MIXED SERRE FIBRATION

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Abstract:- In this paper we introduce and study new concept the mixed Serre fibration (M-Serre fibration) on CW-complex space, and Mixed path lifting property (by short M-PLP). Most of theorems which are valid for Serre fibration be also valid for M-Serre fibration.

Keywords: M-Serre fibration, CW-complex space, M-path lifting property, M-Covering Homotopy Property.

1 Introduction

The following problem is one of the problems in algebraic topology. Let $f\colon \mathrm{E} \to \mathrm{X}$ be a Serre fibration (Jean-Pierre Serre, born 15 September 1926) of CW-complex space. In this study, we looked at Serre fibration at the functions of the numbers of Serre fibration one and two, to become the function $f_i\colon E_i \to X$ (Mixed Serre fibration).

We use the following notation for the closed unit m-disk, the open unit m-disk and the unit (m-1)-sphere

$$D^m = \{x \in \mathbb{R}^m : \| x \| = 1\},$$

$$int(D^m) = \{x \in \mathbb{R}^m : \| x \| < 1\},$$

$$S^{m-1} = \{x \in \mathbb{R}^m : \| x \| = 1\}$$

where $\|\cdot\|$ is the standard norm, $\| (x^1, x^2, \ldots, x^m) \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2}$

2 Preliminaries

Definition 2.1. [9] [8]

An $m - cell$, $m \geq 0$ is a topological space that is homeomorphic to the open m-disk $int(D^m)$.

Definition 2.2. [8][9]

Let X be a topological space and the cell-decomposition of X is a family $\omega = \{e_t | t \in I\}$ of subspace of X such that each e_t is a cell and $X = \sqcup_{t\in I} e_t$ which disjoint union of sets, the m- skeleton of X is the subspace $X = \sqcup_{t\in I, \dim(e_t)\leq m} e_t$

Definition 2.3. [8]

A pair (X, ω) consisting of a Hausdorff space X and a cell-dcomposition ω of X is called a CW-space if the following are satisfied:

• For each $m - cell\ e \in \omega$ there is a map $\psi_e: D^m \to X$ restricting to a homeomorphism $\psi_e | int(D^m): int(D^m) \to e$ and taking $S^{m-1}\ into\ X^{m-1}$, which is called Characteristic Maps.

• For any cell $e \in \omega$ the closure $\bar{e}$ intersects only a finite number of other cells in ω, which is called Closure Finiteness.

• A subset $A \subseteq X$ is closed iff $A \cap \bar{e}$ is closed in X for each $e \in \omega$, which is called Weak Topology

The restrictions $\psi|_{\partial D^m}$ are called the attaching maps.

Notice that we can recover X (up to homeomorphism) from a knowledge of X^0 and the attaching maps. The recovery is described in as follows:

1. If we start with a discrete set X^0 , whose points are regarded as 0-cell.

2. From X^{m-1} by attaching m-cells e_t^m by $maps\ \psi_t : S^{m-1} \to X^{m-1}$, we get form the m-skeleton X^m. The quotient space of the disjoint union $X^{m-1} \sqcup_t D_t^m$ of X^{m-1} with a collection of m-disks D_t^m under the identifications $X \sim \psi_t(x)$ for $x \in \partial D_t^m$. Thus as a set $X^m = X^{m-1} \sqcup_t e_t^m$ where each e_t^m is an open m-cell.

3. Setting X = X^m for some $n < \infty$, setting $X = \cup_m X^m$. In the latter case X is given the weak topology : A set $A \subset X$ is open (or closed) iff $A \cap X^m$ is open (or closed) in X^m for each m .

So we have ,

$$X^0 \subset X^1 \subset X^2 \subset \ldots \subset X^m \subset \ldots$$

If there exist an integer m such that $X^m = X$ then X is called finite dimensional.

Definition 2.4. [3]

A CW-space is said to be regular if all its attaching maps are homeomorphisms .

