

Connectedness In Ideal Approximation Spaces

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ABSTRACT: This paper we study the notion connectedness in approximation spaces and ideal connectedness are introduced, with an explanation of their distinctions.

Keywords: connected, \star -connected, $\star\star$ - connected



1. INTRODUCTION

The development of set theory that addresses the uncertainty of intelligent systems is known as a rough set. Which are defined by incomplete and inaccurate information. Pawlak defines them in an approximation space [1]. The definition of rough sets is as a function of a generally. The equivalence relation R and the universal finite set Y , which is regarded as the principle behind a subset $S \subseteq Y$ upper as well lower approximation. Both are Based on an equivalency relation on Y , An approximation space was described as (Y, R) . Pawlak's rough set was generalized in a number of ways by changing the equivalence relation to any binary relationship. Many authors have studied the relationship between rough sets and topological space [4,6]. A transitive and reflexive relation was used to derive the upper and lower approximation operators, they were demonstrated to represent the topology's closure and interior operators. For the typical case, numerous research studies were presented with rough sets that included some therapeutic uses, as seen in [2,3]. In the paper, assume that Y is a universal set with a finite number of items. A subset of $Y \times Y$ is a relation R , which is a relation on Y from a universe Y to another universe Y . The formula $(a, b) \in R$, which can be shortened to aRb , indicates that a and b are related to each other in R . Additionally, the left and right neighborhoods of $a \in Y$ are represented by $Ra = \{b : bRa\}$ and $aR = \{b : aRb\}$, respectively. The intersection of a set $\langle a \rangle R$ (resp. $R \langle a \rangle$) equals of every neighborhood on the right (resp. left) that has a . The lower ($LR(S)$), the upper $UR(S)$) and the boundary region $BR(S)$ for $S \subseteq Y$ have the following definitions [5,9].

- $LR(S) = \{a \in S : \langle a \rangle R \subseteq S\}$.
- $UR(S) = S \cup \{a \in Y : \langle a \rangle R \cap S \neq \emptyset\}$.
- $BR(S) = UR(S) - LR(S)$.

The terms $LR(S)$, $UR(S)$ and $BR(S)$ denote the lower, upper, and boundary region approximation sets that are linked to the set $S \subseteq Y$ and are predicated on the idea that $\langle a \rangle R$ is an approximation space (Y, R) .

Definition [10]: Let D is a set that is not empty. If $I \subseteq P(D)$ meets these conditions, it is then referred to as an ideal on D :

- $\emptyset \in I$,
- If $S \in I$ as well $N \subseteq S$, then $N \in I$
- If $S, N \in I$, then $S \cup N \in I$.

Definition [11]: Assume that $S \subseteq Y$, I is an ideal defined on Y , and R is a binary relation on Y . Then, $\underline{R}(S)$ and $\overline{R}(S)$ of S , the lower and upper approximations, are outlined as follows:

- $\underline{R}(S) = \{v \in S : \langle v \rangle R \cap S^c \in I\}$,
- $\overline{R}(S) = S \cup \{v \in Y : \langle v \rangle R \cap S \notin I\}$.

Where $\langle v \rangle R$ is the intersection of all right neighborhoods contain v .

Definition [8]: Assume that $S \subseteq Y$, I is an ideal defined on Y , and R is a binary relation on Y . Then, $\underline{R}(S)$ and $\overline{R}(S)$ of S , the lower and upper approximations, are outlined as follows:

- $\underline{R}(S) = \{v \in S : R \langle v \rangle R \cap S^c \in I\}$,
- $\overline{R}(S) = S \cup \{v \in Y : R \langle v \rangle R \cap S \notin I\}$.

Where $R \langle v \rangle R = R \langle v \rangle R \cap \langle v \rangle R$.

