Inner Ideals of the Real Five-Dimensional Lie Algebras with Two-Dimensional Derived, Such That $l' \not\subseteq z$

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1. INTRODUCTION

Georgia Benkart, an American scientist, introduced the notion of inner ideals (see [3]) in 1976. According to Benkart’s definition, one can say that an inner ideal is a subspace $V$ of a Lie algebra $L$, with the property $[V, [V, L]] \subseteq V$, where $[V, [V, L]] = \text{span} \{[v_1, v_2, \ell] : v_1, v_2 \in V, \ell \in L\}$.

An inner ideal $V$ is called commutative if $[V, V] = 0$. She demonstrated that a strong correlation exist between elements in Lie algebras that are ad-nilpotents with inner ideal [4]. An accomplishment of Benkart’s results is done in [5] by Benkart and Fernandiz Lopes and a generalization of there results is done in [6] by Brox, Fernandez Lopez and Gomez Lozano in 2016 to the case of centrally closed prime rings with involution of characteristic not 2, 3 or 5.

It can be seen from [9] and [11] that in Lie algebra the role of inner ideals is equivalent to that of the one-sided ideals in associative algebras. Therefore, Artin’s theory can be generalized if one takes into account the inner ideals of Lie algebras. It was proved in [8] that an Artinian Lie algebra is a Lie algebra that has the property that every decreasing inner ideal chain must be terminates. In [1, Proposition 2], it was proved that every one-sided ideal of the finite dimensional associative algebra $A$ admits Levi decomposition and can be generated by an idempotent if some minimal conditions are met. The same results was obtained for inner ideals by Baranov and Shlaka in [2], where they showed that every inner ideal of the Lie algebra $[A, A]$ admits Levi decomposition and can be generated by idempotent pair (if satisfied some minimal conditions). These results were recently Generalized in [14] for the case of a sub-algebra of finite dimensional Lie algebras. Further generalization is done in [10] and [16] for the infinite dimensional Lie sub-algebras of associative algebras. Abelian non-Jordan Lie inner ideals we also been studied in 2022 (see [15] for more details). Further motivation for studying inner ideals comes from [9], where Fernández L´opez et al showed that when $L$ is an arbitrary non- degenerate Lie algebra over an abelian ring $F$ together with two and three convertible, then for every nonzero commutative inner ideal $V$ of finite length of $L$ is complemented by an commutative inner ideal [9].

**ABSTRACT:** Inner ideal of the five-dimensional non-commutative Lie algebras over the real fields with two-dimensional derived were classified. It is proved that one, two, three and four-dimensional inner ideals are existed in every five-dimensional Lie algebra. It is also proved that five-dimensional Lie algebras contain inner ideals which are neither ideals nor sub-algebras.

**Keywords:** Inner ideal, Lie sub-algebra, five-dimensional Lie algebra.
The classification of the real five-dimensional Lie algebras is given by Schöbel in [13]. He classified them in terms of the derived sub-algebra of these Lie algebras. Keep in mind that a derived sub-algebra \( L' \) of \( L \) is the set

\[
L' = [L, L] = \langle \{[\ell_1, \ell_2] | \ell_1, \ell_2 \in L \} \rangle.
\]

In [12] Saeed and Shlaka studied inner ideals of the four-dimensional Lie algebras over the real fields with two-dimensional derived. They proved that one, two and three-dimensional non-trivial inner ideals exist in every four-dimensional Lie algebra with 2-dimensional derived. Prior to that (see [17]) they classify inner ideals of the two and three-dimensional Lie algebras.

In this paper, we use techniques similar to [12] to study inner ideals of the real five-dimensional Lie algebras with 2-dimensional derived. Suppose that \( L \) is a five-dimensional Lie algebra over the real field with 1-dimensional derived. If \( L \) commutative, then it is easy to see that every 1, 2, 3 and 4-dimensional subspace of \( L \) is an inner ideal. Suppose now that \( L \) is non-commutative, then we get the following results, which is one of our main results:

1.1 Theorem: Let \( L \) be a five-dimensional Lie algebra over the real field \( R \) with 2-dimensional derived \( L' \). Then \( L \) contains a commutative and non-commutative I-ideal.