Definition 2.5. [1]

A triple (E, p, B) called a fiber structure consisting of two space E , B and a continuous onto map $p : E \longrightarrow B$. The total space is E (or fibered) space the projection is p .The base space is called B, the space for each $b_0 \in B$, the set $F = p^{-1}(b_0)$ and F is called fiber over b_0. We refer to (E, p, B) as a fiber structure over B.

Definition 2.6. [4] [1]

Let $p : E \longrightarrow B$ be a map , we say that p has Covering Homotopy Property (C.H.P.by short) with respect to X iff given a map $f: X \rightarrow E$ and $h_t: X \longrightarrow B$ is homotopy such that $p \circ f = h_0$. Then there exist a homotopy h_t^* : X → E such that (1) $h_0^* = f$. (2) $p \circ h_t^* = h_t$, for all $x \in X$ and $t \in I$.

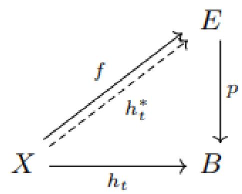


Figure 1: Covering Homotopy Property

Definition 2.7.

The map p is siad to be (Hurewicz) Fibration if it has covering homotopy property w.r.t all space.

3 M-Serre Fibration

In this section we introduce and study a new concept which is namely Mixed Serre fibration M-Serre Fibration . we start with following definition.

Definition 3.1.

(1) Let E_1, E_2 and X be three topological spaces , let $E_i = \{E_1, E_2\}, f_i = \{f_1, f_2\}$ where f_1: $E_1 \rightarrow$ X, f_2 : $E_2 \rightarrow$ X are two maps , and $\alpha : E_2 \longrightarrow E_1$ such that $f_1 \circ \alpha = f_2$ then $\{E_i, f_i, X, \alpha\}$ is a M-fiber space (Mixed-fiber space).

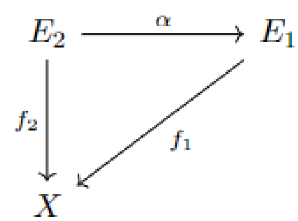


Figure 2: M-fiber space

If E_1= E_2= E, α = identity, = f_1 = f_2 = f then (E, F, X) is the usual fiber space.

(2) Let $\{E_i, f_i$, X, α} be a M-fiber space , let $x_0 \in$ X then $f = \{f_i^{-1}(x_0)\}$ is the M-fiber over x_0.

Definition 3.2. [4] [6]

Let $p: E \longrightarrow B$ be a continuous map of spaces , p has the covering homotopy property (C.H.P) with respect to a CW-complex space X is called Serre Fibration.

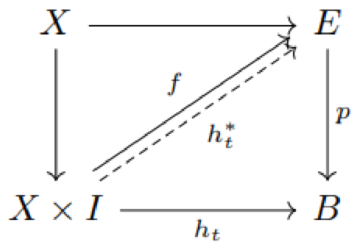


Figure 3: Serre Fibration

Definition 3.3.

Let $\{E_i, f_i, X, \alpha\}$ be a Mixed fiber structure , where $i = 1, 2$. Let X , B be a CW-complex spaces and $h_t: B \rightarrow X$ be map . A continuous $k_1 : B \longrightarrow E_1$ and $k_2: B \longrightarrow E_2$ such that $f_1 \circ k_1 = h_t$ and $f_2 \circ k_2 = h_t$, where $K_i = \{k_1, k_2\}$ is called a Mixed-covering (M-covering) of h_t.

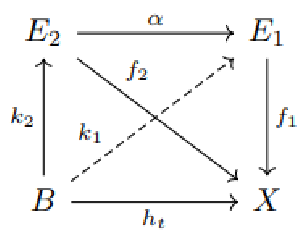


Figure 4: M-covering

Definition 3.4.