Definition: Given an ideal approximation space (Y, R, I) and an arbitrary set $S \subseteq Y$. Thus

1. The set operator $(LC)^* : P(Y) \rightarrow P(Y)$ is known a local function defined as follows:

$$S^* = \cup \{H \subseteq Y : H - S \in I, \underline{R}(H) = H\}. \text{ Also}$$

the operator $cl_R^*(S) : P(Y) \rightarrow P(Y)$ is given as follows: $cl_R^*(S) = S \cup ((S^c)^*)^c$.

2. The set operator $(LC)^{**} : P(Y) \rightarrow P(Y)$ is known a local function defined as follows:

$$S^{**} = \cup \{H \subseteq Y : H - S \in I, \underline{R}(H) = H\}. \text{ Also}$$

the operator $cl_R^{**}(S) : P(Y) \rightarrow P(Y)$ is given as follows: $cl_R^{**}(S) = S \cup ((S^c)^{**})^c$.

2. CONNECTEDNESS OF SPACES

Definition 2.1: Assume that the approximation space is (Y, R) . Then

- 1 Two subsets that are not empty S and V of Y are regarded as separated sets if

$$UR(S) \cap V = S \cap UR(V) = \emptyset.$$

2. (Y, R) is known as disconnected space if there exists separated sets $S, V \subseteq D$

such that $S \cup V = D$. (Y, R) is an approximate space that is considered connected space if it is not disconnected.

Definition 2.2. Assume that the ideal approximation space is (Y, R, I) . Then

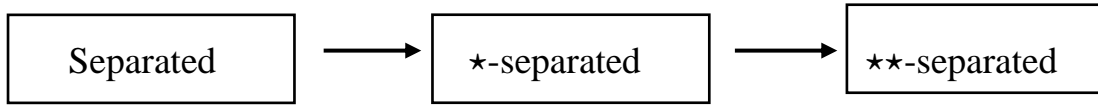
1. Two subsets that are not empty S and V of Y are regarded as \star - separated (resp. $\star\star$ - separated) sets

$$\text{if } cl_R^*(S) \cap V = S \cap cl_R^*(V) = \emptyset \text{ (resp. } cl_R^{**}(S) \cap V = S \cap cl_R^{**}(V) = \emptyset).$$

2. (Y, R, I) is known as \star - disconnected (resp. $\star\star$ - disconnected) space if there

exists \star - separated (resp. $\star\star$ - separated) set $S, V \subseteq D$ such that $S \cup V = D$. (Y, R, I) is an ideal approximation space that is considered \star - connected (resp. $\star\star$ - connected) space if not \star - disconnected (resp. $\star\star$ - disconnected) space.

Remark 2.3: 1. we get the diagrams below:



And hence

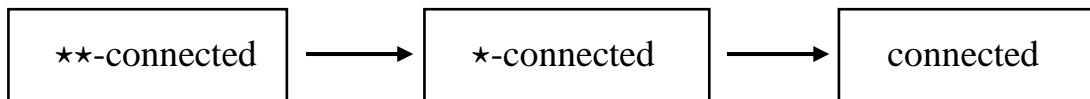


Figure 2.1 Relationships between types of connectedness

2. For $I = \{\emptyset\}$ observe that \star - connected and connected are identical.

The examples that follow demonstrates that the implied reverse meaning is incorrect, as shown in the figure 2.1.

Example 2.4:

1. Let $Y = \{a_1, a_2, a_3, a_4\}$, $R = \{(a_1, a_1), (a_1, a_4), (a_2, a_2), (a_2, a_3), (a_3, a_3), (a_1, a_4)\}$ then $\langle a_1 \rangle_R = \{a_1, a_4\}$, $\langle a_2 \rangle_R = \{a_2, a_3\}$, $\langle a_3 \rangle_R = \{a_3\}$, $\langle a_4 \rangle_R = \{a_4\}$. Also $R \langle a_1 \rangle = \{a_1\}$, $R \langle a_2 \rangle = \{a_2\}$, $R \langle a_3 \rangle = \{a_3\}$, $R \langle a_4 \rangle = \{a_4\}$. Then $I = \{\emptyset, \{a_4\}\}$ and $S = \{a_1\}$, $V = \{a_2, a_4\}$. Then $UR(S) = S \cup \{y \in Y : \langle y \rangle_R \cap S \neq \emptyset\} = \{a_1\}$, and $UR(V) = \{a_1, a_2, a_4\}$. Also $S^* = \{a_1, a_4\}$, $V^* = \{a_4\}$. $SocI_R^*(S) = S \cup ((S^c)^*)^c = \{a_1\}$ and $cl_R^*(V) = V \cup ((V^c)^*)^c = \{a_2, a_4\}$. Thus $cl_R^*(S) \cap V = S \cap cl_R^*(V) = \emptyset$, but $S \cap UR(V) = \{a_1\} \neq \emptyset$. Hence S, V are \star - separated sets but are not separated sets.

2. Consider $I = \{\emptyset, \{a_2\}\}$ and $S = \{a_2\}$, $V = \{a_1, a_3\}$. Then $S^* = \emptyset$, $V^* = \{a_2, a_3\}$. So $cl_R^*(S) = S \cup ((S^c)^*)^c = \{a_2\}$ and $cl_R^*(V) = V \cup ((V^c)^*)^c = \{a_1, a_2, a_3\}$. Also $S^{**} = \{a_2\}$, $V^{**} = \{a_1, a_2, a_3\}$. So $cl_R^{**}(S) = S \cup ((S^c)^{**})^c = \{a_2\}$ and $cl_R^{**}(V) = V \cup ((V^c)^{**})^c = \{a_1, a_3\}$. Thus $cl_R^{**}(S) \cap V = S \cap cl_R^{**}(V) = \emptyset$, but $S \cap cl_R^{**}(V) = \{a_2\} \neq \emptyset$. Hence S, V are $**$ - separated sets however, they are not \star - separated sets.

3. Let $Y = \{b_1, b_2, b_3\}$, $R = \{(b_1, b_1), (b_2, b_3), (b_2, b), (b_3, b_3), (b_3, b_2)\}$ then $\langle b_1 \rangle_R = \{b_1\}$, $\langle b_2 \rangle_R = \{b_2, b_3\}$, $\langle b_3 \rangle_R = \{b_2, b_3\}$ then $I = \{\emptyset, \{b_3\}, \{b_2\}, \{b_2, b_3\}\}$. Here the space Y is connected space because $UR(\{b_2\}) = \{b_2, b_3\}$ and $UR(\{b_1, b_3\}) = Y$. But is not \star - connected space since $Y = \{b_2\} \cup \{b_1, b_3\}$, $cl_R^*(\{b_2\}) \cap \{b_1, b_3\} = \{b_2\} \cap cl_R^*(\{b_1, b_3\}) = \emptyset$.

4. Let $Y = \{b_1, b_2, b_3\}$, $R = \{(b_1, b_1), (b_1, b_3), (b_2, b_2), (b_2, b_1), (b_2, b_3)\}$ then $\langle b_1 \rangle_R = \{b_1, b_3\}$, $\langle b_2 \rangle_R = \{b_1, b_2, b_3\}$, $\langle b_3 \rangle_R = \{b_1, b_3\}$. Also $R \langle b_1 \rangle = \{b_1, b_2\}$, $R \langle b_2 \rangle = \{b_2\}$, $R \langle b_3 \rangle = \emptyset$. Therefore $R \langle b_1 \rangle_R = \{b_1\}$, $R \langle b_2 \rangle_R = \{b_2\}$, $R \langle b_3 \rangle_R = \emptyset$. Then $I = \{\emptyset, \{b_2\}\}$. Here the space Y is \star - connected space because $cl_R^*(\{b_1\}) = cl_R^*(\{b_3\}) = cl_R^*(\{b_1, b_2\}) = cl_R^*(\{b_1, b_3\}) = cl_R^*(\{b_2, b_3\}) = Y$ and $cl_R^*(\{b_2\}) = \{b_2\}$. But is not $**$ - connected space since

$$Y = \{b_2\} \cup \{b_1, b_3\}, cl_R^{**}(\{b_2\}) \cap \{b_1, b_3\} = \{b_2\} \cap cl_R^{**}(\{b_1, b_3\}) = \emptyset.$$

Proposition 2.5: The following statements for an ideal approximation space (Y, R, I) , are equivalent.

