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The Core of Inner ideal of Real Four-Dimensional Lie Algebra with One Dimensional Derived

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ABSTRACT: Suppose that V is any inner ideal of L. The core of V is an inner ideal of L with special requirement. In this paper we prove. If L is a 4-dimension Lie algebra a with 1-dimensional derived, then the core of every inner ideal of L is zero. Moreover L containing a sandwich element if $L' \nsubseteq Z$ and every element in L is sandwich if $L' \subseteq Z$.

Keywords: Lie algebras, The $core_L$(V), Sandwich element, four dimensional

1. INTRODUCTION

The idea of inner ideals was initially introduced in 1976 by American scientist Georgia Benkart [1]. According to Benkart, an inner ideal or (simply $I - ideal$) is a subspace V of a Lie algebra L such that $[V,[V,L]] \subseteq V$. If $[V,V] = 0$, then an inner ideal V is said to be commutative. Inner ideals of the Lie subalgebra of simple associative rings were studied in [1] and fully classified in 2009 by Benkart and Fernandiz Lopez in [2]. In 2016, Brox, Fernandez Lopez, and Gomez Lozano examined the case of centrally closed prime rings with involution of characteristic not equal 2, 3, or 5 [3].

Premet explains in [4] and [5] that the inner ideals of Lie algebras perform a similar role to the one-sided ideal of algebras of associative relations; hence, by taking into account the inner ideals of Lie algebras, one can provide instruction for Artinian theory for Lie algebras. Lie algebra is called Artinian if and only if it possesses a descending chain of inner ideals [6]. Shlaka, and Mousa, in 2023 study inner ideals of the special linear Lie algebras of associative simple finite dimensional algebras (see [7]). It is demonstrated in [8, Proposition 2] that each one-sided ideal of a finite dimensional associative algebra A accepts a Levi decomposition and, under certain minimal circumstances, can be created by an idempotent. Baranov and Shlaka achieve the same results for inner ideals in [8], demonstrating that each inner ideal of the Lie algebra $[A,A]$ permits Levi decomposition and may be created by a pair of idempotent elements (if it satisfies some minimal conditions) [9].

Additional incentive to study inner ideals arises from [10], where Fernández López et al demonstrated that when L is any non-degenerate Lie algebra over an abelian ring F, equipped with two and three invertible elements, then every nonzero commutative inner ideal V of finite length in L is complemented by a commutative inner ideal [10].

In 2023, Shlaka and Saeed [11] studied inner ideal of the four-dimensional Lie algebras depending on Schobel classification of Lie algebras in [15]. Shlaka, and Kareem, described abelian non-Jordan-Lie inner ideals of the orthogonal finite dimensional Lie algebras in [13]. The notion of core of an inner ideal is introduced in [9] by Baranov and Shlaka. Let V be an inner ideal of a Lie algebra L. It is well-known that $[V,[V,L]]$ is also an $I - ideal$ of L. Take $V_0 = V$ and consider the following $I - ideal$ of L : $V_n = [V_{n-1},[V_{n-1},L]] \subseteq V$, for all integers n ≥ 1. Then $V = V_0 \supseteq$

$V_1 \supseteq V_2 \supseteq \cdots$. As L is finite dimension, this series terminates, so there is an integer n such that $V_n = V_{n+1}$. Such V_n is said to be the core of V, is denoted by $core_L(V)$.

In this paper, we study the $core_L(V)$ of the 4- dimensional Lie algebras with 1-dimensional derived. We also study sandwich elements of the four-dimensional Lie algebras with 1-dimensional derived. In section two, we start with some preliminary information about inner ideal (or simply I -ideal), the $cor_L(V)$ of I-ideal, sandwich elements. The third section we state basic concept about 4-dimensional Lie algebra with a 1-dimensional derived. In section four, we showed that $core_L(V) = 0$ for every I-ideal of a real 4-dimensinal Lie algebra with a 1-dimensional derived. Section five we proved that if L is a 4-dimensinal Lie algebra with a 1-diensional derived then L contain sandwich element if $L' \nsubseteq Z$. Moreover, every element in L is sandwich elements if $L' \subseteq Z$.

2. PRELIMINARIES

In this section we state some definitions about I-ideal, the cor_L(V) of inner ideal, extremal and zero divisor or (sandwich)element.

Definition 2.1 [8]: Suppose that V is a subspace of L. Then V is said to be an $I-ideal$ of L when $[V,[V,L]] \subseteq V$. We denoted by I-ideal to be an inner ideal of L. The I-ideal is said to be commutative if $[V,V] = 0$.

Let V be an inner ideal of L. Then $[V,[V,L]] \subseteq V$. It's widely recognized $[V,[V,L]] \subseteq V$ is an I-ideal of L (see [11, Lemma 1.1]). Take $V_0 = V$ and suppose that the following $I-ideals$ of : $V_n = [V_{n-1},[V_{n-1},L]] \subseteq V_{n-1}$ for all $n \geq 1$. Then $= V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$. As L is finite dimension, this series terminates. This motivates the following definition.

