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Some Results of Pseudo Two Absorbing Module

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ABSTRACT: In this entire paper V is any ring with identity and Y be a nitary left − module.The Pseudo Two absorbing module, is a new item of modules that we Presenting a comprehensive study of Pseudo Two absorbing module and giving many new characterizations and properties that is related to this concept. Farther- more, Defining and studying a new class of modules by using this concept

Keywords: Pseudo two absorbing modules; two absorbing module; Socle of module.

1. INTRODUCTION

In this entire paper let V be a commutative ring with identity, and let Y be a unitary left V-module. The concept of a prim e V-module by Saymach in 1979,[1] where A V-module Y is called Prime module if $< 0 >$ is a Prime submodule. Equivalently "a V-module Y is a Prime module if and only if annY = annB for each nonzero submodule B of Y. In 1999, Abdul-Razak [2], presented and studied a quasi-prime module where A V-module Y is said to be a quasi-Prime module if $< 0 >$ is a Prime submodule. Equivalently "a V-module Y is called a quasi-Prime module if and only if $\mathrm{ann}_V B$ is a prime ideal for each nonzero submodule B of Y. Harfash in 2015[3], presented the concept a T-ABSO module where A V-module Y is called 2-absorbing module if $< 0 >$ is a 2-absorbing submodule. In 2022[4], M. W. Allami and W. H. Hanoon, studied and presented the concept of a Socle (Pseudo)-T-Abso submodule of Y, A proper submodule $\mathcal{K}$ of a V – module Y is called Socle(Pseudo) two abso submodule of Y if when $a, b \in V, x \in Y$ such that $abx \in \mathcal{K}$, then $ax \in \mathcal{K} + \mathrm{Soc}(Y)$ or $bx \in \mathcal{K} + \mathrm{Soc}(Y)$ or $ab \in (\mathcal{K} + \mathrm{Soc}(Y) :_V Y)$.

In our work we study and present the concept as a generalization of Some result of Pseudo two absorbing module and give some of this concept quality. Also, many basic properties and characterizations of Pseudo two absorbing module are given.

2 BASIC CONCEPTS

Definition 2.1 [5]: " A proper ideal H of a ring V is called prime ideal if whenever a, b∈ V, ab∈ H then either a∈ H or b∈ H " [1]

Proposition 2.2 [3]: In a multiplication Z-module Z_{pq} is T-ABSO module when p and q are distinct prime numbers.

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Proposition 2.3 [6]: Let A be a submodule of a $V-$ module Y, then $\frac{A+Soc(Y)}{A} \subseteq Soc(\frac{Y}{A})$, and if Y is semi simple, then $\frac{A+Soc(Y)}{A} = Soc(\frac{Y}{A})$.

Definition 2.4 [2]: Let A be a proper submodule of Y. Then A is called a quasi $-$ prime if for $a, b\in$ V, $m\in$Y, $abm\in$ A, implies either $am\in$ A or b$m\in$ A

Definition.2.6[7]: A V-module Y is said to be divisible if $aY = Y$. Moreover, every element x of Y can be divided by a, in the sense that there is an element y in Y such that $x = ay$ for every non-zero element $a \in V$.

Proposition2.7[8]: Every prime submodule is a Pseudo two absorbing submodule

Proposition 2.8[8]: Let $\mathcal{K}$ be aproper submodule of V-module Y, if $\mathcal{K}$ is a quasi-prime submodule of Y, then $\mathcal{K}$ is a Pseudo two absorbing submodule of Y.
Definition2.9[9]: if A be a proper submodule of a V-module Y, A is called $2-$ absorbing submodule of Y if when r, b$\in$ V, x$\in$ Y and rbx$\in$ A, thus rx$\in$ A or bx$\in$ A or rb$\in$ $(A:_V Y)$.
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3 MAIN RESULT

In this section, we are going to present and explore a broader version of the T-ABSO module. There is a lot of coverage of the characteristics and features of the Pseudo two absorbing module concept. The connections between the Pseudo two absorbing module and other unique kinds of modules will also be examined.

Definition 3. 1

AV-module Y is called a Pseudo two absorbing module (for short P-T-ABSO) if the zero submodule $< 0 >$ is a Pseudo two absorbing submodule of Y, that is, whenever $a, b \in V, x \in Y, abx = 0$ implics either $ax \in < 0 > + Soc(Y) = Soc(Y)$ or $bx \in < 0 > + Soc(Y) = Soc(Y)$ or $ab \in (< 0 > + Soc(Y):_V Y)$.