Let Y be a CW-complex space , $f_1: E_1 \rightarrow$ Y, $f_2 : E_2 \rightarrow$ Y, α : $E_2 \rightarrow E_1$ are maps of a spaces such that $f_1 \circ \alpha = f_2$, let $E_i = \{E_1, E_2\}$ where i = 1, 2. $f_i = \{f_1, f_2\}$, the quartic $\{E_i , f_i$, Y, α$\}$ has the Mixed covering homotopy property (M-CHP) w.r.t a CW-space X iff given a map $k: X \rightarrow E_2$ and a homotopy h_t : X → Y such that $f_2\circ k = h_0$, then exists a homotopy g_t: X → E_1 such that (1) $f_1 \circ g_t = h_t$. (2) $\alpha \circ k = g_0$.

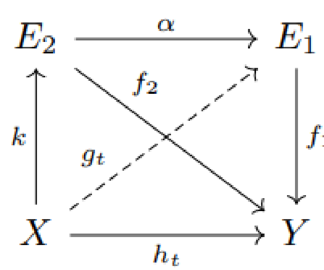


Figure 5: M-Serre Fibration

M-fiber space is called M-Serre fibration ,is it has the (MCHP) with respect to all CW-complex.

Theorem 3.5. Every Serre fibration is a Mixed Serre fibration .

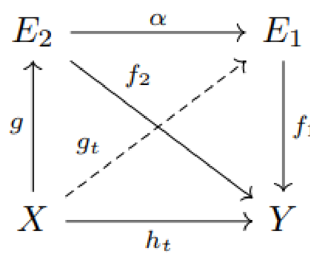
Proof:

Let {E, f, Y, α} is fiber space such that E = E_1 = E_2 , α = I_d (identity), $f = f_1 = f_2$.

Let g : X → E_2 and homotopy h_t : X → Y such that $f_2 \circ g = h_0$, then there exist g_t : X → E_1 such that $g_0 = \alpha \circ g$ and $f_1 \circ g_t = h_t$ for all $x \in$ X and t ∈ I

Then f has M-CHP w.r.t space CW-complex .

Therefore f has M-Serre fibration.

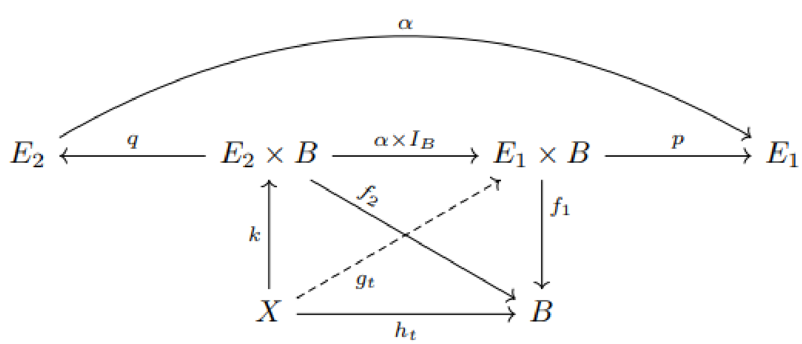


Proposition 3.6.

Mixed Serre fibration may not be a Serre fibration. As show by the following example

Example: Let E_1, E_2, B be a three CW-complex spaces , and $E_1 \neq E_2$, $f_1 : E_1$ × B → B be the projection defined as $f_1(e_1$, b) = b , $f_2 : E_2$ × B → B be the projection defined as $f_2(e_2$, b) = b , α : E_2→ E_1 be any map , $\hat{E_1}$ = E_1×B , $\hat{E_2}$ = E_2×B , $E_i = \{ \hat{E_1}, \hat{E_2} \}$, $f_i = \{f_1, f_2\}$.

Let p: E_1 × B → E_1 be the projection defined as $p(e_1$, b) = e_1 for all (e_1, b) ∈ E_1× B , q: E_2 × B → E_2 be the projection defined as $q(e_2$, b) = e_2 for all (e_2, b) ∈ E_2× B. Let k : X → E_2× B be any map , and h_t: X → B be any homotopy such that $f_2 \circ k = h_0$.