1. Y is \star - connected.
2. For every $S, V \subseteq Y$ with $S \cap V = \emptyset$, $int_R^*(S) = S$, $int_R^*(V) = V$ and $S \cup V = Y$ then $S = \emptyset$ or $V = \emptyset$.
3. For every $S, V \subseteq Y$ with $S \cap V = \emptyset$, $cl_R^*(S) = S$, $cl_R^*(V) = V$ and $S \cup V = Y$ then $S = \emptyset$ or $V = \emptyset$.

Proof.

1. (1) \Rightarrow (2) Let $S, V \subseteq Y$ with \emptyset , $int_R^*(S) = S$, $int_R^*(V) = V$ so that $S \cap V = \emptyset$ and $S \cup V = Y$ then $cl_R^*(S) \subseteq cl_R^*(V^c) = (int_R^*(V))^c = V^c$, $cl_R^*(V) \subseteq cl_R^*(S^c) = (int_R^*(S))^c = S^c$.

Hence $cl_R^*(S) \cap V = S \cap cl_R^*(V) = \emptyset$ that is S, V are \star - separated sets, so that $S \cap V = Y$, but (Y, R, I) is \star - connected implies that $S = \emptyset$ or $V = \emptyset$.

2. (2) \Rightarrow (3) Let $S, V \subseteq Y$ with $S \cap V = \emptyset$, $cl_R^*(S) = S$, $cl_R^*(V) = V$ and $S \cup V = Y$ then $cl_R^*(S) \cap V = S \cap cl_R^*(V) = \emptyset$, thus $int_R^*(S) \cap V = S \cap int_R^*(V) = \emptyset$, so $int_R^*(S) = S$ and $int_R^*(V) = V$. Hence $S = \emptyset$ or $V = \emptyset$.

3. (3) \Rightarrow (1) Directly from (2) there are no two proper \star - separated sets $S, V \subseteq Y$ such that $S \cup V = Y$. Therefore (Y, R, I) is \star - connected.

Corollary 2.6: The following statements for an ideal approximation space (Y, R, I) , are equivalent.

1. Y is connected.
2. For every $S, V \subseteq Y$ with $S \cap V = \emptyset$, $LR(S) = S$, $LR(V) = V$ and $S \cup V = Y$ then $S = \emptyset$ or $V = \emptyset$.
3. For every $S, V \subseteq Y$ with $S \cap V = \emptyset$, $UR(S) = S$, $UR(V) = V$ and $S \cup V = Y$ then $S = \emptyset$ or $V = \emptyset$.

Corollary 2.7: The following statements for an ideal approximation space (Y, R, I) , are equivalent.

1. Y is $\star\star$ - connected.
2. For every $S, V \subseteq Y$ with $S \cap V = \emptyset$, $int_R^{\star\star}(S) = S$, $int_R^{\star\star}(V) = V$ and $S \cup V = Y$ then $S = \emptyset$ or $V = \emptyset$.
3. For every $S, V \subseteq Y$ with $S \cap V = \emptyset$, $cl_R^{\star\star}(S) = S$, $cl_R^{\star\star}(V) = V$ and $S \cup V = Y$ then $S = \emptyset$ or $V = \emptyset$.

Theorem 2.8: Let (Y, R, I) be an ideal approximation space, $N \subseteq Y$ is \star - connected if $S, V \subseteq Y$ are \star - separated sets with $N \subseteq S \cup V$ then either $N \subseteq S$ or $N \subseteq V$.

