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Inner Ideals of the Real Five-Dimensional Lie Algebras with Two-Dimensional Derived, Such That $l' \not\subseteq z$

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ABSTRACT: Inner ideal of the five-dimensional non-commutative Lie alge- bras over the real fields with two-dimensional derived were classified. It is proved that one, two, three and four-dimensional inner ideals are existed in every five-dimensional Lie algebra. It is also proved that five-dimensional Lie algebras contain inner ideals which are neither ideals nor sub-algebras.

Keywords: Inner ideal, Lie sub-algebra, five-dimensional Lie algebra.

1. INTRODUCTION

Georgia Benkart, an American scientist, introduced the notion of inner ideals (see [3]) in 1976. According to Benkart's definition, one can say that an inner ideal is a subspace V of a Lie algebra L, with the property $[V, [V, L]] \subseteq V$, where

$[V, [V, L]] = \text{span}\{[v_1, [v_2, \ell]] : v_1, v_2 \in V, \ell \in L\}$

An inner ideal V is called commutative if $[V, V] = 0$. She demonstrated that a strong correlation exist between elements in Lie algebras that are ad-nilpotents with inner ideal [4]. An accomplishment of Benkart's results is done in [5] by Benkart and Fernandiz Lopes and a generalization of there results is done in [6] by Brox, Fernandez Lopez and Gomez Lozano in 2016 to the case of centrally closed prime rings with involution of characteristic not 2, 3 or 5.

It can be seen from [9] and [11] that in Lie algebra the role of inner ideals is equivalent to that of the one-sided ideals in associative algebras. Therefore, Artin's theory can be generalized if one takes into account the inner ideals of Lie algebras. It was proved in [8] that an Artinian Lie algebra is a Lie algebra that has the property that every decreasing inner ideal chain must be terminates. In [1, Proposition 2], it was proved that every one-sided ideal of the finite dimensional associative algebra A admits Levi decomposition and can be generated by an idempotent if some minimal conditions are met. The same results was obtained for inner ideals by Baranov and Shlaka in [2], where they showed that every inner ideal of the Lie algebra [A, A] admits Levi decomposition and can be generated by idempotent pair (if satisfied some minimal conditions). These results were recently Generalized in [14] for the case of a sub-algebra of finite dimensional algebra. Further generalization is done in [10] and [16] for the infinite dimensional Lie sub-algebras of associative algebras. Abelian non-Jordan Lie inner ideals we also been studied in 2022 (see [15] for more details). Further motivation for studying inner ideals comes from [9], where Fern´andez L´opez et al showed that when L is an arbitrary non- degenerate Lie algebras over an abelian ring F together with two and three convertible, then for every nonzero commutative inner ideal V of finite length of L is complemented by an commutative inner ideal [9].

The classification of the real five-dimensional Lie algebras is given by Schöbel in [13]. He classified them in terms of the derived sub-algebra of these Lie algebras. Keep in mind that a derived sub-algebra L' of L is the set

$L' = [L, L] = \text{span } \{[\ell_1, \ell_2] \mid \ell_1, \ell_2 \in L\}$.

In [12] Saeed and Shlaka studied inner ideals of the four-dimensional Lie algebras over the real fields with two-dimensional derived. They proved that one, two and three- dimensional non-trivial inner ideals exist in every four-dimensional Lie algebra with 2-dimensional derived. Prior to that (see [17]) they classify inner ideals of the two and three-dimensional Lie algebras.

In this paper, we use techniques similar to [12] to study inner ideals of the real five-dimensional Lie algebras with 2-dimensional derived. Suppose that L is a five- dimensional Lie algebra over the real field with 1-dimensional derived. If L commu- tative, then it is easy to see that every 1, 2, 3 and 4-dimensional subspace of L is an inner ideal. Suppose now that L is non-commutative, then we get the following results, which is one of our main results:

1.1Theorem: Let L be a five-dimensional Lie algebra over the real field R with 2- dimensional derived L'. Then L contains a commutative and non-commutative I-ideal.

Recall that if L is 5-dimensional with 2-dimensional derived, then by 2.6Theorem and 2.7Theorem L is either L_ϵ or L_1 or L_2 or L_3 or L_4 or L_5. Thus, to prove the theorem we need to consider all of the cases.

2. Preliminaries

Definition 2.1 [7]: Let L be a vector space over any field F with a bilinear form $L \times L \to L$, where $(\ell_1, \ell_2) \to [\ell_1, \ell_2]$, for all $\ell_1, \ell_2 \in L$. Then L is called a Lie algebra over F , if the following conditions are satisfied:

(1) $[\ell_1, \ell_2] = 0$ for all $\ell_1, \ell_2 \in L$.

(2) $[\ell_1, [\ell_2, \ell_3]] + [\ell_2, [\ell_3, \ell_1]] + [\ell_3, [\ell_1, \ell_2]] = 0$ for all $\ell_1, \ell_2, \ell_3 \in L$.

2.2Definition [7]: The subspace B of a Lie algebra L is said to be a Lie sub-algebra of L, if $[b_1, b_2] \in B$ for all $b_1, b_2 \in B$.

2.3Definition [7]: The derived of a Lie algebra L is the set $L' = \text{span}\{[a, b] \mid a, b \in L\}$, where L' is a Lie sub algebra of L.

2.4Definition [7]: The center of L is the set $Z = \{x \in L \mid [x, y] = 0, \forall y \in L\}$.

2.5Definition [2]: Let V be a subspace of L. Then V is said to be an inner ideal of L when $[V, [V, L]] \subseteq V$. We denote by I-ideal to be an inner ideal of L. The inner ideal V is said to be commutative if $[V, V] = 0$.

Note that in every Lie algebra L, we have L, {0} are inner ideals of L called the trivial inner ideals. Recall that every ideal I of L is inner ideal, because $[I, [I, L]] = [I, I] \subseteq I$, but the inverse is not true. Since L is five-dimensional, the dimension of L' may be 1, 2, 3, 4 or 5. In [13] Schöbel classified the real n-dimensional Lie algebras by relating the dimensional of L'. For the dimensional of L' is 2, we have the following result, for the proof see [13, Theorem 1].