Define g_t : X → E_1 × B as follows $g_t(x) = \{\alpha \circ q \circ k(x), h_t(x)\}$, the g_t satisfy (1)$f_1 \circ g_t = h_t$ $\forall t \in I$ (2) $g_0 = (\alpha \times I_B) \circ k$ Therefore f_i : E_i× B → B is M-Serre fibration , which is not Serre fibration .

Definition 3.7.

Let (X_i , f_i , Y, α) be M-fiber structure X_i be a CW-complex , and $g : Y' \to$ Y be any continuous map into base Y .

Let X_1'= {(x_1, y')∈ X_1× Y' : $f_1(x_1) = g(y')$} , and X_2'= {(x_2, y')∈ X_2× Y' : $f_2(x_2) = g(y')$} , then

X_i'={X_1', X_2'} is called a M-pullback of f_i by g and f_i'= {f_1', f_2'} : $X' \to Y'$ is called induced M-function of f_i by g.

Define α': $X_2' \to X_1'$ by $\alpha'(x_2,y')$=(α(x_2),y').

To show α' is continuous .

Since α'= α × $I_{y'}$, α is continuous and $I_{y'}$ is continuous then α' is continuous .

To show is commutative .

$f_1' \circ \alpha'(x_2,y')$ = f_1'(α(x_2),y') =y'. $f_2'(x_2,y')$ =y'.

Therefore $f_1' \circ \alpha'$=f_2'.

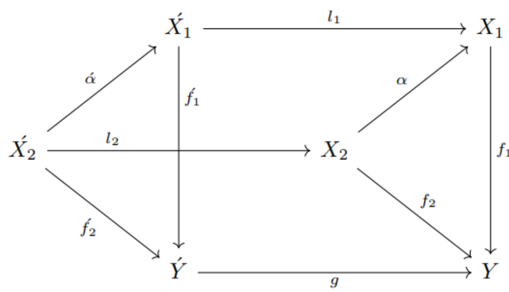


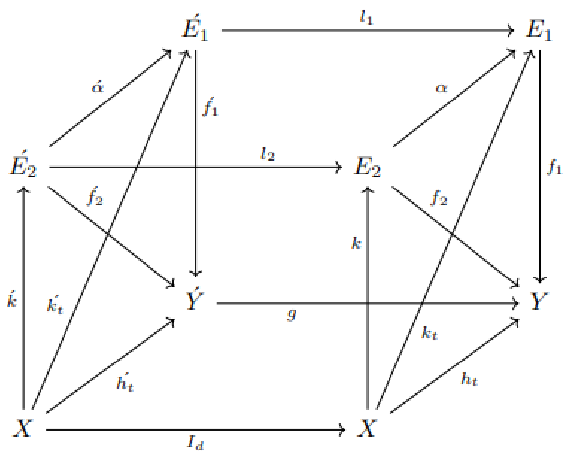
Figure 6: M-Pullback

Theorem 3.8.

The M-pullback of M-Serre fibration is also M-Serre fibration.

proof:

Let k': X →E_2' and k : X →E_2 . Define a homotopy h_t : X → Y such that h_0= $f_2 \circ k$, since f_i is M-Serre fibration ,



then there exist k_t : X → E_1 such that $f_1 \circ k_t = h_t$ and $k_0 = \alpha \circ k$.

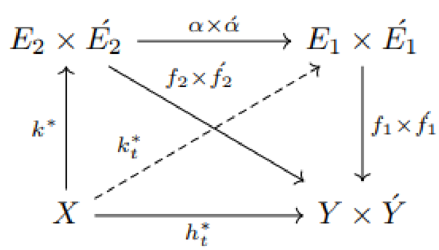
Define h'_t : X → Y' as $g \circ h'_t = f_1 \circ k_t$ and $h'_0 = f'_2 \circ k'$, then there exist k'_t : X → E'_1 , where k'_t (x)= $(k_t$ (x) , $h'_t(x))$. Hence $f'_1 \circ$ k'_t= h'_t and k'_0 $= \alpha' \circ k'$.
there for $f_i{}'$: $E_i \rightarrow Y'$ is M-Serre fibration .