Remark and Example 3. 2

1. Every T-ABSO module is P-T-ABSO module but the converse not true for example in Z-module Z_{12} $Soc(Z_{12}) = < 2 >$ then for all $abx = 0$ either $ax \in < 2 >$ or $bx \in < 2 >$ or $ab \in (< 2 >:_Z Z_{12}) = 2Z$, but Z_{12} is not T-ABSO module since $2 \cdot 2 \cdot 3 = 0 \in < 0 >$, but $2 \cdot 3 = 6 \notin < 0 >$ and $2 \cdot 2 = 4 \notin (< 0 >:_Z Z_{12}) = 12Z$

2. Since a multiplication Z-module Z_{pq} is T-ABSO module when p and q are distinct prime numbers. Then that is P-T-ABSO module. In particular each of the following Z-modules Z_{10}, Z_{21}, Z_{33} is Pseudo two absorbing module.

3. For each prime number P, Z_{p^2} is P-T-ABSO module.

4. For each prime number p, Z_p is a two absorbing Z-module, so it is P-T-ABSO module

5. It is not necessary that a proper submodule of a Pseudo two absorbing module is a P-T-ABSO module. For example in the Z-module Z_{12}, Z_{12} is a P-T-ABSO module since $< \bar{0} >$is a Pseudo two absorbing submodule, where as $< \bar{0} > + Soc(Z_{12}) =< \bar{0} > + < \bar{2} > = < \bar{2} >$,so if $a, b \in Z, x \in Z_{12}$ with $abx \in < \bar{0} >$ then at least two of a, b, x are even, but $< \bar{6} >$ is not a Pseudo two absorbing submodule, since $2 \cdot 2 \cdot \bar{3} = 0$, then $2 \cdot \bar{3} \notin < \bar{0} > + Soc(< \bar{6} >) =< \bar{0} > + < \bar{0} > = < \bar{0} >$ and $2 \cdot 2 \notin (< \bar{0} >:_Z < \bar{6} >) = 6Z$.

Proposition 3. 3

If a proper submodule Soc(Y) is a Pseudo two absorbing submodule of Y, then Y is a Pseudo two absorbing module

Proof:

Let $abx = 0$, then $abx \in Soc(Y)$ for $a, b \in V, x \in Y$ and hence either ax $\in < 0 > +Soc(Y)$ or bx $\in < 0 > +Soc(Y)$ or $ab \in (< 0 > +Soc(Y):_V Y)$, then $< 0 >$ is Pseudo two absorbing submodule of Y. Thus, Y is P-T-ABSO module. ■

Proposition 3. 4

Let Y be a V-module. Then, the following statements are equivalent:

1- Y is a P-T-ABSO module.
2- $(Soc(Y):_V bA) = (Soc(Y):_V A)$ or $(Soc(Y):_V bA) = (Soc(Y):_V bY)$ for any non-zero submodule A with $bA \nsubseteq Soc(Y), b \in V$.
3- $(Soc(Y):_V bx) = (Soc(Y):_V x)$ or $(Soc(Y):_V bx) = (Soc(Y):_V bY)$ for any $x \in Y$ with $bx \notin Soc(Y)$ and $b \in V$.

Proof:

$(1) \Rightarrow (2)$ Since $bA \subseteq A$ then $(Soc(Y):_V A) \subseteq (Soc(Y):_V bA)$. Let $abA = 0$ with $a \in (Soc(Y):_V bA)$, for $a \in V$ and since Y is a P-T-ABSO module, then $< 0 >$ is a Pseudo two absorbing submodule of Y, but $bA \nsubseteq Soc(Y)$, then either $aA \subseteq < 0 > +Soc(Y)$ or $ab \in (Soc(Y):_V Y)$.
If $aA \subseteq Soc(Y)$, then $a \in (Soc(Y):_V A)$. Hence, $(Soc(Y):_V bA) \subseteq (Soc(Y):_V A)$.
If $ab \in (Soc(Y):_V Y)$, then $abY \subseteq Soc(Y)$ and hence $a \in (Soc(Y):_V bY)$. Thus, $(Soc(Y):_V bA) \subseteq (Soc(Y):_V bY)$

