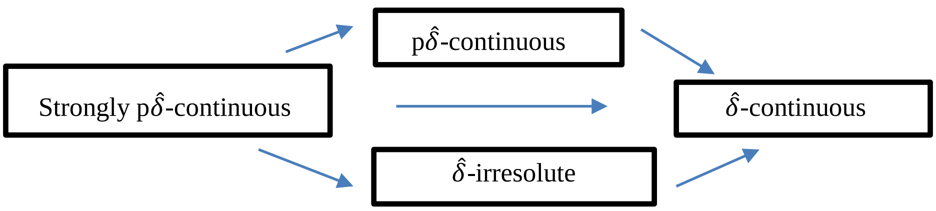

Figure 3.1: Relationships among the function that defined in Definition (3.1).

Proposition 3.1.
If the function $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ is a strongly p$\hat{\delta}$-continuous function then f is a p$\hat{\delta}$-continuous function.
Proof. Let f be a strongly p$\hat{\delta}$-continuous function.
Let $\mathcal{V} \in \mathbb{T}'$ and $\mathcal{V}$ is a pre-$\hat{\delta}$-open set, Since every pre-$\hat{\delta}$-open is a $\hat{\delta}$-open.
Then $\mathcal{V}$ is a $\hat{\delta}$-open set in $\mathbb{Y}$.This implies $f^{-1}(\mathcal{V}) \in \mathbb{T}$,
and $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open. so f is a p$\hat{\delta}$-continuous function.

Proposition 3.2.
If the function f is a strongly p$\hat{\delta}$-continuous function then f is a $\hat{\delta}$-irresolute function.
Proof. Let f be a strongly p$\hat{\delta}$-continuous function.

Let $\mathcal{V}$ be a $\hat{\delta}$-open set in $\mathbb{Y}$. Since f is a strongly p$\hat{\delta}$-continuous function,
so $f^{-1}(\mathcal{V}) \in \mathbb{T}$ and $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set.
Since every pre-$\hat{\delta}$-open set is a $\hat{\delta}$-open set.
This implies $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$.
Then f is a $\hat{\delta}$-irresolute function.

Proposition 3.3.
If the function f is a $\hat{\delta}$-irresolute function then f is a $\hat{\delta}$-continuous function.
Proof. Let f be a $\hat{\delta}$-irresolute function. Let $\mathcal{V}$ be a pre-$\hat{\delta}$-open set in $\mathbb{Y}$.
Then $\mathcal{V}$ be a $\hat{\delta}$-open in $\mathbb{Y}$, since every pre-$\hat{\delta}$-open is a $\hat{\delta}$-open.
Then $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$. Then f is a $\hat{\delta}$-continuous function.

Proposition 3.4.
If the function f is a p$\hat{\delta}$-continuous function then f is a $\hat{\delta}$-continuous function.
Proof. Let f be a p$\hat{\delta}$-continuous function.
Let $\mathcal{V}$ be a pre-$\hat{\delta}$-open set in $\mathbb{Y}$.
Then $f^{-1}(\mathcal{V}) \in \mathbb{T}$ and $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set.
Since every pre-$\hat{\delta}$-open is a $\hat{\delta}$-open set.
Then $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$.
Hence f is a $\hat{\delta}$-continuous function.

Corollary 3.1.
If the function f is a strongly p$\hat{\delta}$-continuous function then f is a $\hat{\delta}$-continuous function.
Proof. Let f be a strongly p$\hat{\delta}$-continuous function.
Let $\mathcal{V}$ be a pre-$\hat{\delta}$-open set in $\mathbb{Y}$. Since every pre-$\hat{\delta}$-open is a $\hat{\delta}$-open set.
Since f is a strongly p$\hat{\delta}$-continuous function.
Therefore $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set in $\mathbb{X}$.
so $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$ .Then f is a $\hat{\delta}$-continuous function.

Proposition 3.5.
If $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ and $g: (\mathbb{Y}, \mathbb{T}', \mathbb{I}') \to (\mathbb{W}, \mathbb{T}'', \mathbb{I}'')$ are both p$\hat{\delta}$-continuous function then $gof: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{W}, \mathbb{T}'', \mathbb{I}'')$ is a p$\hat{\delta}$-continuous function.
Proof. If $\mathcal{A} \subseteq \mathbb{W}$ is a pre-$\hat{\delta}$-open. Then $g^{-1}(\mathcal{A})$ is a pre-$\hat{\delta}$-open in $\mathbb{Y}$.
Then $f^{-1}(g^{-1}(\mathcal{A}))$ is a pre-$\hat{\delta}$-open in $\mathbb{X}$,
since g and f are p$\hat{\delta}$-continuous.
Thus $(gof)^{-1}(\mathcal{A}) = f^{-1}(g^{-1}(\mathcal{A}))$ is a pre-$\hat{\delta}$-open in $\mathbb{X}$.
Then gof is a p$\hat{\delta}$-continuous function.