Recall that if \( L \) is 5-dimensional with 2-dimensional derived, then by 2.6 Theorem and 2.7 Theorem \( L \) is either \( L_1 \) or \( L_2 \) or \( L_3 \) or \( L_4 \) or \( L_5 \). Thus, to prove the theorem we need to consider all of the cases.

2. Preliminaries

Definition 2.1 [7]: Let \( L \) be a vector space over any field \( F \) with a bilinear form \( L \times L \rightarrow L \), where \( (\ell_1, \ell_2) \rightarrow [\ell_1, \ell_2] \), for all \( \ell_1, \ell_2 \in L \). Then \( L \) is called a Lie algebra over \( F \), if the following conditions are satisfied:

\[
\begin{align*}
(1) & \; [\ell_1, \ell_2] = 0 \text{ for all } \ell_1, \ell_2 \in L. \\
(2) & \; [\ell_1, [\ell_2, \ell_3]] + [\ell_3, [\ell_1, \ell_2]] + [\ell_2, [\ell_3, \ell_1]] = 0 \text{ for all } \ell_1, \ell_2, \ell_3 \in L.
\end{align*}
\]

Definition 2.2 [7]: The subspace \( B \) of a Lie algebra \( L \) is said to be a Lie sub-algebra of \( L \), if \( [b_1, b_2] \in B \) for all \( b_1, b_2 \in B \).

Definition 2.3 [7]: The derived of a Lie algebra \( L \) is the set \( L' = \langle \{[a, b] | a, b \in L \} \rangle \), where \( L' \) is a Lie sub algebra of \( L \).

4. Definition [7]: The center of \( L \) is the set \( Z = \{x \in L | [x, y] = 0, \forall y \in L \} \).

2.5 Definition [2]: Let \( V \) be a subspace of \( L \). Then \( V \) is said to be an inner ideal of \( L \) when \( [V, [V, L]] \subseteq V \). We denote by \( V' \) the ideal of \( L \) that \( V \) is an inner ideal of \( L \). The inner ideal \( V' \) is said to be commutative if \( [V, V'] = 0 \).

Note that in every Lie algebra \( L \), we have \( L', \{0\} \) are inner ideals of \( L \) called the trivial inner ideals. Recall that every ideal \( I \) of \( L \) is inner ideal, because \( [I, [1, L]] = [I, I] \subseteq I \), but the inverse is not true. Since \( L \) is 5-dimensional, the dimension of \( L \) may be 1, 2, 3, 4 or 5. In [13] Schöbel classified the real \( n \)-dimensional Lie algebras by relating the dimension of \( L \). For the dimension of \( L \) is 2, we have the following result, for the proof see [13, Theorem 1].

2.6 Theorem [13]: Suppose that \( L \) is a real \( n \)-dimensional Lie algebra with a two-dimensional derived \( L' \), such that \( L \cap Z = \{0\} \) Then \( L = L_4 \oplus Z_1 \), where \( L_4 \) is a 4-dimensional real Lie algebra with 2-dimensional derived algebra and \( L \nsubseteq Z_2, Z_2 \) is the n-dimensional center of \( L \).

2.7 Theorem [13]: Suppose that \( L \) is a real 4-dimensional Lie algebra with a two-dimensional derived \( \cdot \), such that \( L \nsubseteq Z, Z \) and let \( \{x_1, x_2, x_3, x_4\} \) be a basis of \( L \). Then \( L \) is one of the following six standard forms.

- \( L_1: [x_1, x_2] = c x_2, [x_2, x_3] = x_1, \) and otherwise is zero, where \( c = \pm 3 \).
- \( L_2: [x_1, x_2] = x_1, [x_2, x_3] = x_2, \) and otherwise is zero.
- \( L_3: [x_1, x_2] = x_1, [x_2, x_3] = x_2, \) and otherwise is zero.
- \( L_4: [x_1, x_2] = x_1, [x_2, x_3] = x_1, [x_2, x_4] = x_1, [x_3, x_4] = x_2, \) and otherwise is zero.
- \( L_5: [x_1, x_2] = x_1, [x_3, x_4] = x_1, [x_2, x_4] = x_2, \) and otherwise is zero.