Proposition 3.9.

Let $f_i{}'$: E'_i→ Y be two M-Serre fibration then f_i×$f_i{}'$: E_i×E'_i → Y ×Y' is also M-Serre fibration.

Proof:

Let X be a CW-complex space Let K^*: X → E_2 × E'_2 be a map, where $K^*(x)$ = ($k(x)$, $k'(x)$) such that k : X → E_2 and k' : X → E'_2 and h^*_t : X → Y × Y' define as $h^*_t(x)$ = {$h_t(x)$, $h'_t(x)$} and $(f_2$×$f'_2) \circ k^* = h_0$.



Such that h_t : X → Y and h'_t: X → Y' since f_i, $f_i{}'$are M-Serre fibretion , then there exists a homotopy k_t : X → E_1 such that $f_1 \circ k_t$= h_t , k_0 = $\alpha \circ k$ and a homotopy k'_t: X → E'_1 such that $f'_1 \circ k'_t$=h'_t, k'_0= $\alpha' \circ k'$.
Now , for h^*_t there exist K^*_t: X → E_1 × E'_1 define as $K^*_t(x)$= { $k_1(x)$, $k'_t(x)$ } such that $(f_i$ ×$f_i{}')$ $\circ K^*_t(x)$ = $h^*_t(x)$ and K^*_0 = (α × α^1) ∘ K^* since X be a CW-complex .
Therefore f_i×$f_i{}'$: E_i×E'_i → Y × Y' is M-Serre fibration .

Definition 3.10. [4]

Let p : E → B be a map is said to be have the bundle property (BP) for each $b \in$ B, if there exists a space X such that , there is an open neighborhood V of b in B together with a homeomorphism,

$$Q_V: \mathrm{V} \times \mathrm{X} \rightarrow p^{-1} (\mathrm{V})$$

satisfying the condition $P_{Q_V}(v, x) = v$, ($v \in$ V, $x \in$ X)

In this case , "the space E is called a bundle space over the base space B relative to the projection p: E → B" . "The space X will be called a director space . The open sets V and the homeomorphisms Q_V will be called the decomposing neighborhoods and the decomposing functions respectively" . If p : E → B has the bundle property (BP), then it has the paraCHP.

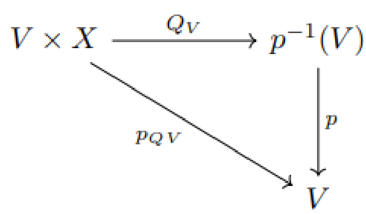


Figure 7: Bundle Property

Definition 3.11.

Let $p : E_1 \rightarrow$ A and q: $E_2 \rightarrow$ B be a maps said to have the M-bundle property (MBP) if there exists a space X such that , for each $a \in$ A and $b \in$ B , have an open neighborhood U,V of b in B together with a homeomorphism ,

$$Q_U = \mathrm{U} \times \mathrm{X} \rightarrow p^{-1} (\mathrm{U})$$

satisfying the condition, $P_{Q_U}(\mu, x) = \mu$, (μ ∈ U, $x \in$ X).

And $N_V = \mathrm{V} \times \mathrm{X} \rightarrow q^{-1} (\mathrm{V})$

satisfying the condition $G_{N_V}(\varepsilon, x) = \varepsilon$, (ε ∈ V, $x \in$ X).