2.6Theorem [13]: Suppose that L is a real n-dimensional Lie algebra with a two- dimensional derived L', such that $L' \cap Z = \{0\}$ Then $L = L_4 \oplus Z_1$, where L_4 is a 4-dimensional real Lie algebra with 2-dimensional derived algebra and $L' \nsubseteq Z_4$, Z_n is the n-dimensional center of L.

2.7Theorem [13]: Suppose that L is a real 4-dimensional Lie algebra with a two- dimensional derived L', such that $L' \nsubseteq Z$, and let $\{x_1, x_2, x_3, x_4\}$ be a basis of L. Then L is one of the following six standard forms.

$L_\epsilon : [x_1, x_4] = \epsilon x_2, [x_2, x_4] = x_1$, and otherwise is zero, where $\epsilon = \mp$.

$L_1 : [x_1, x_4] = x_1, [x_2, x_4] = x_2$, and otherwise is zero.

$L_2 : [x_1, x_4] = -x_1 + px_2, [x_2, x_4] = x_1$, and otherwise is zero, where $p \in R$.

$L_3 : [x_1, x_3] = x_1, [x_2, x_4] = x_2$, and otherwise is zero.

$L_4 : [x_1, x_3] = -x_2, [x_1, x_4] = x_1, [x_2, x_3] = x_1, [x_2, x_4] = x_2$, and otherwise is zero.

$L_5 : [x_1, x_4] = x_1, [x_2, x_3] = x_1, [x_2, x_4] = x_2$, and otherwise is zero.

3. INNER IDEALS OF THE FIVE-DIMENSIONAL LIE ALGEBRA

Throughout this section, we prove some results related to inner ideal of the 5- dimensional real Lie algebra with 2-dimensional derived. Our aim is to prove the following theorem.

3.1Proposition: Suppose that $L = L_\epsilon$ and $L' \nsubseteq Z$. Then the following is hold.

1. L contains a 1-dimensional I-ideal which is not ideal.
2. L contains a 2-dimensional commutative I-ideal which is not ideal.
3. L contains a 3-dimensional commutative I-ideal which is not ideal.
4. L contains a 3-dimensional non-commutative I-ideal.
5. L contains a 4-dimensional commutative I-ideal.
6. L contains a 4-dimensional non-commutative I-ideal.

Proof: By 2.6Theorems and 2.7, there is a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_ϵ with the Lie multiplication $[x_1, x_4] = \epsilon x_2$, $[x_2, x_4] = x_1$ and otherwise is zero, where $\epsilon = \mp$. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_2\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2$, $y = \alpha_2 x_2$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_2, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]] = [\alpha_1 x_2, \alpha_2\beta_4 x_1] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_2, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4 x_1 \notin V$,

V is not ideal of L, as required.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_2, x_3\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_3$, $y = \alpha_3 x_2 + \alpha_4 x_3$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_2 + \alpha_4 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_2 + \alpha_2 x_3, \alpha_3\beta_4 x_1] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_2 + \alpha_2 x_3, \alpha_3 x_2 + \alpha_4 x_3] = 0$. Therefore, V is a commutative I-ideal of L.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_2 + \alpha_2 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4 x_1 \notin V$,

Therefore, V is not ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z$, $y = \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \alpha_4\beta_4\epsilon x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z] = 0$.

Therefore, V is a commutative I-ideal of L.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4\epsilon x_2 \notin V$,

Therefore, V is not ideal of L.

4) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_2, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4$, $y = \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4, \alpha_4\beta_4\epsilon x_2 + \alpha_5\beta_4 x_1 + \alpha_6\beta_1\epsilon x_2 - \alpha_6\beta_2 x_1]$

$= (\alpha_3\alpha_4\beta_4\epsilon + \alpha_3\alpha_6\beta_1\epsilon)x_1 + (\alpha_3\alpha_5\beta_4\epsilon + \alpha_3\alpha_6\beta_2\epsilon)x_2 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4, \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4]$

$= (\alpha_1\alpha_6\epsilon + \alpha_3\alpha_4\epsilon)x_2 + (\alpha_2\alpha_6 - \alpha_3\alpha_5)x_1 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

5) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$.

Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_5\beta_4\epsilon x_2 + \alpha_6\beta_4 x_1] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z] = 0$

Therefore, V is a commutative I-ideal of L, as required.

6) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4, \alpha_5\beta_4\epsilon x_2 + \alpha_6\beta_4 x_1 + \alpha_8\beta_1\epsilon x_2 - \alpha_8\beta_2 x_1]$

$= (\alpha_4\alpha_5\beta_4\epsilon + \alpha_4\alpha_8\beta_1\epsilon)x_1 + (\alpha_4\alpha_6\beta_4\epsilon + \alpha_4\alpha_8\beta_2\epsilon)x_2 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4, \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4]$

$= (\alpha_1\alpha_8\epsilon + \alpha_4\alpha_5\epsilon)x_2 + (\alpha_2\alpha_8 - \alpha_4\alpha_6)x_1 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required. □

3.2Remark: 3.1Theorem is not true if we state that every 1, 2, 3and4-dimensional subspace is an I-ideal because L = L_ϵ contains a 1, 2, 3and4-dimensional subspace which is not I-ideal. As one can see in the following examples.

3.3Example: Recall that we fix a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_ϵ with the Lie multiplication $[x_1, x_4] = \epsilon x_2$, $[x_2, x_4] = x_1$ and otherwise is zero, where $\epsilon = \mp$. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_4$, $y = \alpha_2 x_4$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_4, \alpha_2\beta_1\epsilon x_2 - \alpha_2\beta_2 x_1] = \alpha_1\alpha_2\beta_1\epsilon x_1 + \alpha_1\alpha_2\beta_2\epsilon x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_3 + \alpha_2 x_4$, $y = \alpha_3 x_3 + \alpha_4 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_3 + \alpha_4 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_3 + \alpha_2 x_4, \alpha_4\beta_1\epsilon x_2 - \alpha_4\beta_2 x_1] = \alpha_2\alpha_4\beta_1\epsilon x_1 + \alpha_2\alpha_4\beta_2\epsilon x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4$, $y = \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4, \alpha_4\beta_4\epsilon x_2 + \alpha_6\beta_1\epsilon x_2 - \alpha_6\beta_2 x_1]$

$= (\alpha_3\alpha_4\beta_4\epsilon + \alpha_3\alpha_6\beta_1\epsilon)x_1 + \alpha_3\alpha_6\beta_2\epsilon x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

4) We claim that the 4-dimensional subspace $V = \text{span}\{x_2, x_3, x_4, z\}$ is not an I- ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_3 + \alpha_4 z$, $y = \alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \alpha_5\beta_4 x_1 + \alpha_7\beta_1\epsilon x_2 - \alpha_7\beta_2 x_1]$

$= (\alpha_3\alpha_5\beta_4\epsilon + \alpha_3\alpha_7\beta_2\epsilon)x_2 + \alpha_3\alpha_7\beta_1\epsilon x_1 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3.4 Proposition: Suppose that $L = L_1$ and $L' \nsubseteq Z$. Then the following is hold.