3. INNER IDEALS OF THE FIVE-DIMENSIONAL LIE ALGEBRA

Throughout this section, we prove some results related to inner ideal of the 5-dimensional real Lie algebra with 2-dimensional derived. Our aim is to prove the following theorem.

3.1 Proposition: Suppose that \( L = L_6 \) and \( L \nsubseteq Z \). Then the following is hold.

1. \( L \) contains a 1-dimensional I-ideal which is not ideal.
2. \( L \) contains a 2-dimensional commutative I-ideal which is not ideal.
3. \( L \) contains a 3-dimensional commutative I-ideal which is not ideal.
4. \( L \) contains a 3-dimensional non-commutative I-ideal.
5. \( L \) contains a 4-dimensional commutative I-ideal.
6. \( L \) contains a 4-dimensional non-commutative I-ideal.

Proof: By 2.6 Theorems and 2.7, there is a basis \( \{x_1, x_2, x_3, x_4, z\} \) of \( L_6 \) with the Lie multiplication \([x_1, x_4] = c x_2, [x_2, x_4] = x_1\) and otherwise is zero, where \( c = \pm 3 \). Let \( \ell \in L \). Then \( \ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta z \) for some \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R \).

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1) We claim that the 1-dimensional subspace $V = \text{span} \{x_2\}$ is an I-ideal of $L$. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = a_1x_2, y = a_2x_2$ for some $a_1, a_2 \in R$. Since 
\[ [x, [y, L]] = [x, [a_2x_2, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z]] = \{a_1x_2, a_2b_1x_1 = 0 \in V, [V, [V, L]] \subseteq V. \] Therefore, $V$ is an I-ideal of $L$.

Now we need to show that $V$ is not ideal. Since 
\[ [x, L] = [a_1x_2, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z] = a_1b_1x_1 \notin V, \]
$V$ is not ideal of $L$, as required.

2) We claim that the 2-dimensional subspace $V = \text{span} \{x_2, x_3\}$ is an I-ideal of $L$. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = a_1x_2 + a_3x_3, y = a_4x_2 + a_5x_3$ for some $a_1, a_2, a_3, a_4, a_5 \in R$. Since 
\[ [x, [y, L]] = [x, [a_2x_2 + a_3x_3, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z]] = [a_1x_2 + a_3x_3, a_2b_1x_1 = 0 \in V, [V, [V, L]] \subseteq V. \] Therefore, $V$ is an I-ideal of $L$.

Now we need to show that $V$ is commutative. Since 
\[ [x, y] = [a_1x_2 + a_3x_3, a_4x_2 + a_5x_3] = 0. \] Therefore, $V$ is a commutative I-ideal of $L$.

It remains to us show that $V$ is not ideal. Since 
\[ [x, L] = [a_1x_2 + a_3x_3, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z] = a_1b_1x_1 \notin V, \] Therefore, $V$ is not ideal of $L$.

3) We claim that the 3-dimensional subspace $V = \text{span} \{x_1, x_3, z\}$ is an I-ideal of $L$. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = a_1x_1 + a_2x_2 + a_3z, y = a_4x_1 + a_5x_3 + a_6z$ for some $a_1, a_2, a_3, a_4, a_5, a_6 \in R$. Since 
\[ [x, [y, L]] = [x, [a_2x_2 + a_3x_3, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z]] = [a_1x_1 + a_3x_3, a_2b_1x_1 = 0 \in V, [V, [V, L]] \subseteq V. \] Therefore, $V$ is an I-ideal of $L$.

Now we need to show that $V$ is not ideal. Since 
\[ [x, y] = [a_1x_1 + a_3x_3, a_4x_1 + a_5x_3 + a_6z] = 0. \] Therefore, $V$ is a commutative I-ideal of $L$.

It remains to us show that $V$ is not ideal. Since 
\[ [x, L] = [a_1x_1 + a_3x_3, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z] = a_1b_1x_1 \notin V, \] Therefore, $V$ is not ideal of $L$.