Proposition 3.6.
If $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ and $g: (\mathbb{Y}, \mathbb{T}', \mathbb{I}') \to (\mathbb{W}, \mathbb{T}'', \mathbb{I}'')$ are both $\hat{\delta}$-irresolute functions then $gof: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{W}, \mathbb{T}'', \mathbb{I}'')$ is a $\hat{\delta}$-irresolute functions.

Proof. If $\mathcal{A} \subseteq \mathbb{W}$ is a $\hat{\delta}$-open. Then $g^{-1}(\mathcal{A})$ is a $\hat{\delta}$-open in $\mathbb{Y}$.
Then $f^{-1}(g^{-1}(\mathcal{A}))$ is a $\hat{\delta}$-open in $\mathbb{X}$,
since g and f are $\hat{\delta}$-irresolute functions.
Thus $(gof)^{-1}(\mathcal{A}) = f^{-1}(g^{-1}(\mathcal{A}))$ is a $\hat{\delta}$-open in $\mathbb{X}$.
Then gof is a $\hat{\delta}$-irresolute function.

Proposition 3.7.
Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{W}$ be any ideal spaces for any p$\hat{\delta}$-continuous function $f: \mathbb{X} \to \mathbb{Y}$ and any strongly p$\hat{\delta}$-continuous function $g: \mathbb{Y} \to \mathbb{W}$ then composition $gof: \mathbb{X} \to \mathbb{W}$ is strongly p$\hat{\delta}$-continuous function.
Proof. Let $\mathcal{A} \subseteq \mathbb{W}$ be a $\hat{\delta}$-open. Then $g^{-1}(\mathcal{A})$ is a pre-$\hat{\delta}$-open in $\mathbb{Y}$,
since g is strongly p$\hat{\delta}$-continuous function, by the p$\hat{\delta}$-continuous of f.
$f^{-1}(g^{-1}(\mathcal{A}))$ is a pre-$\hat{\delta}$-open in $\mathbb{X}$. But $(gof)^{-1}(\mathcal{A}) = f^{-1}(g^{-1}(\mathcal{A}))$.
Hence $(gof)^{-1}(\mathcal{A})$ is a pre-$\hat{\delta}$-open in $\mathbb{X}$.
Then gof is a strongly p$\hat{\delta}$-continuous function.

Proposition 3.8.
Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{W}$ be any ideal spaces for any $\hat{\delta}$-irresolute function $f: \mathbb{X} \to \mathbb{Y}$ and any $\hat{\delta}$-continuous function $g: \mathbb{Y} \to \mathbb{W}$ then composition $gof: \mathbb{X} \to \mathbb{W}$ is a $\hat{\delta}$-continuous function.
Proof. Let $\mathcal{A} \subseteq \mathbb{W}$ be a pre-$\hat{\delta}$-open. Then $g^{-1}(\mathcal{A})$ is a $\hat{\delta}$-open in $\mathbb{Y}$,
since g is a $\hat{\delta}$-continuous function. By the $\hat{\delta}$-irresolute function of f.
$f^{-1}(g^{-1}(\mathcal{A}))$ is a $\hat{\delta}$-open in $\mathbb{X}$. But $(gof)^{-1}(\mathcal{A}) = f^{-1}(g^{-1}(\mathcal{A}))$.
Hence $(gof)^{-1}(\mathcal{A})$ is a $\hat{\delta}$-open in $\mathbb{X}$.
Then gof is a $\hat{\delta}$-continuous function.

Remark 3.1.[18]
Let ($\mathbb{X}$, $\mathbb{T}$) be a topological space if every dense is open then every pre-open set is open set.

Remark 3.2.
Let ($\mathbb{X}$,$\mathbb{T}$,$\mathbb{I}$) be an ideal space, and since every pre-open set is open set by (Remark 3.1) then every $\hat{\delta}$-closed is pre-$\hat{\delta}$-closed.

Definition 3.2.
Let ($\mathbb{X}$,$\mathbb{T}$,$\mathbb{I}$) be an ideal space is said to be a $\mathbb{T}_{P\hat{\delta}}$-space if every $\hat{\delta}$-open subset of $\mathbb{X}$ is pre-$\hat{\delta}$-open.