1. L contains a 1-dimensional Ideal.
2. L contains a 2-dimensional commutative I-ideal.
3. L contains a 3-dimensional commutative I-ideal.
4. L contains a 3-dimensional non-commutative I-ideal.
5. L contains a 4-dimensional commutative I-ideal.
6. L contains a 4-dimensional non-commutative I-ideal.

Proof. By 2.6Theorems and 2.7, there is a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_1 with the Lie multiplication $[x_1, x_4] = x_1$, $[x_2, x_4] = x_2$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_2\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2$, $y = \alpha_2 x_2$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_2, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]] = [\alpha_1 x_2, \alpha_2\beta_4 x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_2, x_3\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$ Then $x = \alpha_1 x_2 + \alpha_2 x_3$, $y = \alpha_3 x_2 + \alpha_4 x_3$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_2 + \alpha_4 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_2 + \alpha_2 x_3, \alpha_3\beta_4 x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_2 + \alpha_2 x_3, \alpha_3 x_2 + \alpha_4 x_3] = 0$. Therefore, V is a commutative I-ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z$, $y = \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \alpha_4\beta_4 x_1] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z] = 0$

Therefore, V is a commutative I-ideal of L.

4) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_2, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4$, $y = \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4, \alpha_4\beta_4 x_1 + \alpha_5\beta_4 x_2 - \alpha_6\beta_1 x_1 - \alpha_6\beta_2 x_2]$

$= (-\alpha_3\alpha_4\beta_4 + \alpha_3\alpha_6\beta_1)x_1 + (-\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_6\beta_2)x_2 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y]\neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4, \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4]$

$= (\alpha_1\alpha_6 - \alpha_3\alpha_4)x_1 + (\alpha_2\alpha_6 - \alpha_3\alpha_5)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

5) We claim that the 4-dimensional subspace V = span $\{x_1, x_2, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3+\alpha_4 z$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_5\beta_4 x_1 + \alpha_6\beta_4 x_2] = 0 \in V,$

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z] = 0$

Therefore, V is a commutative I-ideal of L, as required.

6) We claim that the 4-dimensional subspace V = span $\{x_1, x_2, x_3, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4, \alpha_5\beta_4 x_1 + \alpha_6\beta_4 x_2 - \alpha_8\beta_1 x_1 - \alpha_8\beta_2 x_2]$

$= (-\alpha_4\alpha_5\beta_4 + \alpha_4\alpha_8\beta_1)x_1 + (-\alpha_4\alpha_6\beta_4 + \alpha_4\alpha_8\beta_2)x_2 \in V,$

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y]\neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4, \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4]$

$= (\alpha_1\alpha_8 - \alpha_4\alpha_5)x_1 + (\alpha_2\alpha_8 - \alpha_4\alpha_6)x_2\neq 0$

Therefore, V is a non-commutative I-ideal of L, as required. □

3.5 Remark: 3.4Theorem is not true if we state that every 1, 2, 3 and 4-dimensional subspace is an I-ideal because $L = L_1$ contains a 1, 2, 3and4-dimensional subspace which is not I-ideal. As one can see in the following examples.

3.6 Example: Recall that we fix a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_1 with the Lie mul- tiplication $[x_1, x_4] = x_1$,$[x_2, x_4] = x_2$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace V = span $\{x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_4$, $y = \alpha_2 x_4$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_4, -\alpha_2\beta_1 x_1 - \alpha_2\beta_2 x_2] = \alpha_1\alpha_2\beta_1 x_1 + \alpha_1\alpha_2\beta_2 x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

2) We claim that the 2-dimensional subspace V = span $\{x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_3 + \alpha_2 x_4$, $y = \alpha_3 x_3 + \alpha_4 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_3 + \alpha_4 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_3 + \alpha_2 x_4, -\alpha_4\beta_1 x_1 - \alpha_4\beta_2 x_2] = \alpha_2\alpha_4\beta_1 x_1 + \alpha_2\alpha_4\beta_2 x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3) We claim that the 3-dimensional subspace V = span $\{x_1, x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4$, $y = \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4, \alpha_4\beta_4 x_1 - \alpha_6\beta_1 x_1 - \alpha_6\beta_2 x_2]$

$= (-\alpha_3\alpha_4\beta_4 + \alpha_3\alpha_6\beta_1)x_1 + \alpha_3\alpha_6\beta_2 x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

4) We claim that the 4-dimensional subspace V = span $\{x_2, x_3, x_4, z\}$ is not an I- ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_2 +\alpha_2 x_3 +\alpha_3 x_3 +\alpha_4 z$, $y = \alpha_5 x_2 +\alpha_6 x_3 +\alpha_7 x_4 +\alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \alpha_5\beta_4 x_2 - \alpha_7\beta_1 x_1 - \alpha_7\beta_2 x_2]$

$= (-\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_7\beta_2)x_2 + \alpha_3\alpha_7\beta_1 x_1 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3.7Proposition: Suppose that $L = L_2$ and $L' \nsubseteq Z$. Then the following is hold.

1. L contains a 1-dimensional I-ideal which is not ideal.
2. L contains a 2-dimensional commutative I-ideal which is not ideal.
3. L contains a 3-dimensional commutative I-ideal which is not ideal.
4. L contains a 3-dimensional non-commutative I-ideal.
5. L contains a 4-dimensional commutative I-ideal.
6. L contains a 4-dimensional non-commutative I-ideal.