4) We claim that the 3-dimensional subspace $V = \text{span} \{x_1, x_2, x_3\}$ is an I-ideal of $L$. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = a_1x_1 + a_2x_2 + a_3x_3, y = a_4x_1 + a_5x_2 + a_6x_3$ for some $a_1, a_2, a_3, a_4, a_5, a_6 \in R$. Since 
\[ [x, [y, L]] = [x, [a_2x_2 + a_3x_3, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z]] = [a_1x_1 + a_3x_3, a_2b_1x_1 = 0 \in V, [V, [V, L]] \subseteq V. \] Therefore, $V$ is an I-ideal of $L$.

It remains to show that $[x, y] \neq 0$ since 
\[ [x, y] = [a_1x_1 + a_3x_3, a_4x_1 + a_5x_2 + a_6x_3] = (a_5a_4 - a_4a_5)x_1 \neq 0 \]
Therefore, $V$ is a non-commutative I-ideal of $L$, as required.

5) We claim that the 4-dimensional subspace $V = \text{span} \{x_1, x_2, x_3, z\}$ is an I-ideal of $L$. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = a_1x_1 + a_2x_2 + a_3x_3 + a_4z, y = a_5x_1 + a_6x_2 + a_7x_2 + a_8x_3 + a_9z$ for some $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \in R$. Since 
\[ [x, [y, L]] = [x, [a_2x_2 + a_3x_3 + a_4z, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z]] = [a_1x_1 + a_3x_3 + a_4z, a_2b_1x_1 = 0 \in V, [V, [V, L]] \subseteq V. \] Therefore, $V$ is an I-ideal of $L$.

Now we need to show that $V$ is commutative. Since 
\[ [x, y] = [a_1x_1 + a_3x_3 + a_4z, a_5x_1 + a_6x_2 + a_7x_3 + a_8x_3 + a_9z] = 0. \] Therefore, $V$ is a commutative I-ideal of $L$, as required.

6) We claim that the 4-dimensional subspace $V = \text{span} \{x_1, x_2, x_3, x_4\}$ is an I-ideal of $L$. We need to show that $[V, [V, L]] \subseteq V$. Let $x, y \in V$. Then $x = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4, y = a_5x_1 + a_6x_2 + a_7x_3 + a_8x_4$ for some $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12} \in R$. Since 
\[ [x, [y, L]] = [x, [a_2x_2 + a_3x_3 + a_4x_4, b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5z]] = [a_1x_1 + a_3x_3 + a_4x_4, a_2b_1x_1 + a_3b_1x_1 + a_4b_1x_1] = (a_1a_2b_1z + a_1a_3b_1z + a_1a_4b_1z)x_2 \in V, [V, [V, L]] \subseteq V. \] Therefore, $V$ is an I-ideal of $L$.

It remains to show that $[x, y] \neq 0$ since 
\[ [x, y] = [a_1x_1 + a_3x_3 + a_4x_4, a_5x_1 + a_6x_2 + a_7x_3 + a_8x_4 + a_9z] = (a_5a_4 - a_4a_5)x_1 \neq 0 \]
Therefore, $V$ is a non-commutative I-ideal of $L$, as required.
3.2 Remark: 3.1 Theorem is not true if we state that every 1, 2, 3 and 4-dimensional subspace is an I-ideal because \(L = L\) contains a 1, 2, 3 and 4-dimensional subspace which is not I-ideal. As one can see in the following examples.

3.3 Example: Recall that we fix a basis \([x_1, x_2, x_3, x_4]\) of \(L\) with the Lie multiplication \([x_1, x_4] = x_2, [x_2, x_4] = x_1\) and otherwise is zero, where \(e = \mathbb{F}\). Let \(\ell \in L\). Then \(\ell = [\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta z]\) for some \(\beta_1, \beta_2, \beta_3, \beta_4, \beta \in R\).