$(2) \Rightarrow (3)$ It is clear

$(3) \Rightarrow (1)$ Let $abx = 0$ with $bx \notin Soc(Y)$ by condition (3) either $(Soc(Y):_V bx) = (Soc(Y):_V x)$ or $(Soc(Y):_V bx) = (Soc(Y):_V bY)$, but $a \in (< 0 >:_V bx)$ hence $a \in (< 0 > +Soc(Y):_V bx)$, So either $a \in (Soc(Y):_V x)$ or $a \in (Soc(Y):_V bY)$. It follows that either $ax \in Soc(Y)$ or $abY \subseteq Soc(Y)$, that is $ax \in Soc(Y)$ or $ab \in (Soc(Y):_V Y)$. Therefore (0) is Pseudo two absorbing submodule of Y. Hence Y is a P-T-ABSO module. ■

Proposition 3. 5

If $ann_V A = ann_V Y$ for each non-zero submodule A of Y, then a module Y is a P-T-ABSO module.

Proof:

Let $abx = 0$ for some $a, b \in V, x \in Y$

If $x \neq 0$, then $ab \in ann_V < x > = ann_V Y$ and hence $ab \in ann_V Y = (< 0 >:_V Y)$. Thus, $ab \in (< \bar{0} > +Soc(Y):_V Y)$.

If $x = 0$, then $ax = 0$. Therefore $ax \in < 0 > +Soc(Y)$, then $< 0 >$ is a Pseudo two absorbing submodule. Thus, Y is P-T-ABSO module ■

Remark 3. 6

If $ann_V B \neq ann_V Y$ for a non-zero submodule B of Y, then Y is not necessary is not a P-T-ABSO module as the following example show, in $Y = Z_8$ as Z-module, we see that $< \bar{2} >$ is a non-zero submodule of Z_8 and $ann_Z < \bar{2} >$ =4Z $\neq ann_Z Z_8 = 8Z$ and Z_8 is a Pseudo two absorbing module because $< 0 >$ is a Pseudo two absorbing submodule of Z_8.

Proposition 3. 7

Let Y be a V-module and B a proper submodule of Y. If B is a Pseudo two absorbing submodule of Y, then $\frac{Y}{B}$ is a P-T-ABSO module.

Proof:

Let $ab(x+B) = abx + B \in B = 0_{\frac{Y}{B}}$, where $a, b \in V$, $x + B \in \frac{Y}{B}, x \in Y$. Then $abx \in B = 0_{\frac{Y}{B}}$, but B is Pseudo two absorbing submodule of Y, implies that either $ax \in B + Soc(Y)$ or $bx \in B + Soc(Y)$ or $abY \subseteq B + Soc(Y)$. It follows that either $a(x+B) \in \frac{B+Soc(Y)}{B}$ or $b(x+B) \in \frac{B+Soc(Y)}{B}$ or $ab\frac{Y}{B} \subseteq \frac{B+Soc(Y)}{B}$, that is either $ax + B \in \frac{B}{B} + \frac{B+Soc(Y)}{B} \subseteq B + Soc\left(\frac{Y}{B}\right)$ or $bx + B \in \frac{B}{B} + \frac{B+Soc(Y)}{B} \subseteq B + Soc\left(\frac{Y}{B}\right)$ or $ab\frac{Y}{B} \subseteq \frac{B}{B} + \frac{B+Soc(Y)}{B} \subseteq B + Soc\left(\frac{Y}{B}\right)$(by Proposition2. 3). Hence $B = 0_{\frac{Y}{B}}$ is a Pseudo two absorbing submodule of $\frac{Y}{B}$. Thus, $\frac{Y}{B}$ is a P-T-ABSO module ■

Proposition 3. 8

Let Y be a V-module. If $ann_V B = ann_V\, bB$ or $ann_V\, bB = ann_V\, bY$ for any nonzero sunmodule B with $bB \nsubseteq Soc(Y), b \in V$, then B is P-T-ABSO module.

Proof:

Let $abB = 0$ where $a, b \in V, B \leq Y$, suppose that $bB \nsubseteq Soc(Y)$ and either $ann_V bB = ann_V\, B$ or $ann_V\, bB = ann_V\, bY$, but $a \in ann\,_V bB$ so either $a \in ann\,_V B$ or $a \in ann_V\, bY$. It follows that either $aB = 0 \in < 0 > +Soc(Y)$ or $abY = 0 \in < 0 > +Soc(Y)$, then $< 0 >$ is a Pseudo two absorbing submodule of Y. Thus, Y is P-T-ABSO module. ■

Corollary 3. 9

Let Y be a V-module and $ann_V x = ann_V bx$ or $ann\,_V bx = ann\,_V bY$ for any $x \in Y$ with $bx \notin Soc(Y)$ where $b \in V$, then Y is a P-T-ABSO module.