Proof: By 2.6Theorems and 2.7, there is a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_2 with the Lie multiplication $[x_1, x_4] = -x_1 + px_2$, $[x_2, x_4] = x_1$ and otherwise is zero, where $p \in R$. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_2\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$ Then $x = \alpha_1 x_2$, $y = \alpha_2 x_2$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_2, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]] = [\alpha_1 x_2, \alpha_2\beta_4 x_1] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_2, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4 x_1 \notin V$,

V is not ideal of L, as required.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_2, x_3\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_3$, $y = \alpha_3 x_2 + \alpha_4 x_3$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_2 + \alpha_4 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_2 + \alpha_2 x_3, \alpha_3\beta_4 x_1] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_2 + \alpha_2 x_3, \alpha_3 x_2 + \alpha_4 x_3] = 0$. Therefore, V is a commutative I-ideal of L.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_2 + \alpha_2 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z] = \alpha_1\beta_4 x_1 \notin V$,

Therefore, V is not ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z$, $y = \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, -\alpha_4\beta_4 x_1 + \alpha_4\beta_4 p x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 z] = 0$

Therefore, V is a commutative I-ideal of L.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]$

$= -\alpha_1\beta_4 x_1 + \alpha_1\beta_4 p x_2 \notin V$,

Therefore, V is not ideal of L.

4) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_2, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4$, $y = \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4, -\alpha_4\beta_4 x_1 + \alpha_4\beta_4 p x_2 + \alpha_5\beta_4 x_1 + \alpha_6\beta_1 x_1 - \alpha_6\beta_1 p x_2 - \alpha_6\beta_2 x_1]$

$= (-\alpha_3\alpha_4\beta_4 - \alpha_3\alpha_4\beta_4 p + \alpha_3\alpha_5\beta_4 + \alpha_3\alpha_6\beta_1 + \alpha_3\alpha_6\beta_1 p - \alpha_3\alpha_6\beta_2)x_1$

$+p(\alpha_3\alpha_4\beta_4 - \alpha_3\alpha_5\beta_4 - \alpha_3\alpha_6\beta_1 + \alpha_3\alpha_6\beta_2)x_2 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4, \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4]$

$= (-\alpha_1\alpha_6 + \alpha_2\alpha_6 + \alpha_3\alpha_4 - \alpha_3\alpha_5)x_1 + p(\alpha_1\alpha_6 - \alpha_3\alpha_4)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

5) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, -\alpha_5\beta_4 x_1 + \alpha_5\beta_4 p x_2 + \alpha_6\beta_4 x_1] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 z, \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 z] = 0$

Therefore, V is a commutative I-ideal of L, as required.

6) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4, -\alpha_5\beta_4 x_1 + \alpha_5\beta_4 p x_2 + \alpha_6\beta_4 x_1 + \alpha_8\beta_1 x_1 - \alpha_8\beta_1 p x_2$

$-\alpha_8\beta_2 x_1] = (-\alpha_4\alpha_5\beta_4 - \alpha_4\alpha_5\beta_4 p + \alpha_4\alpha_6\beta_4 + \alpha_4\alpha_8\beta_1 + \alpha_4\alpha_8\beta_1 p - \alpha_4\alpha_8\beta_2)x_1$

$+p(\alpha_4\alpha_5\beta_4 - \alpha_4\alpha_6\beta_4 - \alpha_4\alpha_8\beta_1 + \alpha_4\alpha_8\beta_2)x_2 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4, \alpha_5x_1 + \alpha_6x_2 + \alpha_7x_3 + \alpha_8x_4]$
$= (-\alpha_1\alpha_8 + \alpha_2\alpha_8 + \alpha_4\alpha_5 - \alpha_4\alpha_6)x_1 + p(\alpha_1\alpha_8 - \alpha_4\alpha_5)x_2 \neq 0$
Therefore, V is a non-commutative I-ideal of L, as required.
□

3.8 Remark: 3.7 Theorem is not true if we state that every 1, 2, 3and4-dimensional subspace is an I-ideal because L = L_2 contains a 1, 2, 3and4-dimensional subspace which is not I-ideal. As one can see in the following examples.

3.9 Example: Recall that we fix a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_2 with the Lie multi- plication $[x_1, x_4] = (-x_1 + px_2)$, $[x_2, x_4] = x_1$ and otherwise is zero, where p ∈ R. Let ℓ ∈ L. Then $\ell = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace V = span $\{x_4\}$ is not an I-ideal of L. Let x, y ∈ V. Then $x = \alpha_1x_4$, $y = \alpha_2x_4$ for some $\alpha_1, \alpha_2 \in R$. Since
$[x, [y, \ell]] = [x, [\alpha_2x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$
$= [\alpha_1x_4, \alpha_2\beta_1x_1 - \alpha_2\beta_1px_2 - \alpha_2\beta_2x_1]$
$= (\alpha_1\alpha_2\beta_1 + \alpha_1\alpha_2\beta_1p - \alpha_1\alpha_2\beta_2)x_1 + p(\alpha_1\alpha_2\beta_1 + \alpha_1\alpha_2\beta_2)x_2 \notin V$,
$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

2) We claim that the 2-dimensional subspace V = span $\{x_3, x_4\}$ is not an I-ideal of L. Let x, y ∈ V . Then $x = \alpha_1x_3 + \alpha_2x_4$, $y = \alpha_3x_3 + \alpha_4x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since
$[x, [y, \ell]] = [x, [\alpha_3x_3 + \alpha_4x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$
$= [\alpha_1x_3 + \alpha_2x_4, \alpha_4\beta_1x_1 - \alpha_4\beta_1px_2 - \alpha_4\beta_2x_1]$
$= (\alpha_2\alpha_4\beta_1 + \alpha_2\alpha_4\beta_1p - \alpha_2\alpha_4\beta_2)x_1 + p(-\alpha_2\alpha_4\beta_1 + \alpha_2\alpha_4\beta_2)x_2 \notin V$,
$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3) We claim that the 3-dimensional subspace V = span $\{x_1, x_3, x_4\}$ is not an I-ideal of L. Let x, y ∈ V. Then $x = \alpha_1x_1 + \alpha_2x_3 + \alpha_3x_4$, $y = \alpha_4x_1 + \alpha_5x_3 + \alpha_6x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since
$[x, [y, \ell]] = [x, [\alpha_4x_1 + \alpha_5x_3 + \alpha_6x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$
$= [\alpha_1x_1 + \alpha_2x_3 + \alpha_3x_4, -\alpha_4\beta_4x_1 + \alpha_4\beta_4px_2 + \alpha_6\beta_1x_1 - \alpha_6\beta_1px_2 - \alpha_6\beta_2x_1]$
$= (-\alpha_3\alpha_4\beta_4 - \alpha_3\alpha_4\beta_4p + \alpha_3\alpha_6\beta_1 + \alpha_3\alpha_6\beta_1p - \alpha_3\alpha_6\beta_2)x_1$
$+p(\alpha_3\alpha_4\beta_4 - \alpha_3\alpha_6\beta_1 + \alpha_3\alpha_6\beta_2)x_2 \notin V$,
$[V, [V, L]] \nsubseteq V$, Therefore *V* is not an *I*-ideal of *L*.