Definition 2.2. [9]: let V be an I-ideal of L and L as a finite dimension Lie algebra. Then there is an integer n such that $V_n = V_{n-1}$. Thus V_n is called the core of V, denoted by $core_L(V)$.

Definition 2.3.[14]: Let $x \in L$ be non zero element, then x is called extremal if $[x,[x,L]] \subseteq Fx$, where $Fx = span\{x \mid x \in L\}$.
The element x is said to be zero-divisor (sandwich) if $\{x \in L \mid [x,[x,L]] = 0\}$.

3. BASIC CONCEPT OF REAL FOUR-DIMENSIONAL LIE ALGEBRA WITH 1-DIMENSIONAL DERIVED

In this paper $p \geq 0$, L, V and Z are field of any characteristic, the characteristic of R. Lie algebra over R. Inner ideal of L, and the center of L, respectively. Recall that the center of L. Z_n n-dimensional center. We denoted by U_2 is two-dimensional Lie algebra with the following property $[u_1,u_2] = u_1$, where is a basis for U_2. H_j is j-dimensional Lie algebra with the following property $[h_2,h_3] = [h_4,h_5] = \cdots = [h_{j-1},h_j] = h_1$, and $[h_n,h_m] = 0$ otherwise, where $h_1,h_2,\ldots h_j$ are a basis for H_j.
Let L be a four-dimensional. The dimension of L' may be 1,2 3 or 4 . In [15] Schöbel classified the real four-dimensional Lie algebra by relating the dimension of L'. Suppose that the dimensional of L' is 1. Then we have the following result. For the proof see [15].

Theorem 3.1. *[15]: Consider L as a real n-dimensional Lie algebra with a 1-dimensional derived. Then L is one of the following :*

1. If $L' \nsubseteq Z$, then $L = U_2 \oplus Z_{n-2}$.
2. If $L' \subseteq Z$, then $L = H_j \oplus Z_{n-j}$ $(j = 2m-1, m \geq 2)$.

Where in the case (1) we have $L' \nsubseteq Z$, while in the case (2) we have $L' \subseteq Z$,

Proposition 3.4. *[11]: If $L = U_2 \oplus Z_2$ and $L' \nsubseteq Z$, then*

1. L contains a 1-dimensional I-ideal.
2. L contains a 2-dimensional non-commutative I-ideal.

3. L contains a 3-dimensional non commutative I-ideal.

Remark 3.5. According to [11], if L is a real 4-dimensinal with 1-dimensional derived and $L' \nsubseteq Z$, then L has basis $\{ u_1, u_2, z_1, z_2 \}$ and the Lie multiplication of this basis is $[u_1, u_2] = u_1$ and otherwise is zero. Thus

1. $span\{u_1\}$, $span\{z_1\}$ and $span\{z_2\}$ are all 1-dimensional I-ideal. However, $span\{u_2\}$ is un I-ideal as proved in [11, Remark 3.3].
2. The only non-commutative I-ideal of a 2-dimensional I-ideal of L is $span\{u_1, u_2\}$.
3. The only non-commutative I-ideal of L are span $\{u_1, u_2, z_1\}$ and $span\{u_1, u_2, z_2\}$.

Proposition 3.6. *[11]: Let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 1-dimensional subspace of L is an I-ideal.*

Proposition 3.7. *[11] Let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 2-dimensional subspace of L is an I-ideal.*

Proposition 3.8. *[11] Let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 3-dimensional subspace of L is an I-ideal.*

4. THE CORE OF THE REAL FOUR-DIMENSIONAL LIE ALGEBRA WITH ONE-DIMINSIONAL DERIVED

In this section we show that, if L is a real 4-dimensitional with 1-dimensional derived. Then L has a $core_L(V) = 0$ for every I-ideal.

Proposition 4.1. *Let $L = U_2 \oplus Z_2$ and $L' \nsubseteq Z$, then*

1. $cor_L(V) = 0$ for every 1-dimensional I-ideal.
2. $cor_L(V) = 0$ for every 2-dimensional I-ideal.
3. $cor_L(V) = 0$ for every 3-dimensional I-ideal.

Proof. By Theorem 3.1 L has basis $\{ u_1, u_2, z_1, z_2 \}$ such that $[u_1, u_2] = u_1$ and otherwise is zero, and by Proposition 3.2, if $L = U_2 \oplus Z_2$ and $L' \nsubseteq Z$, then L contain a 1, 2 and 3-dimensional I-ideal. Suppose that $y \in L$. Then $y = \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2$ for some $\lambda, \mu, \alpha, \beta \in R$.

1. Let V be a 1-dimensional I-ideal of L. Then by Remark 3.5 $V = span\{u_1\}$ or $V = span\{z_1\}$ or $V = span\{z_2\}$. Since $z_1, z_2 \in Z(L)$, it is clear that $core_L(V) = 0$ if $V = span\{z_1\}$ or $span\{z_2\}$. It remains to $V = span\{u_1\}$. We need to show $cor_L(V) = 0$. Let $x \in cor_L(V)$, $a, b \in V$. Then $a = \lambda_1 u_1$ and $b = \mu_1 u_1$ for some $\lambda_1, \mu_1 \in R$.