Recall that a proper submodule B of a V-module Y, is said to be a semi prime submodule if $a \in V$, $x \in Y$, with $a^2x \in B$ implies that $ax \in B$.[10]

Proposition 3. 10

Let Y be a divisible and $< 0 >$be a semi prime submodule of Y. Then Y is a P-T-ABSO module where $ab \notin (< 0 > +Soc(Y) :_V Y), \forall a, b \in V$

Proof:

Let $abx = 0, a, b \in V, x \in Y$ since $ab \notin (< 0 > +Soc(Y) :_V Y)$, then $abY \nsubseteq < 0 > +Soc(Y)$. Hence $abY \neq 0$, so $abY = bY = Y$, because Y is divisible. Thus $bx = abx_1$ for some $x_1 \in Y$. Since $abx = a^2(bx_1) = 0$ which implies that $abx_1 = 0$, since $< 0 >$ is semi prime submodule of Y, thuse $bx = 0$.Hence $bx \in < 0 > +Soc(Y)$.Then$< 0 >$ is Pseudo two absorbing submosule. Thus, the proof is complete. ■

Proposition 3. 11

Let Y be a P-T-ABSO module and $(Soc(Y) :_V Y)$ be a prime ideal. If Soc(Y) is a proper submodule of Y, then Soc(Y) is a qusi-prime submodule of Y.

Proof:

Let $abx \in Soc(Y)$ for $a, b \in V, x \in Y$, suppose that $bx \notin Soc(Y)$
Since $< 0 >$ is a Pseudo two absorbing submodule then by Proposition 3. 4 $(Soc(Y):_V bx) = (Soc(Y):_V x)$ or $(Soc(Y):_V bx) = (Soc(Y):_V bY)$, but $a \in (Soc(Y):_V bx)$ hence either $a \in (Soc(Y):_V x)$ or $a \in (Soc(Y):_V bY)$
If $a \in (Soc(Y):_V x)$, then $ax \in Soc(Y)$ we are done.
If $a \in (Soc(Y):_V bY)$, then $abY \subseteq Soc(Y)$ hence $ab \in (Soc(Y):_V Y)$
Since $(Soc(Y):_V Y)$ is a prime ideal and $bx \notin Soc(Y)$, then $ax \in Soc(Y)$. Thus, Soc(Y) is quasi-prime submodule of Y. ∎

Corollary 3. 12

Let Y be a V-module with $(Soc(Y):_V Y)$ be a prime ideal of V. Then Y is a P-T-ABSO module if and only if Soc(Y) is a quasi − prime submodule of Y.

Proof:

⟹) Since Soc(Y) is quasi prime submodule of Y, then by Proposition2.8, we get Soc(Y) is a Pseudo two absorbing submodule of Y and by Proposition 3. 3 we get Y is a P-T-ABSO module.

⟸)it is clear by Proposition 3. 11. ∎

Proposition 3.13

Let Y be a multiplication V-module. Then the follwing statement are equivalent where Soc(Y) is a proper submodule of Y with $(Soc(Y):_V Y)$ is prime ideal

1- Y is a P-T-ABSO module
2- Soc(Y) is a quas-prime submodule
3- Soc(Y) is a prime submodule.

Proof:

(1) → (2) it follows by Corollary 3. 12.

(2) → (3) let $ax \in Soc(Y)$ for $a \in V, x \in Y$ so $a < x > \subseteq Soc(Y)$, but Y is multiplication V-module So there exist ideal I of V such that $< x > = IY$. Hence $aIY \subseteq Soc(Y)$ so $aI \subseteq (Soc(Y):_V Y)$, but $(Soc(Y):_V Y)$ is a prime ideal, then either $a \in (Soc(Y):_V Y)$ or $I \subseteq (Soc(Y):_V Y)$, which means either $a \in (Soc(Y):_V Y)$ or $IY \subseteq Soc(Y)$ so either $a \in (Soc(Y):_V Y)$ or $< x > \subseteq Soc(Y)$.Thuse, Soc(Y) is a prime submodule of Y.

(3) → (1) since Soc(Y) is a prime submodule then by Proposition 2.7, we get Soc(Y) is Pseudo two absorbing submodule of Y , thus by Proposition 3.3 , we have Y is P-T-ABSO module. ∎

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