4) We claim that the 4-dimensional subspace V = span $\{x_2, x_3, x_4, z\}$ is not an I- ideal of L. Let x, y ∈ V. Then $x = \alpha_1x_2 + \alpha_2x_3 + \alpha_3x_4 + \alpha_4z$, $y = \alpha_5x_2 + \alpha_6x_3 + \alpha_7x_4 + \alpha_8z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since
$[x, [y, \ell]] = [x, [\alpha_5x_2 + \alpha_6x_3 + \alpha_7x_4 + \alpha_8z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$
$= [\alpha_1x_2 + \alpha_2x_3 + \alpha_3x_4 + \alpha_4z, \alpha_5\beta_4x_1 + \alpha_7\beta_1x_1 - \alpha_7\beta_1px_2 - \alpha_7\beta_2x_1]$
$= (\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_7\beta_1 + \alpha_3\alpha_7\beta_1p - \alpha_3\alpha_7\beta_2)x_1 + p(-\alpha_3\alpha_5\beta_4 - \alpha_3\alpha_7\beta_1 + \alpha_3\alpha_7\beta_2)x_2 \notin V$,
$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3.10 Proposition: Suppose that L = L_3 and $L' \nsubseteq Z$. Then the following is hold.

1. L contains a 1-dimensional I-ideal .
2. L contains a 2-dimensional commutative I-ideal.
3. L contains a 2-dimensional non-commutative I-ideal.
4. L contains a 3-dimensional commutative I-ideal.
5. L contains a 3-dimensional non-commutative I-ideal.
6. L contains a 4-dimensional non-commutative I-ideal.

Proof: By 2.6Theorems and 2.7, there is a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_3 with the Lie multiplication $[x_1, x_3] = x_1$,$[x_2, x_4] = x_2$ and otherwise is zero. Let ℓ ∈ L. Then $\ell = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace V = span $\{x_2\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let x, y ∈ V. Then $x = \alpha_1x_2$, $y = \alpha_2x_2$ for some $\alpha_1, \alpha_2 \in R$. Since
$[x, [y, \ell]] = [x, [\alpha_2x_2, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]] = [\alpha_1x_2, \alpha_2\beta_4x_2] = 0 \in V$,
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

2) We claim that the 2-dimensional subspace V = span $\{x_2, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let x, y ∈ V. Then $x = \alpha_1x_2 + \alpha_2z$, $y = \alpha_3x_2 + \alpha_4z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since
$[x, [y, \ell]] = [x, [\alpha_3x_2 + \alpha_4z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$
$= [\alpha_1x_2 + \alpha_2z, \alpha_3\beta_4x_2] = 0 \in V$,
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.
Now we need to show that V is commutative. Since
$[x, y] = [\alpha_1x_2 + \alpha_2z, \alpha_3x_2 + \alpha_4z] = 0$. Therefore, V is a commutative I-ideal of L.

3) We claim that the 2-dimensional subspace V = span $\{x_1, x_3\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let x, y ∈ V. Then $x = \alpha_1x_1 + \alpha_2x_3$, $y = \alpha_3x_1 + \alpha_4x_3$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since
$[x, [y, \ell]] = [x, [\alpha_3x_1 + \alpha_4x_3, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$
$= [\alpha_1x_1 + \alpha_2x_3, \alpha_3\beta_3x_1 - \alpha_4\beta_1x_1] = (-\alpha_2\alpha_3\beta_3 + \alpha_2\alpha_4\beta_1)x_1 \in V$,
$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_3, \alpha_3 x_1 + \alpha_4 x_3] = (\alpha_1\alpha_4 - \alpha_2\alpha_3)x_1 \neq 0$

Therefore, V is a non-commutative I-ideal of L.

4) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_2, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 z$, $y = \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 z, \alpha_4\beta_3 x_1 + \alpha_5\beta_4 x_2] = 0 \in V,$

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 z, \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 z] = 0$

Therefore, V is a commutative I-ideal of L.

5) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_2, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4$, $y = \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4, \alpha_4\beta_3 x_1 + \alpha_5\beta_4 x_2 - \alpha_6\beta_2 x_2] = (-\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_6\beta_2)x_2 \in V,$

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4, \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 x_4] = (\alpha_2\alpha_6 - \alpha_3\alpha_5)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

6) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, $y = \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4, \alpha_5\beta_3 x_1 + \alpha_6\beta_4 x_2 - \alpha_7\beta_1 x_1 - \alpha_8\beta_2 x_2]$

$= (-\alpha_3\alpha_5\beta_3 + \alpha_3\alpha_7\beta_1)x_1 + (-\alpha_4\alpha_6\beta_4 + \alpha_4\alpha_8\beta_2)x_2 \in V,$

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4, \alpha_5 x_1 + \alpha_6 x_2 + \alpha_7 x_3 + \alpha_8 x_4]$

$= (\alpha_1\alpha_7 - \alpha_3\alpha_5)x_1 + (\alpha_2\alpha_8 - \alpha_4\alpha_6)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required. □

3.11Remark: 3.10Theorem is not true if we state that every 1, 2, 3and4-dimensional subspace is an I-ideal because $L = L_3$ contains a 1, 2, 3and4-dimensional subspace which is not I-ideal. As one can see in the following examples.