Since $x = [a, [b, y]] = [\lambda_1 u_1, [\mu_1 u_1, \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2]]$

$$= [\lambda_1 u_1, \mu_1 \lambda [u_1, u_1] + \mu_1 \mu [u_1, u_2] + \mu_1 \alpha [u_1, z_1] + \mu_1 \beta [u_1, z_2]]$$

$$= [\lambda_1 u_1, \mu_1 \mu u_1] = \lambda_1 \mu_1 \mu [u_1, u_1] = 0,$$

so $cor_L(V) = 0$.

2. Let V be a 2-dimensional I-ideal of L. Then by Remark 3.5 $V = span\{u_1, u_2\}$ is only non-commutative I-ideal of L. We claim that $cor_L(V) = span\{u_1\}$ and $cor_L(V) \subseteq V_1$, we need to show $V_1 \subseteq cor_L(V)$.
Since
$$V_1 = [V, [V, L]]$$
let $x \in V_1$, then there exists $a, b \in V$ and $y \in L$. Then $a = \alpha_1 u_1 + \beta_1 u_2$, $b = \alpha_2 u_1 + \beta_2 u_2$ and $y = \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda, \mu, \alpha, \beta \in R$.

Thus
$$x = [a, [b, y]]$$

$= [\, a, [\, \alpha_2\, u_1 \,+\, \beta_2\, u_2 \,, \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2 \,]\,]$
$= [\, \alpha_1\, u_1 \,+\, \beta_1\, u_2 \,, \alpha_2\, \mu\, u_1 \,-\, \beta_2\, \lambda\, u_1 \,]$
$= \alpha_1\, \alpha_2\, [\, u_1 \,, u_1 \,] \,-\, \alpha_1\, \beta_2 [\, u_1 \,, u_1 \,] \,+\, \beta_1\, \alpha_2\, [\, u_2 \,, u_1 \,] \,-\, \beta_1\, \beta_2\, \lambda\, [\, u_2 \,, u_1 \,]$
$= -\, \beta_1\, \alpha_2\, u_1 \,+\, \beta_1\, \beta_2\, \lambda\, u_1 \,=\, (\, \beta_1\, \beta_2\, \lambda \,-\, \beta_1\, \alpha_2 \,) u_1 \,\in\, span\{\, u_1 \,\}$.
and

$$V_2 \,=\, [\, V_1 \,, [\, V_1 \,, L \,]\,]\,.$$

Let $x_1 \,\in\, V_2$, then there exists $a_1 \,, b_1 \,\in V_1$ and $y \,\in\, L$. Then $a_1 \,=\, \alpha_1\, u_1, b_1 \,=\, \alpha_2\, u_1$ and $y = \,\lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2$ for some $\alpha_1 \,, \beta_1 \,, \lambda \,, \mu \,, \alpha \,, \beta \,\in\, R$.
Thus
$x_1 = [\, a_1 \,, [\, b_1 \,, y \,]\,]$
$= [\, \alpha_1\, u_1 \,, [\, \beta_1\, u_1 \,, \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2 \,]\,]$
$= [\, \alpha_1\, u_1 \,, \beta_1\, \mu\, u_1 \,] \,=\, \alpha_1\, \beta_1\, [\, u_1 \,, u_1 \,] \,= 0,$
so $co\, r_L\, (\, V \,) \,= 0.$

3. Let V be a 3-dimensional I-ideal of L. Then by Remark 3.5 $V = span\{\, u_1 \,, u_2 \,, z_1 \,\}$ and $V = span\{\, u_1 \,, u_2 \,, z_2 \,\}$ are only non-commutative I-ideal of L . We claim that $co\, r_L\, (\, V \,) \,=\, span\{\, u_1 \,\}$ and $co\, r_L\, (\, V \,) \,\subseteq\, V_1$, we need to show $V_1 \,\subseteq\, co\, r_L\, (\, V \,)$. Since