Theorem 3.1.
Let $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ be a function if $\mathbb{Y}$ is a $\mathbb{T}_{P\hat{\delta}}$-space then f is a strongly p$\hat{\delta}$-continuous function if and only if it is a p$\hat{\delta}$-continuous function.
Proof. Suppose $\mathbb{Y}$ is a $\mathbb{T}_{P\hat{\delta}}$-space and f is a p$\hat{\delta}$-continuous function.
Let $\mathcal{V}$ be a $\hat{\delta}$-open set in $\mathbb{Y}$. Then $\mathcal{V}$ is a pre-$\hat{\delta}$-open set,

since $\mathbb{Y}$ is a $\mathbb{T}_{P\hat{\delta}}$-space. Then $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set in $\mathbb{X}$,
since f is a p$\hat{\delta}$-continuous function.
Then f is a strongly p$\hat{\delta}$-continuous function.
Converse is obvious (Proposition 3.1).

Theorem 3.2.
Let $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ be a function if X is a $\mathbb{T}_{P\hat{\delta}}$-space then f is a strongly p$\hat{\delta}$-continuous function if and only if it is a $\hat{\delta}$-irresolute function.
Proof. Suppose $\mathbb{X}$ is a $\mathbb{T}_{P\hat{\delta}}$-space and f is a $\hat{\delta}$-irresolute function.
Let $\mathcal{V}$ be a $\hat{\delta}$-open set in $\mathbb{Y}$. Then $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$.
Then $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set in $\mathbb{X}$.
Then f is a strongly p$\hat{\delta}$-continuous function.
Converse is obvious (Proposition 3.2).

Theorem 3.3.
Let $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ be a function if $\mathbb{Y}$ is a $\mathbb{T}_{P\hat{\delta}}$-space then f is a $\hat{\delta}$-irresolute function if and only if it is a $\hat{\delta}$-continuous function.
Proof. Suppose $\mathbb{Y}$ is a $\mathbb{T}_{P\hat{\delta}}$-space and f is a $\hat{\delta}$-continuous function.
Let $\mathcal{V}$ be a $\hat{\delta}$-open set in $\mathbb{Y}$. Then $\mathcal{V}$ is a pre-$\hat{\delta}$-open set in $\mathbb{Y}$.
Then $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$. Then f is a $\hat{\delta}$-irresolute function.
Converse is obvious (Proposition 3.3).

Theorem 3.4.
Let $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ be a function if $\mathbb{X}$ is a $\mathbb{T}_{P\hat{\delta}}$-space then f is a p$\hat{\delta}$-continuous function if and only if it is a $\hat{\delta}$-continuous function.
Proof. Suppose $\mathbb{X}$ is a $\mathbb{T}_{P\hat{\delta}}$-space and f is a $\hat{\delta}$-continuous function.
Let $\mathcal{V}$ be a pre-$\hat{\delta}$-open set in $\mathbb{Y}$.
Then $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$. Then $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set in $\mathbb{X}$.
Then f is a p$\hat{\delta}$-continuous function.
Converse is obvious (Proposition 3.4).

Theorem 3.5.
Let $f: (\mathbb{X}, \mathbb{T}, \mathbb{I}) \to (\mathbb{Y}, \mathbb{T}', \mathbb{I}')$ be a function if $\mathbb{X}$ and $\mathbb{Y}$ are a $\mathbb{T}_{P\hat{\delta}}$-spaces then f is a strongly p$\hat{\delta}$-continuous function if and only if it is a $\hat{\delta}$-continuous function.
Proof. Suppose $\mathbb{X}$ and $\mathbb{Y}$ are a $\mathbb{T}_{P\hat{\delta}}$-spaces and f is a $\hat{\delta}$-continuous function. Let $\mathcal{V}$ be a $\hat{\delta}$-open set in $\mathbb{Y}$.
Let $\mathcal{V}$ be a pre-$\hat{\delta}$-open set in $\mathbb{Y}$. Then $f^{-1}(\mathcal{V})$ is a $\hat{\delta}$-open set in $\mathbb{X}$.
Then $f^{-1}(\mathcal{V})$ is a pre-$\hat{\delta}$-open set in $\mathbb{X}$.
Then f is a strongly p$\hat{\delta}$-continuous function.
Converse is obvious (Corollary 3.1) .

4 Conclusion

In this work, several properties of p$\hat{\delta}$-continuous, $\hat{\delta}$-irresolute, $\hat{\delta}$-continuous, strongly p$\hat{\delta}$-continuous functions were studied and the relationship between p$\hat{\delta}$-continuous and $\hat{\delta}$-continuous, strongly p$\hat{\delta}$-continuous and (p$\hat{\delta}$-continuous, $\hat{\delta}$-continuous, $\hat{\delta}$-irresolute) also between $\hat{\delta}$-irresolute and $\hat{\delta}$-continuous in ideal topological spaces, as well as the relationship between them in $\mathbb{T}_{p\hat{\delta}}$ –space.

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