3.12Example: Recall that we fix a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_3 with the Lie mul- tiplication $[x_1, x_3] = x_1, [x_2, x_4] = x_2$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_4$, $y = \alpha_2 x_4$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_4, -\alpha_2\beta_2 x_2] = \alpha_1\alpha_2\beta_2 x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_3 + \alpha_2 x_4$, $y = \alpha_3 x_3 + \alpha_4 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3 x_3 + \alpha_4 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_3 + \alpha_2 x_4, -\alpha_3\beta_1 x_1 - \alpha_4\beta_2 x_2] = \alpha_1\alpha_3\beta_1 x_1 + \alpha_2\alpha_4\beta_2 x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4$, $y = \alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4 x_1 + \alpha_5 x_3 + \alpha_6 x_4, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_1 + \alpha_2 x_3 + \alpha_3 x_4, \alpha_4\beta_3 x_1 - \alpha_5\beta_1 x_1 - \alpha_6\beta_2 x_2]$

$= (-\alpha_2\alpha_4\beta_3 + \alpha_2\alpha_5\beta_1)x_1 + \alpha_3\alpha_6\beta_2 x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

4) We claim that the 4-dimensional subspace $V = \text{span}\{x_2, x_3, x_4, z\}$ is not an I- ideal of L. Let $x, y \in V$. Then $x = \alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z$, $y = \alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4 + \alpha_8 z, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 z]]$

$= [\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_4 + \alpha_4 z, \alpha_5\beta_4 x_2 - \alpha_6\beta_1 x_1 - \alpha_7\beta_2 x_2]$

$= \alpha_2\alpha_6\beta_1 x_1 + (-\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_7\beta_2)x_2 \notin V,$

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3.13 Proposition: Suppose that $L = L_4$ and $L' \nsubseteq Z$. Then the following is hold.

1. L contains a 1-dimensional I-ideal which is not ideal.
2. L contains a 2-dimensional commutative I-ideal which is not ideal .
3. L contains a 3-dimensional commutative I-ideal .

4. L contains a 3-dimensional non-commutative I-ideal.
5. L contains a 4-dimensional non-commutative I-ideal.

Proof: By 2.6Theorems and 2.7, there is a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_4 with the Lie multiplication $[x_1, x_3] = -x_2$, $[x_1, x_4] = x_1$, $[x_2, x_3] = x_1$, $[x_2, x_4] = x_2$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_2\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1x_2$, $y = \alpha_2x_2$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2x_2, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_2, \alpha_2\beta_3x_1 + \alpha_2\beta_4x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is not ideal. Since

$[x, \ell] = [\alpha_1x_2, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]$

$= \alpha_1\beta_3x_1 + \alpha_1\beta_4x_2 \notin V$,

V is not ideal of L, as required.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_2, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1x_2 + \alpha_2z$, $y = \alpha_3x_2 + \alpha_4z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3x_2 + \alpha_4z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_2 + \alpha_2z, \alpha_3\beta_3x_1 + \alpha_3\beta_4x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1x_2 + \alpha_2z, \alpha_3x_2 + \alpha_4z] = 0$. Therefore, V is a commutative I-ideal of L.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1x_2 + \alpha_2z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z] = \alpha_1\beta_3x_1 + \alpha_1\beta_4x_2 \notin V$,

Therefore, V is not ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_2, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1x_1 + \alpha_2x_2 + \alpha_3z$, $y = \alpha_4x_1 + \alpha_5x_2 + \alpha_6z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4x_1 + \alpha_5x_2 + \alpha_6z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_2 + \alpha_3z, -\alpha_4\beta_3x_2 + \alpha_4\beta_4x_1 + \alpha_5\beta_3x_1 + \alpha_5\beta_4x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1x_1 + \alpha_2x_2 + \alpha_3z, \alpha_4x_1 + \alpha_5x_2 + \alpha_6z] = 0$

Therefore, V is a commutative I-ideal of L.

4) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_2, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_4$, $y = \alpha_4x_1 + \alpha_5x_2 + \alpha_6x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4x_1 + \alpha_5x_2 + \alpha_6x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_4, -\alpha_4\beta_3x_2 + \alpha_4\beta_4x_1 + \alpha_5\beta_3x_1 + \alpha_5\beta_4x_2 - \alpha_6\beta_1x_1 - \alpha_6\beta_2x_2]$

$= (-\alpha_3\alpha_4\beta_4 - \alpha_3\alpha_5\beta_3 + \alpha_3\alpha_6\beta_1)x_1 + (\alpha_3\alpha_4\beta_3 - \alpha_3\alpha_5\beta_4 + \alpha_3\alpha_6\beta_2)x_2 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_4, \alpha_4x_1 + \alpha_5x_2 + \alpha_6x_4]$

$= (\alpha_1\alpha_6 - \alpha_3\alpha_4)x_1 + (\alpha_2\alpha_6 - \alpha_3\alpha_5)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

5) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4z$, $y = \alpha_5x_1 + \alpha_6x_2 + \alpha_7x_3 + \alpha_8z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5x_1 + \alpha_6x_2 + \alpha_7x_3 + \alpha_8z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4z, -\alpha_5\beta_3x_2 + \alpha_5\beta_4x_1 + \alpha_6\beta_3x_1 + \alpha_6\beta_4x_2 + \alpha_7\beta_1x_2$

$-\alpha_7\beta_2x_1] = (\alpha_3\alpha_5\beta_3 - \alpha_3\alpha_6\beta_4 - \alpha_3\alpha_7\beta_1)x_1 + (\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_6\beta_3 - \alpha_3\alpha_7\beta_2)x_2 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4z, \alpha_5x_1 + \alpha_6x_2 + \alpha_7x_3 + \alpha_8z]$

$= (\alpha_2\alpha_7 - \alpha_3\alpha_6)x_1 + (-\alpha_1\alpha_7 + \alpha_3\alpha_5)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required. □

3.14 Remark: 3.13Theorem is not true if we state that every 1, 2, 3and4-dimensional subspace is an I-ideal because $L = L_4$ contains a 1, 2, 3and4-dimensional subspace which is not I-ideal. As one can see in the following examples.