$$V_1 \,=\, [\, V, [\, V, L \,]\,]$$

let $x \,\in\, V_1$, then there exists $a, b \,\in\, V$ and $y \,\in\, L$. Then $a = \,\alpha_1\, u_1 \,+\, \beta_1\, u_2 \,+\, \gamma_1\, z_1 \,, b = \,\alpha_2\, u_1 \,+\, \beta_2\, u_2 \,+\, \gamma_2\, z_1$ and $y = \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2$ for some $\alpha_1 \,, \alpha_2 \,, \beta_1 \,, \beta_2 \,, \gamma_1 \,, \gamma_2 \,, \lambda \,, \mu \,, \alpha \,, \beta \,\in\, R$.
Thus
$x = [\, a, [\, b, y \,]\,]$
$= [\, a, [\, \alpha_2\, u_1 \,+\, \beta_2\, u_2 \,+\, \gamma_2\, z_1 \,, \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2 \,]\,]$
$= [a, \alpha_2\;\lambda\, [\, u_1 \,, u_1 \,] \,+\, \alpha_2\;\mu\, [\, u_1 \,, u_2 \,] \,+\, \alpha_2\;\alpha\, [\, u_1 \,, u_1 \,] \,+\, \alpha_2\;\beta\, [\, u_1 \,, z_2 \,] \,+\, \beta_2\;\lambda\, [\, u_2 \,, v_1 \,]$
$+\, \beta_2\;\mu\, [\, u_2 \,, v_2 \,] \,+\, \beta_2\;\alpha\, [\, u_2 \,, z_1 \,] \,+\, \beta_2\;\beta\, [\, u_2 \,, z_2 \,] \,+\, \gamma_2\;\lambda\, [\, u \,, v_1 \,] \,+\, \gamma_2\;\mu\, [\, u_1 \,, u_2 \,]$
$+\, \gamma_2\;\alpha\, [\, z_1 \,, z_1 \,] \,+\, \gamma_2\;\beta\, [\, z_1 \,, z_2 \,]$
$= [\, \alpha_1\, u_1 \,+\, \beta_1\, u_2 \,+\, \gamma_1\, z_1 \,, \alpha_2\, \mu\, u_1 \,-\, \beta_2\, \lambda\, u_1 \,]$
$= -\, \beta_1\, \alpha_2\, \mu\, u_1 \,+\, \beta_1\, \beta_2\, \lambda\, u_1 \,=\, (\, \beta_1\, \beta_2\, \lambda \,-\, \beta_1\, \alpha_2\, \mu) u_1 \,\in\, span\{\, u_1 \,\}$.
and

$$V_2 \,=\, [\, V_1 \,, [\, V_1 \,, L \,]\,]\,.$$

Let $x_1 \,\in\, V_2$, then there exists $a_1 \,, b_1 \,\in\, V$ and $y \,\in\, L$. Then $a_1 \,=\, \alpha_1\, u_1 \,, b_1 \,=\, \alpha_2\, u_1$ and $y = \,\lambda\, v_1 \,+\, \mu\, v_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2$ for some $\alpha_1 \,, \beta_1 \,, \lambda \,, \mu \,, \alpha \,, \beta \,\in\, R$.
Thus
$x_1 \,= [\, a_1 \,, [\, b_1 \,, y \,]\,]$
$= [\, \alpha_1\, u_1 \,, [\, \beta_1\, u_1 \,, \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2 \,]\,]$
$= [\, \alpha_1\, u_1 \,, \beta_1\, \mu\, u_1 \,] \,=\, \alpha_1\, \beta_1\, [\, u_1 \,, u_1 \,] \,= 0,$
so $co\, r_L\, (\, V \,) \,= 0.$

Proposition 4.2. *Let* $L = \, H_3 \,\oplus\, Z_1$ *and* $L' \,\subseteq\, Z$. *Then every 1-dimensional subspace of* L *has* $cor\, e_L\, (\, V \,) = 0$ *for every I-ideal.*

Proof. Let V be a 1-dimensional subspace of L and let $v \,\in\, V$ be a nonzero. Then $\{v\}$ from a basis of V. We extend $\{v\}$ to form a basis $\{\, h_1 \,, h_2 \,, h_3 \,, v\}$, where $h_1 \,, h_2 \,, h_3 \in H_3$ such that the Lie multiplication of this basis satisfy the condition of Theorem 3.1 and by Proposition 3.4 let $L = \, H_3 \,\oplus\, Z_1$ and $L' \,\subseteq\, Z$. Then every 1-dimensional subspace of L is an I-ideal. we need to show $cor\, e_L\, (\, V \,) \,= 0$. Let $x \,\in\, core_L\, (\, v \,)$, $a, b \,\in\, V$ and $y \,\in\, L$. Then $a = \,\lambda\, v, b = \,\mu\, v$ and $y = \,\alpha\, h_1 \,+\, \beta\, h_2 \,+\, \gamma\, h_3 \,+\, \delta\, v$ fore some $\lambda \,, \mu \,, \alpha \,, \beta \,, \gamma \,, \delta \,\in\, R$. Then

$$x = [a, [\, b, y \,] \; (1)$$

By Theorem 3.1 we need to consider four cases dependent on the multiplication of $h_1 \,, h_2 \,, h_3$ and v whether $[\, h_1 \,, h_2 \,] \,=\, h_1 \,, v \,\in\, Z_1$, or $[\, h_2 \,, h_3 \,] \,= v$; $h_1 \,\in\, Z_1$ or $[\, v, h_3 \,] \,=\, h_1 \,, h_2 \,\in\, Z_1$ or $[\, h_2 \,, v \,] \,=\, h_1 \,, h_3 \,\in\, Z_1$.