Proof.

Let S, V are \star - separated sets with $N \subseteq S \cup V$, thus $cl_R^*(S) \cap V = S \cap cl_R^*(V) = \emptyset$ and $N = (N \cap S) \cup (N \cap V)$ since $cl_R^*(N \subseteq S) \cap (N \subseteq V) \subseteq cl_R^*(N) \cap cl_R^*(S) \cap (N \cap V) = cl_R^*(N) \cap N \cap cl_R^*(S) \cap V = N \cap \emptyset = \emptyset$.

By similar way since $cl_R^*(N \subseteq V) \cap (N \subseteq S) \subseteq cl_R^*(N) \cap cl_R^*(V) \cap (N \cap S) = cl_R^*(N) \cap N \cap cl_R^*(V) \cap S = N \cap \emptyset = \emptyset$. Then $(N \cap S)$ and $(N \cap V)$ are \star - separated sets with $N = (N \cap S) \cup (N \cap V)$.

But N is \star - connected implies that $N \subseteq S$ or $N \subseteq V$.

Corollary 2.9: Let (Y, R, I) be an ideal approximation space and $N \subseteq Y$ is connected if $S, V \subseteq Y$ are separated sets with $N \subseteq S \cup V$ then either $N \subseteq S$ or $N \subseteq V$.

Corollary 2.10: Let (Y, R, I) be an ideal approximation space and $N \subseteq Y$ is $\star\star$ - connected if $S, V \subseteq Y$ are $\star\star$ - separated sets with $N \subseteq S \cup V$ then either $N \subseteq S$ or $N \subseteq V$.

Theorem 2.11: If N is \star - connected subset of (Y, R, I) as well $N \subset M \subset N^*$ then M is also a \star - connected subset of Y .

Proof.

Assume M isn't a \star -connected subset of (Y, R, I) thus there are \star -separated sets S and V such that $M = S \cup V$. This that suggests S and V are not empty and $cl_R^*(V) \cap S = V \cap cl_R^*(S) = \emptyset$ by theorem 2.8, we have that either $N \subset S$ or $N \subset V$. Suppose that $N \subset S$ then $N^* \subset S^*$ this implies that $V \subset M \subset N^*$ and $V = N^* \cap V = \emptyset$, thus V is an empty set. Since V is not empty this is a contradiction. Assume that $N \subseteq V$ in similar manner it follows that S is empty. This contradiction. Hence M is \star -connected.

Corollary 2.12: If N is $\star\star$ -connected subset of (Y, R, I) as well $N \subset M \subseteq N^{**}$ then M is also a $\star\star$ -connected subset of Y .

Corollary 2.13.

1. If N is \star -connected subset of (Y, R, I) then N^* is \star -connected.
2. If N is $\star\star$ -connected subset of (Y, R, I) then N^{**} is $\star\star$ -connected.

Theorem 2.14: If $\{Z_i : i \in I\}$ is not empty family of \star -connected sets of an ideal approximation space with $\bigcap_{i \in I} Z_i \neq \emptyset$, then $\bigcup_{i \in I} Z_i$ is \star -connected.

Proof.

Assume that $\bigcup_{i \in I} Z_i$ is not \star -connected. Then we obtain $\bigcup_{i \in I} Z_i = S \cup V$ where S and V are \star -separated sets in Y . Since $\bigcap_{i \in I} Z_i \neq \emptyset$ we've a point $y \in \bigcap_{i \in I} Z_i$ since $y \in \bigcup_{i \in I} Z_i$ either $y \in S$ or $y \in V$. Assume that $y \in S$ since $y \in Z_i$ for all $i \in I$ then Z_i and S intersect for every $i \in I$ according to Theorem 2.8 $Z_i \subset S$ or $Z_i \subset V$ since S and V are disjoint $Z_i \subset S$ for every $i \in I$ consequently $\bigcup_{i \in I} Z_i \subset S$. This means that V is empty. This is a contradiction. Assume that $y \in V$ in a similar manner, we have that S is empty. This is a contradiction thus $\bigcup_{i \in I} Z_i$ is \star -connected.