3.15 Example: Recall that we fix a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_4 with the Lie multi- plication $[x_1, x_3] = -x_2$, $[x_1, x_4] = x_1$, $[x_2, x_3] = x_1$, $[x_2, x_4] = x_2$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_4$, $y = \alpha_2x_4$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_4, -\alpha_2\beta_1x_1 - \alpha_2\beta_2x_2] = \alpha_1\alpha_2\beta_1x_1 + \alpha_1\alpha_2\beta_2x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

2) We claim that the 2-dimensional subspace V = span $\{x_3, x_4\}$ is not an I-ideal of L. Let x, y ∈ V. Then $x = \alpha_1x_3 + \alpha_2x_4$, $y = \alpha_3x_3 + \alpha_4x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3x_3 + \alpha_4x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_3 + \alpha_2x_4, \alpha_3\beta_1x_2 - \alpha_3\beta_2x_1 - \alpha_4\beta_1x_1 - \alpha_4\beta_2x_2]$

$=(-\alpha_1\alpha_3\beta_1 + \alpha_1\alpha_4\beta_2 + \alpha_2\alpha_3\beta_2 + \alpha_2\alpha_4\beta_1)x_1+(-\alpha_1\alpha_3\beta_2 - \alpha_1\alpha_4\beta_1 - \alpha_2\alpha_3\beta_1 + \alpha_2\alpha_4\beta_2)x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3) We claim that the 3-dimensional subspace V = span $\{x_1, x_4, z\}$ is not an I-ideal of L. Let x, y ∈ V. Then $x = \alpha_1x_1 + \alpha_2x_4 + \alpha_3z$, $y = \alpha_4x_1 + \alpha_5x_4 + \alpha_6z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4x_1 + \alpha_5x_4 + \alpha_6z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_4 + \alpha_3z, -\alpha_4\beta_3x_2 + \alpha_4\beta_4x_1 - \alpha_5\beta_1x_1 - \alpha_5\beta_2x_2]$

$= (-\alpha_2\alpha_4\beta_4 + \alpha_2\alpha_5\beta_1)x_1 + (\alpha_2\alpha_4\beta_3 + \alpha_2\alpha_5\beta_2)x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

4) We claim that the 4-dimensional subspace V = span $\{x_2, x_3, x_4, z\}$ is not an I- ideal of L. Let x, y ∈ V. Then $x = \alpha_1x_2 +\alpha_2x_3 +\alpha_3x_4 +\alpha_4z$, $y = \alpha_5x_2 +\alpha_6x_3 +\alpha_7x_4 +\alpha_8z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5x_2 + \alpha_6x_3 + \alpha_7x_4 + \alpha_8z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_2 + \alpha_2x_3 + \alpha_3x_4 + \alpha_4z, \alpha_5\beta_3x_1 + \alpha_5\beta_4x_2 + \alpha_6\beta_1x_2 - \alpha_6\beta_2x_1 - \alpha_7\beta_1x_1$

$-\alpha_7\beta_2x_2] = (-\alpha_2\alpha_5\beta_4 - \alpha_2\alpha_6\beta_1 + \alpha_2\alpha_7\beta_2 - \alpha_3\alpha_5\beta_3 + \alpha_3\alpha_6\beta_2 + \alpha_3\alpha_7\beta_1)x_1$

$+(\alpha_2\alpha_5\beta_3 - \alpha_2\alpha_6\beta_2 - \alpha_2\alpha_7\beta_1 - \alpha_3\alpha_5\beta_4 - \alpha_3\alpha_6\beta_1 + \alpha_3\alpha_7\beta_2)x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3.16 Proposition: Suppose that $L = L_5$ and $L' \nsubseteq Z$. Then the following is hold.

1. L contains a 1-dimensional I-ideal which is not ideal.
2. L contains a 2-dimensional commutative I-ideal which is not ideal.
3. L contains a 3-dimensional commutative I-ideal.
4. L contains a 3-dimensional non-commutative I-ideal.
5. L contains a 4-dimensional non-commutative I-ideal.

Proof: By 2.6Theorems and 2.7, there is a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_5 with the Lie multiplication $[x_1, x_4] = x_1$, $[x_2, x_3] = x_1$,$[x_2, x_4] = x_2$ and otherwise is zero. Let ℓ ∈ L. Then $\ell = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace V = span $\{x_2\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let x, y ∈ V. Then $x = \alpha_1x_2$, $y = \alpha_2x_2$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] =[x, [\alpha_2x_2, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]=[\alpha_1x_2, \alpha_2\beta_3x_1 + \alpha_2\beta_4x_2] = 0\in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is not ideal. Since

$[x, \ell] = [\alpha_1x_2, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z] = \alpha_1\beta_3x_1 + \alpha_1\beta_4x_2 \notin V$,

V is not ideal of L, as required.

2) We claim that the 2-dimensional subspace V = span $\{x_2, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let x, y ∈ V. Then $x = \alpha_1x_2 +\alpha_2z$, $y = \alpha_3x_2 +\alpha_4z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3x_2 + \alpha_4z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_2 + \alpha_2z, \alpha_3\beta_3x_1 + \alpha_3\beta_4x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1x_2 + \alpha_2z, \alpha_3x_2 + \alpha_4z] = 0$. Therefore, V is a commutative I-ideal of L.

It remains to us show that V is not ideal. Since

$[x, \ell] = [\alpha_1x_2 + \alpha_2z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z] = \alpha_1\beta_3x_1 + \alpha_1\beta_4x_2 \notin V$,

Therefore, V is not ideal of L.

3) We claim that the 3-dimensional subspace V = span $\{x_1, x_2, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let x, y ∈ V. Then $x = \alpha_1x_1 + \alpha_2x_2 + \alpha_3z$, $y = \alpha_4x_1 + \alpha_5x_2 + \alpha_6z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4x_1 + \alpha_5x_2 + \alpha_6z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_2 + \alpha_3z, \alpha_4\beta_4x_1 + \alpha_5\beta_3x_1 + \alpha_5\beta_4x_2] = 0 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

Now we need to show that V is commutative. Since

$[x, y] = [\alpha_1x_1 + \alpha_2x_2 + \alpha_3z, \alpha_4x_1 + \alpha_5x_2 + \alpha_6z] = 0$

Therefore, V is a commutative I-ideal of L.