Case 1. Suppose initially that $V \,\in\, Z_1$ and $[\, h_2 \,, h_3 \,] \,= h_1$, otherwise is zero. By Equation 1
$x = [\, a, [\, b, y \,]\,] \,= [\, a, [\mu\, v, \alpha\, h_1 \,+\, \beta\, h_2 \,+\, \gamma\, h_3 \,+\, \delta\, v \,]\,]$
$= [\, a, \mu\, \alpha\, [\, v, h_1 \,] \,+\, \mu\, \beta\, [\, v, h_2 \,] \,+\, \mu\, \gamma\, [\, v, h_3 \,] \,+\, \mu\, \delta\, [\, v, v \,]\,]$
$= [\, a, 0 \,] \,= 0, so\; co\, r\, L\, (\, V \,) \,= 0.$

Suppose now that $h_1 \in Z_1$ and $[\, h_2 \,, h_3 \,] \,= v$, otherwise is zero. By Equation 1
$x = [\, a, [\, b, y \,]\,] \,= [\, a, 0 \,] \,= 0,$
$so\; cor\; e\; L\, (\, V \,) \,= 0.$
Suppose Next that $h_2 \,\in Z_3$ and $[\, v, h_3 \,] \,=\, h_1$, otherwise is zero. By Equation1

$$x = [\, a, [\, b, y\,]\,] = [\, a, [\, \mu\, v, \alpha\, h_1 + \beta\, h_2 + \gamma\, h_3 + \delta\, v\,]\,]$$
$$= [\, \lambda\, v, \mu\, \gamma\, [\, v, h\, 3\,] = [\, \lambda\, v, \mu\, \gamma\, h\, 1\,] = \lambda\, \mu\, \gamma\, [\, v, h\, 1\,] = 0$$
$$so\ cor\ e\ L\ (\, V\,) = 0\,.$$

Finally let $h_3 \in Z_1$ and $[\, h_2, v\,] = h_1$, otherwise is zero. By Equation1
$$x = [\, a, [\, b, y\,]\,] = [\, \lambda\, v, [\, \mu\, v, \alpha\, h_1 + \beta\, h_2 + \gamma\, h_3 + \delta\, v\,]\,]$$
$$= [\, \lambda\, v, -\mu\, \beta\, h_1\,] = -\lambda\, \mu\, \beta\, [\, v, h\, 1\,] = 0,$$
so $cor\ e_L\ (\, V\,) = 0$.

Proposition 4.3. *Let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 2-dimensional subspace of L has $cor\ e_L\ (\, V\,) = 0$ for every I-ideal.*

Proof. Let V be a two-dimension subspace of L, and let $v_1, v_2 \in V$ be a non-zero. Then $\{\, v_1, v_2\,\}$ from a basis of V. We extend $\{\, v_1, v_2\,\}$ to form a basis $\{\, h_1, h_2, v_1, v_2\,\}$ such that the Lie multiplication of this basis satisfies the condition of Theorem 3.1 and by Proposition 3.5 let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 2-dimensional subspace of L is an I-ideal. We need to show that $cor\ e_L\ (\, v\,) = 0$.
Let $x \in cor\ e_L\ (\, V\,)$, $a, b \in V$ and $y \in L$. Then $a = \lambda_1\, v_1 + \mu_1\, v_2$, $b = \lambda_2\, v_1 + \mu_2\, v_2$ and $y = \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2$ for $some\ \lambda_1, \lambda_2, \mu_1, \mu_2, \alpha, \beta, \gamma, \delta \in R$.
Then
$$x = [a, [\, b, y\,]\ (2)$$
By Theorem 3.1 we need to consider six cases depending on the multiplication of h_1, h_2, v_1 and v_2 whether $[\, v_2, h_1\,] = v_1$; $h_2 \in Z_1$ or $[\, h_1, v_2\,] = v_1$; $h_2 \in Z_1$ or $[\, h_1, h_2\,] = v_1$; $v_2 \in Z_1$ or $[\, v_1, v_2\,] = h_1$; $h_2 \in Z_1$ or $[\, v_1, h_2\,] = h_1$; $v_2 \in Z_1$ or $[\, h_2, v_1\,] = h_1$; $v_2 \in Z_1$

Case 1. Suppose first that $h_2 \in Z_1$ and $[\, v_2, h_1\,] = v_1$, otherwise is zero. By Equation 2
$$x = [a, [\, b, y\,]$$
$$= [\, a, [\, \lambda_2\, v_1 + \mu_2\, v_2, \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2\,]\,]$$
$$= \lambda\, 1\, \mu\, 1\, [\, v_1, v_1\,] + \lambda_1\, \mu_2\, \alpha\, [\, v_2, v_1\,] = 0,$$
$$so\ co\ r_L\ (\, V\,) = 0.$$

Case 2. Suppose now that $h_2 \in Z_1$ and $[\, h_1, v_2\,] = v_1$, otherwise is zero. By Equation 2
$$x = [a, [\, b, y\,] = [\, a, [\, \lambda_2\, v_1 + \mu_2\, v_2, \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2\,]\,]$$
$$= [\, a, -\mu_2\, \alpha\, v_1\,] = [\, \lambda_1\, v_1 + \mu_1\, v_2, -\mu_2\, \alpha\, v_1\,]$$
$$= -\lambda_1\, \mu_2\, \alpha\, [\, v_1, v_1\,] - \mu_1\, \mu_2\, \alpha\, [\, v_2, v_1\,] = 0,$$
$$so\ cor\ e_L\ (\, V\,) = 0.$$