Corollary 2.15: If $\{Z_i : i \in I\}$ is not empty family of $\star\star$ -connected sets of an ideal approximation space with $\bigcap_{i \in I} Z_i \neq \emptyset$, then $\bigcup_{i \in I} Z_i$ is $\star\star$ -connected.

Corollary 2.16:

1. If N is a subset of the ideal approximation space (Y, R, I) that is a \star -connected and $N \cap N^* \neq \emptyset$ then $cl_R^*(N)$ is a \star -connected.
2. If N is a subset of the ideal approximation space (Y, R, I) that is a $\star\star$ -connected and $N \cap N^{**} \neq \emptyset$ then $cl_R^{**}(N)$ is a $\star\star$ -connected.

Theorem 2.17: Let (Y, R, I) be an ideal approximation space, $\{N_\beta : \beta \in \Delta\}$ be a family of \star -connected subset of Y and N be a \star -connected sub set of Y . If $N \cap N_\beta \neq \emptyset$ for all β then $N \cup (\bigcup N_\beta)$ is \star -connected.

Proof.

Since $N \cap N_\beta \neq \emptyset$ for every $\beta \in \Delta$ according to Theorem 2.14 $N \cap N_\beta$ is \star -connected for each $\beta \in \Delta$. Moreover $N \cup (\bigcup N_\beta) = \bigcup (N \cup N_\beta)$ and $\bigcap (N \cup N_\beta) \supseteq N \neq \emptyset$. Thus, by Theorem 2.14 $N \cup (\bigcup N_\beta)$ is \star -connected.

Corollary 2.18: Let (Y, R, I) be an ideal approximation space, $\{N_\beta : \beta \in \Delta\}$ be a family of $\star\star$ -connected subset of Y and N be a $\star\star$ -connected sub set of Y . If $N \cap N_\beta \neq \emptyset$ for all β then $N \cup (\bigcup N_\beta)$ is $\star\star$ -connected.

Theorem 2.19: Assume that $h: (Y, R_1, I) \rightarrow (W, R_2)$ be an \star -continuous function. Then $h(D) \subseteq W$ is connected set if D is \star -connected in Y .

Proof.

Let $h(D)$ be disconnected then there exists two separated sets $S, V \subseteq W$ with $h(D) \subseteq S \cup V$. That is $U_{R_2}(S) \cap V = S \cap U_{R_2}(V) = \emptyset$. Then $D \subseteq h^{-1}(S) \cap h^{-1}(V)$ and since h is \star -continuous we get that:

$$cl_{R_1}^*(h^{-1}(S)) \cap h^{-1}(V) \subseteq h^{-1}(U_{R_2}(S)) \cap h^{-1}(V) = h^{-1}(U_{R_2}(S) \cap V) = h^{-1}(\emptyset) = \emptyset.$$

and in similar way we have $cl_{R_1}^*(h^{-1}(V)) \cap h^{-1}(S) \subseteq h^{-1}(U_{R_2}(V)) \cap h^{-1}(S) =$

$$h^{-1}(U_{R_2}(V) \cap S) = h^{-1}(\emptyset) = \emptyset. \text{ Hence } h^{-1}(S) \text{ and } h^{-1}(V) \text{ are } \star\text{-separated sets in } Y \text{ so that } D \subseteq h^{-1}(S) \cap h^{-1}(V).$$

So D is \star -disconnected which contradicts that D is \star -connected

this is because of the incorrect assumption that $h(D)$ is disconnected and so $h(D)$ is connected set.

Corollary 2.20: Assume that $h: (Y, R_1, I) \rightarrow (W, R_2)$ be an $\star\star$ -continuous function. Then $h(D) \subseteq W$ is connected set if D is $\star\star$ -connected in Y .

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