4) We claim that the 3-dimensional subspace V = span $\{x_1, x_2, x_4\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let x, y ∈ V. Then $x = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_4$, $y = \alpha_4x_1 + \alpha_5x_2 + \alpha_6x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4x_1 + \alpha_5x_2 + \alpha_6x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_4, \alpha_4\beta_4x_1 + \alpha_5\beta_3x_1 + \alpha_5\beta_4x_2 - \alpha_6\beta_1x_1 - \alpha_6\beta_2x_2]$

$= (-\alpha_3\alpha_4\beta_4 - \alpha_3\alpha_5\beta_3 + \alpha_3\alpha_6\beta_1)x_1 + (-\alpha_3\alpha_5\beta_4 + \alpha_3\alpha_6\beta_2)x_2 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_4, \alpha_4x_1 + \alpha_5x_2 + \alpha_6x_4]$

$= (\alpha_1\alpha_6 - \alpha_3\alpha_4)x_1 + (\alpha_2\alpha_6 - \alpha_3\alpha_5)x_2 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

5) We claim that the 4-dimensional subspace $V = \text{span}\{x_1, x_2, x_3, z\}$ is an I-ideal of L. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = \alpha_1x_1+\alpha_2x_2+\alpha_3x_3+\alpha_4z$, $y = \alpha_5x_1 + \alpha_6x_2 + \alpha_7x_3 + \alpha_8z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5x_1 + \alpha_6x_2 + \alpha_7x_3 + \alpha_8z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4z, \alpha_5\beta_4x_1 + \alpha_6\beta_3x_1 + \alpha_6\beta_4x_2 - \alpha_7\beta_2x_1] = -\alpha_3\alpha_6\beta_4x_1 \in V$,

$[V, [V, L]] \subseteq V$. Therefore, V is an I-ideal of L.

It remains to show that $[x, y] \neq 0$ Since

$[x, y] = [\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4z, \alpha_5x_1 + \alpha_6x_2 + \alpha_7x_3 + \alpha_8z] = (\alpha_2\alpha_7 - \alpha_3\alpha_6)x_1 \neq 0$

Therefore, V is a non-commutative I-ideal of L, as required.

□

3.17 Remark: 3.16Theorem is not true if we state that every 1, 2, 3and4-dimensional subspace is an I-ideal because $L = L_5$ contains a 1, 2, 3and4-dimensional subspace which is not I-ideal. As one can see in the following examples.

3.18 Example: Recall that we fix a basis $\{x_1, x_2, x_3, x_4, z\}$ of L_5 with the Lie multiplication $[x_1, x_4] = x_1$, $[x_2, x_3] = x_1$,$[x_2, x_4] = x_2$ and otherwise is zero. Let $\ell \in L$. Then $\ell = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z$ for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R$.

1) We claim that the 1-dimensional subspace $V = \text{span}\{x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_4$, $y = \alpha_2x_4$ for some $\alpha_1, \alpha_2 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_2x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_4, -\alpha_2\beta_1x_1 - \alpha_2\beta_2x_2] = \alpha_1\alpha_2\beta_1x_1 + \alpha_1\alpha_2\beta_2x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

2) We claim that the 2-dimensional subspace $V = \text{span}\{x_3, x_4\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_3 + \alpha_2x_4$, $y = \alpha_3x_3 + \alpha_4x_4$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_3x_3 + \alpha_4x_4, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_3 + \alpha_2x_4, -\alpha_3\beta_2x_1 - \alpha_4\beta_1x_1 - \alpha_4\beta_2x_2]$

$= (\alpha_1\alpha_4\beta_2 + \alpha_2\alpha_3\beta_2 + \alpha_2\alpha_4\beta_1)x_1 + \alpha_2\alpha_4\beta_2x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

3) We claim that the 3-dimensional subspace $V = \text{span}\{x_1, x_4, z\}$ is not an I-ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_1 + \alpha_2x_4 + \alpha_3z$, $y = \alpha_4x_1 + \alpha_5x_4 + \alpha_6z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_4x_1 + \alpha_5x_4 + \alpha_6z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_1 + \alpha_2x_4 + \alpha_3z, \alpha_4\beta_4x_1 - \alpha_5\beta_1x_1 - \alpha_5\beta_2x_2]$

$= (-\alpha_2\alpha_4\beta_4 + \alpha_2\alpha_5\beta_1)x_1 + \alpha_2\alpha_5\beta_2x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

4) We claim that the 4-dimensional subspace $V = \text{span}\{x_2, x_3, x_4, z\}$ is not an I- ideal of L. Let $x, y \in V$. Then $x = \alpha_1x_2 +\alpha_2x_3 +\alpha_3x_4 +\alpha_4z$, $y = \alpha_5x_2 +\alpha_6x_3 +\alpha_7x_4 +\alpha_8z$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \in R$. Since

$[x, [y, \ell]] = [x, [\alpha_5x_2 + \alpha_6x_3 + \alpha_7x_4 + \alpha_8z, \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_4x_4 + \beta_5z]]$

$= [\alpha_1x_2 + \alpha_2x_3 + \alpha_3x_4 + \alpha_4z, \alpha_5\beta_3x_1 + \alpha_5\beta_4x_2 - \alpha_6\beta_2x_1 - \alpha_7\beta_1x_1 - \alpha_7\beta_2x_2]$

$= (-\alpha_2\alpha_5\beta_4 + \alpha_2\alpha_7\beta_2 - \alpha_3\alpha_5\beta_3 + \alpha_3\alpha_6\beta_2 + \alpha_3\alpha_7\beta_1)x_1$

$+(-\alpha_3\alpha_5\beta_4 - \alpha_3\alpha_6\beta_1 + \alpha_3\alpha_7\beta_2)x_2 \notin V$,

$[V, [V, L]] \nsubseteq V$, Therefore V is not an I-ideal of L.

Now we are ready to prove 1.1Theorem . Recall that L is either L_ϵ or L_1 or L_2 or L_3 or L_4 or L_5 or L_6 or L_7. We need to show that L contains a commutative and non-commutative I-ideal.

Proof: for**1.1 Theorem** If $L = L_\epsilon$, by the 3.1Proposition, L contains a commutative and non-commutative I-ideal.

If $L = L_1$, by the 3.4Proposition L contains a commutative and non- commutative I-ideal.

If $L = L_2$, by the3.7 Proposition L contains a commutative and non- commutative I-ideal.

If $L = L_3$, by the 3.10 Proposition L contains a commutative and non-commutative I-ideal.

If $L = L_4$, by the 3.13 Proposition L contains a commutative and non-commutative I-ideal.

If $L = L_5$, by the 3.16 Proposition L contains a commutative and non-commutative I-ideal.

4. CONCLUSION

In this paper, we proved that where L is 4-dimensional real Lie algebras with 2-dimensional derived. Then L contains a commutative and non-commutative I-ideal.

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