Case 3. Suppose that $v_2 \in Z_1$ and $[\, h_1, h_2\,] = v_1$, otherwise is zero. By Equation 2
$$x = [a, [\, b, y\,] = [\, a, 0\,] = 0,$$
$$so\ cor\ e\ L\ (\, V\,) = 0.$$

Case 4. Suppose that $h_2 \in Z_1$ and $[\, v_1, v_2\,] = h_1$, otherwise is zero. By Equation 2
$$x = [a, [\, b, y\,] = [\, a, [\, \lambda_2\, v_1 + \mu_2\, v_2, \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2\,]\,]$$
$$= \lambda_1\, \mu_2\, [\, v_1, h_1\,] - \lambda_1\, \mu_2\, [\, v_1, h_1\,] - \mu_1\, \mu_2\, \delta\, [\, v_2, h_1\,] - \mu_1\, \mu_2\, \gamma\, [\, v_2, h_1\,] = 0,$$
$$so\ cor\ e_L\ (\, V\,) = 0.$$

case 5. Suppose that $v_2 \in Z_1$ and $[\, v_1, h_2\,] = h_1$, otherwise is zero. By Equation2
$$x = [a, [\, b, y\,] = [\, a, [\, \lambda_2\, v_1 + \mu_2\, v_2, \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2\,]\,]$$
$$= [a, \lambda_2\, \alpha\, [\, v_1, h_1\,] + \lambda_2\, \beta\, [\, v_1, h_2\,] + \lambda_2\, \gamma\, [\, v_1, v_1\,] + \mu_2\, \delta\, [\, v_1, v_2\,] + \mu_2\, \alpha\, [\, v_2, h_1\,] + \mu_2\, \beta\, [\, v_2, h_2\,]$$
$$+ \mu_2\, \gamma\, [\, v_2, v_1\,] + \mu_2\, \delta\, [\, v_2, v_2\,]$$
$$= [\, \lambda_1\, v_1 + \mu_1\, v_2, \lambda_2\, \beta\, h_1\,]$$
$$= \lambda_1\, \lambda_2\, \beta\, [\, v_1, h_1\,] + \mu_1\, \lambda_2\, \beta\, [\, v_2, h_1\,] = 0,$$
$$so\ cor\ e\ L\ (\, V\,) = 0.$$

Case 6. Suppose that $v_2 \in Z_1$ and $[\, h_2, v_1\,] = h_1$, otherwise is zero. By Equation 2
$$x = [a, [\, b, y\,] = [\, a, [\, \lambda\, 2\, v\, 1 + \mu\, 2\, v\, 2, \alpha\, h\, 1 + \beta\, h\, 2 + \gamma\, v\, 1 + \delta\, v\, 2\,]\,]$$
$$= [a, \lambda_2\, \alpha\, [\, v_1, h_1\,] + \lambda_2\, \beta\, [\, v_1, h_2\,] + \lambda_2\, \gamma\, [\, v_1, v_1\,] + \mu_2\, \delta\, [\, v_1, v_2\,] + \mu_2\, \alpha\, [\, v_2, h_1\,] + \mu_2\, \beta\, [\, v_2, h_2\,]$$
$$+ \mu\, 2\, \gamma\, [\, v\, 2, v\, 1\,] + \mu\, 2\, \delta\, [\, v\, 2, v\, 2\,]$$
$$= [\, \lambda_1\, v_1 + \mu_1\, v_2, -\lambda_2\, \beta\, h_1\,]$$
$$= -\lambda_1\, \lambda_2\, \beta\, [\, v_1, h_1\,] - \mu_1\, \lambda_2\, \beta\, [\, v_2, h_1\,] = 0,$$

so $cor\ e\ L\ (V) = 0.$

Proposition 4.4. *Let* $L = H_3 \oplus Z_1$ *and* $L' \subseteq Z$. *Then every 3-dimensional subspace of* L *has* $cor\ e_L\ (V) = 0$ *for every I-ideal.*

Proof. Let V be a 3-dimensinal subspace of L and $\{v_1, v_2, v_3\}$ be a basis of V. we extend $\{v_1, v_2, v_3\}$ to form a basis $\{h_1, v_1, v_2, v_3\}$ such that the Lie multiplication of this basis satisfies the condition Theorem 3.1 and by Proposition prop 3.6 let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 3-dimensional subspace of L is an I-ideal.
We need to show $cor\ e_L\ (V) = 0$. Let $x \in cor\ e_L\ (v)$, $a, b \in V$ and $y \in L$. Then $a = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, $b = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$ and $y = \lambda_1 h_1 + \lambda_2 v_1 + \lambda_3 v_2 + \lambda_4 v_3$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in R$. Then

$$x = [a, [b, y]\ (3)$$

By Theorem 3.1 we need to consider four cases depending on the multiplication of h_1, v_1, v_2 and v_3 whether $[v_2, v_3] = v_1$; $h_1 \in Z_1$ or $[v_2, h_1] = v_1$; $v_3 \in Z_1$ or $[h_1, v_2] = v_1$; $v_3 \in Z_1$ or $[v_1, v_2] = h_1$; $v_3 \in Z_1$.