The space E_i, where i = 1, 2 is called a M-bundle space over the base spaces B, A relative to the projection $p : E_1 \rightarrow$ A and q: $E_2 \rightarrow$ B. The space D will be called a director space . The open sets U , V and the homeomorphisms Q_U, N_V will be called the decomposing neighborhoods and the decomposing functions respectively .If the maps have the M-BP , then its have the M-paraCHP

4 The path lifting propert

In this section we speaking , let p: E → B be a map and is said to have the path lifting property (PLP by short) if, each path f: I → B with $f(0) = p(e)$, for each $e \in$ E, there exists a path ω : I → E such that ω(0) = e, pω = f, and that ω depends continuously on e and f.
For a precise definition , let Ω_p denote the subspace of the product space E×B^I defined by
Ω_p = {(e, f) ∈ E ×B^I | p (e) = f (0) }.

Define a map $q: E^I \rightarrow \Omega_p$. By taking $q(\omega) = (\omega(0), p\omega)$ for each $\omega : I \rightarrow E$ in E^I. Then $p: E \rightarrow B$ is said to have the PLP if there exists a map $\lambda : \Omega_p \rightarrow E^I$ such that $q\lambda$ is the identity map on Ω_p . It is well-known that a map $p: E \rightarrow B$ has the PLP iff it has the ACHP . The map λ in the above definition is called a lifting function for $p: E \rightarrow B$. If λ lifts constant paths to constant paths , then it is called a regular lifting function for $p: E \rightarrow B$ and the triple $\xi = (E, p, B)$ is called a regular serre fiber space .

Definition 4.1.

Let (E_i, f_i, Y, α) be M-fiber structure and $Y^I = \{\omega : I \rightarrow Y\}$, $\Omega_{f_i} \subseteq E_i \times Y^I$ be the subspace , $\Omega_f = (e, \omega) \in E_i \times Y^I / f_i(e) = \omega(0)$. A M-lifting function for (E_i, f_i, Y, α) is continuous map $\lambda_i : \Omega_f \rightarrow E_i^I$ such that $\lambda_i(e, \omega)(0) = e$ and $f \circ \lambda_i(e, \omega)(t) = \omega(t)$ for each $(e, \omega) \in \Omega_f$ and $t \in I$
thus $\lambda_i = \{\lambda_1, \lambda_2\}$ and $\Omega_{f_i} = \{\Omega_{f_1}, \Omega_{f_2}\}$, where $\lambda_1: \Omega_{f_1} \rightarrow E_1^I$ and $\lambda_2: \Omega_{f_2} \rightarrow E_2^I$ defind as
$\lambda_1(e_1, \omega)(0) = e_1$, $f_1 \circ \lambda_1(e_1, \omega)(t) = \omega(t)$ and $\lambda_2(e_2, \omega)(0) = e_2$, $f_2 \circ \lambda_2(e_2, \omega)(t) = \omega(t)$.
Thus a M-lifting function therefore associates .

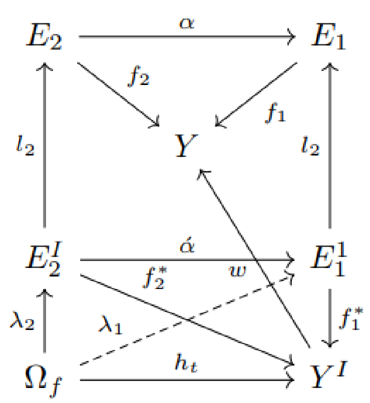


Figure 8: M-Lifting Function

With each $e \in E$, and each bath ω in Y starting at $f_i(e)$ a path $\lambda_1(e_1, \omega)$ in E_1 and $\lambda_2(e_2, \omega)$ in E_2 , starting at e_2 and e_1 ,and is M-cover of ω since the c-topology used in E^I , the continuity of λ is equivalent to that of associated $\lambda_i: \Omega_f \times I \rightarrow E_i$.

Example 4.2.