Suppose first that $v_1, v_2, v_3 \notin Z$, and $[v_2, v_3] = v_1$ otherwise is zero, and $x_1 \in Z_1$. By Equation 3

$$x = [a, [b, y]]$$
$$= [a, [\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3, \lambda_1 h_1 + \lambda_2 v_1 + \lambda_3 v_2 + \lambda_4 v_3]$$
$$= [a, \beta_1 \lambda_1 [v_1, h_1] + \beta_1 \lambda_2 [v_1, v_1] + \beta_1 \lambda_3 [v_1, v_2] + \beta_1 \lambda_4 [v_1, v_3] + \beta_2 \lambda_1 [v_2, h_1] + \beta_2 \lambda_2 [v_2, v_1] + \beta_2 \lambda_3 [v_2, v_2] + \beta_2 \lambda_4 [v_2, v_3] + \beta_3 \lambda_1 [v_3, h_1] + \beta_3 \lambda_2 [v_3, v_1] + \beta_3 \lambda_3 [v_3, v_2] + \beta_3 \lambda 4 [v_3, v_3]$$
$$= [\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \beta_2 \lambda_4 v_1 - \beta_3 \lambda_3 v_1]$$
$$= \alpha_1 \beta_2 \lambda_4 [v_1, v_1] - \alpha_1 \beta_3 \lambda_3 [v_1, v_1] + \alpha_2 \beta_2 \lambda_4 [v_2, v_1] - \alpha_2 \beta_3 \lambda_3 [v_1, v_1] + \alpha_3 \beta_3 \lambda_3 [v_3, v_1] = 0,$$
$$so\ co\ r_L\ (V) = 0.$$

Suppose now that $v_1, v_2 \notin Z$, and $[v_2, h_1] = v_1$ otherwise is zero, and $v_3 \in Z_1$. Then

$$x = [a, [b, y]] = [\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \beta_2 \lambda_1 v_1]$$
$$= \alpha_1 \beta_2 \lambda_1 [v_1, v_1] + \alpha_2 \beta_2 \lambda_1 [v_2, v_1] + \alpha_3 \beta_2 \lambda_1 [v_3, v_1] = 0,$$
$$so\ co\ r_L\ (V) = 0.$$

Suppose next that $v_1, v_2 \notin Z$, and $[h_1, v_2] = v_1$ otherwise is zero, and $v_3 \in Z_1$. By Equation 3

$$x = [a, [b, y]]$$
$$= [\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, -\beta_2 \lambda_1 v_1]$$
$$= [-\alpha_1 \beta_2 \lambda_1 [v_1, v_1] - \alpha_1 \beta_2 \lambda_1 [v_2, v_1] - \alpha_3 \beta_2 \lambda_1 [v_3, v_1] = 0,$$
$$so\ cor\ e_L\ (V) = 0.$$

Finally let $v_1, v_2 \notin Z$, and $[v_1, v_2] = h_1$ otherwise is zero, and $[v_1, v_2] = h_1$. By Equation 3

$$x = [a, [b, y]]$$
$$= [\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \beta_1 \lambda_3 h_1 - \beta_2 \lambda_2 h_1]$$
$$= \alpha_1 \beta_1 \lambda_3 [v_1, h_1] - \alpha_1 \beta_2 \lambda_2 [v_1, h_1] + \alpha_2 \beta_1 \lambda_3 [v_2, h_1] - \alpha_2 \beta_2 \lambda_2 [v_2, h_1] + \alpha_3 \beta_1 \lambda_3 [v_1, h_1] - \alpha_1 \beta_2 \lambda_2 [v_2, h_1] = 0,$$
$$so\ cor\ e_L\ (V) = 0.$$

Theorem 4.5. Suppose that L is a real 4-dimansional Lie algebra with a 1-dimensional derived, then $cor_L(V) = 0$ for every I-ideal.

Proof. Since L is 4-dimension with a 1-dimensional derived. Then by Theorem 3.1 either $L = U_2 \oplus Z_2$ or $L = H_3 \oplus Z_1$.
Suppose first that $L = U_2 \oplus Z_2$. Then by proposition 4.1. $cor_L(V) = 0$ for every I-ideal.
Suppose now that $L = H_3 \oplus Z_1$.
Thus

1- By Proposition 4.2 every 1-dimensional subspace has $core_L(V) = 0$ for every I-ideal.
2- By Proposition 4.3 every 2-dimensional subspace has $core_L(V) = 0$ for every I-ideal.
3- By Proposition 4.4 every 3-dimensional subspace has $core_L(V) = 0$ for every I-ideal.

5. SANDWICH ELEMENTS OF A REAL FOUR DIMENSIONAL LIE ALGEBRA WITH 1-DIMENSIONAL DERIVED

In this section we proved that if L is a 4-dimensinal Lie algebra with a 1-diensional derived then L contain sandwich element if $L' \nsubseteq Z$. Moreover every elements in L is sandwich elements if $L' \subseteq Z$.

Proposition 5.1. *Let* $L = U_2 \oplus Z_2$ *and* $L' \nsubseteq Z$, *then* L *contain a sandwich element.*

Proof. By Theorem 3.1 L has basis $\{ u_1, u_2, z_1, z_2 \}$ such that $[u_1, u_2] = u_1$ and otherwise is zero .
Suppose that $x \in L$, and $y \in L$.
Then $x = u_1$ and $y = \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2$ for some $\lambda, \mu, \alpha, \beta \in R$.
Since

$$[u_1,[u_1,y]] = [u_1,[u_1,\lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2]]$$
$$= [u_1, \lambda [u_1,u_1] + \mu [u_1,u_2] + \alpha [u_1 z_1] + \beta [u_1, z_2]]$$
$$= [u_1, \mu u_1]$$
$$= \mu [u_1, u_1] = 0,$$

so u_1 is a sandwich elements.

Remark 5.2. Proposition 5.1 is not true if we state *let* $L = U_2 \oplus Z_2$ *and* $L' \nsubseteq Z$, *then every elements in L is a sandwich element.* As one can see in the following example.

Example 5.3. Consider $x \in L$ and $y \in L$. Then $x = u_2$ and $y = \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2$.
Since

$$[u_2,[u_2,y]] = [u_2,[u_2,\lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2]]$$
$$= [u_2, \lambda [u_2,v_1] + \mu [u_2,v_2] + \alpha [u_2,z_1] + \beta [u_2,z_2]]$$
$$= [u_2, -\lambda u_1] = -\lambda [u_2,u_1]$$
$$= \lambda u_1.$$

Thus, u_2 is non-sandwich elements.

Proposition 5.4. *Let* $L = H_3 \oplus Z_1$ *and* $L' \subseteq Z$. *Then every basis of* L *is a sandwich elements .*

Proof. By Theorem 3.1 L has basis $\{h_1, h_2, h_3, z\}$ such that $[h_2, h_3] = h_1$ and otherwise is zero. Let $y \in L$. Then $y = \alpha h_1 + \beta h_2 + \gamma h_3 + \delta z$, we need to show that each basis element is sandwich. Since $z \in Z(L)$ it is clear that z is sandwich element. It remain to show that h_1, h_2 and h_3 are all sandwich elements.

Suppose first that $h_1 \in L$ and $y \in L$. Then $y = \alpha h_1 + \beta h_2 + \gamma h_3 + \delta z$ fore some $\lambda, \mu, \alpha, \beta, \gamma, \delta \in R$.
Since

$$[h_1,[h_1,y]] = [h_1,[h_1,\alpha h_1 + \beta h_2 + \gamma h_3 + \delta z]$$
$$= [h_1, 0] = 0,$$

so h_1 is sandwich element.

Suppose now that $h_2 \in L$ and $y \in L$. Then $y = \alpha h 1 + \beta h 2 + \gamma h 3 + \delta z$ fore some $\lambda, \mu, \alpha, \beta, \gamma, \delta \in R$.
Since

$$[h_2,[h_2,y]] = [h_2,[h_2,\alpha h_1 + \beta h_2 + \gamma h_3 + \delta z]]$$
$$= [h_2, \alpha [h_2,h_1] + \beta [h_2,h_2] + \gamma [h_2,h_3] + \delta [h_2,z]]$$
$$= [h_2, \gamma h_1] = \gamma [h_2,h_1] = 0,$$

so h_2 is sandwich elements and let

Finally let $h_3 \in V$ and $y \in L$. Then $y = \alpha h_1 + \beta h_2 + \gamma h_3 + \delta z$ fore some $\lambda, \mu, \alpha, \beta, \gamma, \delta \in R$.
Since

$$[h_3,[h_3,y]] = [h_3,[h_3,\alpha h_1 + \beta h_2 + \gamma h_3 + \delta z]$$
$$= [h_3, \alpha [h_3,h_1] + \beta [h_3,h_2] + \gamma [h_3,h_3] + \delta [h_3,z]]$$
$$= [h_3, -\beta h_1] = -\beta [h_3,h_1] = 0,$$

so h_3 is sandwich elements.

Theorem 5.5. Suppose that L is a real 4-dimansional Lie algebra with a1-dimensional derived, then L contain a sandwich element if $L' \nsubseteq Z$ and every element in L is a sandwich if $L' \subseteq Z$.

Proof. Since L is 4-dimensional Lie algebra with a 1-dimensional derived. Then by Theorem 3.1 either $L = U_2 \oplus Z_2$ or $L = H_3 \oplus Z_1$.

Suppose first that $L = U_2 \oplus Z_2$. Then by proposition 5.1. L contain a sandwich element.

Suppose now that $L = H_3 \oplus Z_1$. Then by Proposition 5.4. L has a sandwich element for every basis of L.

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