A well-known example of a M-Serre fiber space is the (E_i, f_i, Y, α), $Y^I = \{\omega : I \rightarrow Y\}$ and $f_1(\omega) = \omega(1)$, $f_2(\omega') = \omega'(1)$, where $\omega, \omega' : I \rightarrow Y$. A lifting functions $\lambda_1: \Omega_{f_1} \rightarrow E_1^I$ for $f_1 : E_1 \rightarrow Y$ and $\lambda_2: \Omega_{f_2} \rightarrow E_2^I$ for $f_2 : E_2 \rightarrow Y$, are defined as follows:

$$\lambda_1(\sigma, \omega)(t)(s) = \begin{cases} \sigma\left(\frac{4s}{1+t}\right) & if\ 0 \leq s \leq \frac{1+t}{4} \\ \omega(4s - t - 1) & if\ \frac{1+t}{4} \leq s \leq \frac{2+t}{4} \end{cases}$$

$$\lambda_2(\sigma', \omega')(t)(s) = \begin{cases} \sigma'(4s - t - 2) & if\ \frac{2+t}{4} \leq s \leq \frac{3+t}{4} \\ \omega'\left(\frac{4s-t-1}{1-t}\right) & if\ \frac{3+t}{4} \leq s \leq 1 \end{cases}$$

Note that this particular λ_1 , λ_2 are not regular.

Definition 4.3.

We say that a space Y admit ϕ -function , if there exists a function ϕ : Y^I → I such that ϕ(ω) = 0 iff ω is a constant path .

Proposition 4.4.

If a space Y admit a ϕ- function , then every M-Serre fiber space ξ = (E_i, f_i , Y) is regular.

Proof:

Since Y admit a ϕ -function , there exists a function ϕ_1: Y^I → I such that ϕ_1(ω) = 0 iff ω is constant , and ϕ_2: Y^I → I such that ϕ_2(σ) = 0 iff σ is constant .

Define a functions $g : Y^I \to Y^I$ by g(ω)(t) = u($\frac{t}{\phi_1(\omega)}$) for t < $\phi_1(\omega)$ and g(ω)(t) = ω(1) for $\phi_1(\omega)$ ≤ t ≤ 1 , h : $Y^I \to Y^I$ by h(σ)(t) = v($\frac{t}{\phi_2(\sigma)}$) for t < ϕ_2(σ) and h(σ)(t) = σ(1) for ϕ_2(σ) ≤ t ≤ 1 .

Now if λ_i are any lifting function for (E_i, f_i , Y) , where $i = 1,2$ define : λ_1': $\Omega_{f_1} \to E_1^I$ as follows:

$$\lambda_1' (e_1, \omega)(t) = \lambda_1(e_1, g(\omega))(\phi_1(\omega) \cdot t) .$$

And define : λ_2': $\Omega_{f_2} \to E_2^I$ as follows:

$$\lambda_2' (e_2, \sigma)(t) = \lambda_2(e_2, h(\sigma))(\phi_2(\sigma) \cdot t) .$$

Then λ_i' are an regular lifting function for (E_i , p_i , X) , where $i = 1,2$ hence ξ = (E_i , p_i , X) is a regular M-Serre fiber space .

Corollary 4.5.

If X is metric space ,then every M-Serre fiber space (E_i , p_i , X) is regular.

Proof:

Define ϕ : X^I→ I as follows: ϕ(u) = diam($u(I)$) where u : I → X and diam means diameter. It is easy to see that ϕ(u) = 0 iff u is constant , hence B admit a ϕ-function , so by the above proposition , every M-serre fiber space (E_i , p_i, X) is regular .

5 Concolusion

- Every Serre fibration is a Mixed Serre fibration .
- Mixed Serre fibration may not be a Serre fibration.
- The M-pullback of M-Serre fibration is also M-Serre fibration.
- Two M-Serre fibration is also M-Serre fibration
- If the space Y admit a ϕ- function , then every M-Serre fiber space ξ = (E_i, f_i , Y) is regular.
- If X is metric ,then every M-Serre fiber space (E_i , p_i , X) is regular.

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