1) We claim that the 1-dimensional subspace \(V = \{x_1\}\) is not an I-ideal of \(L\). Let \(x, y \in V\). Then \(x = \alpha x_1, y = \alpha x_2\) for some \(\alpha, \alpha \in R\). Since \([x, [y, \ell]] = [\alpha x_2, [\alpha x_1 + \beta x_1 + \beta x_2 + \beta x_3 + \beta x_4 + \beta z]]\)
2) We claim that the 2-dimensional subspace \(V = \{x_1, x_2\}\) is not an I-ideal of \(L\). Let \(x, y \in V\). Then \(x = \alpha x_1 + \alpha x_2, y = \alpha x_1 + \alpha x_2 + \alpha x_4 + \alpha x_4\) for some \(\alpha, \alpha, \alpha, \alpha \in R\). Since \([x, [y, \ell]] = [\alpha x_2 + \alpha x_3, [\alpha x_1 + \alpha x_3 + \beta x_3 + \beta x_3 + \beta x_4 + \beta z]]\)
3) We claim that the 3-dimensional subspace \(V = \{x_1, x_3, x_4\}\) is not an I-ideal of \(L\). Let \(x, y \in V\). Then \(x = \alpha x_1 + \alpha x_3 + \alpha x_4, y = \alpha x_1 + \alpha x_3 + \alpha x_4 + \alpha x_4\) for some \(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha \in R\). Since \([x, [y, \ell]] = [\alpha x_2 + \alpha x_3 + \alpha x_4, [\alpha x_1 + \alpha x_3 + \beta x_3 + \beta x_3 + \beta x_4 + \beta z]]\)
4) We claim that the 4-dimensional subspace \(V = \{x_1, x_2, x_3, x_4\}\) is not an I-ideal of \(L\). Let \(x, y \in V\). Then \(x = \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha x_4, y = \alpha x_1 + \alpha x_3 + \alpha x_4 + \alpha x_4\) for some \(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha \in R\). Since \([x, [y, \ell]] = [\alpha x_2 + \alpha x_3 + \alpha x_4, [\alpha x_1 + \alpha x_3 + \beta x_3 + \beta x_3 + \beta x_4 + \beta z]]\)

3.4 Proposition: Suppose that \(L = L_1\) and \(L \not\subseteq \mathbb{Z}\). Then the following is hold.

1. \(L\) contains a 1-dimensional I-ideal.
2. \(L\) contains a 2-dimensional commutative I-ideal.
3. \(L\) contains a 3-dimensional commutative I-ideal.
4. \(L\) contains a 3-dimensional non-commutative I-ideal.
5. \(L\) contains a 4-dimensional commutative I-ideal.
6. \(L\) contains a 4-dimensional non-commutative I-ideal.

Proof. By 2.6 Theorems and 2.7, there is a basis \([x_1, x_2, x_3, x_4]\) of \(L\) with the Lie multiplication \([x_1, x_4] = x_2, [x_2, x_4] = x_1\) and otherwise is zero. Let \(\ell \in L\). Then \(\ell = [\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta x_4 + \beta z]\) for some \(\beta_1, \beta_2, \beta_3, \beta, \beta \in R\).

1) We claim that the 1-dimensional subspace \(V = \{x_1\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\). Then \(x = \alpha x_1, y = \alpha x_2\) for some \(\alpha, \alpha \in R\). Since \([x, [y, \ell]] = [\alpha x_2, [\alpha x_1 + \beta x_1 + \beta x_2 + \beta x_3 + \beta x_4 + \beta z]]\)
2) We claim that the 2-dimensional subspace \(V = \{x_1, x_2\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\). Then \(x = \alpha x_1 + \alpha x_3 + \alpha x_4, y = \alpha x_1 + \alpha x_3 + \alpha x_4 + \alpha x_4\) for some \(\alpha, \alpha, \alpha, \alpha \in R\). Since \([x, [y, \ell]] = [\alpha x_2 + \alpha x_3 + \alpha x_4, [\alpha x_1 + \alpha x_3 + \beta x_3 + \beta x_3 + \beta x_4 + \beta z]]\)
3) We claim that the 3-dimensional subspace \(V = \{x_1, x_3, x_4\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\). Then \(x = \alpha x_1 + \alpha x_3 + \alpha x_4, y = \alpha x_1 + \alpha x_3 + \alpha x_4\) for some \(\alpha, \alpha, \alpha, \alpha, \alpha \in R\). Since \([x, [y, \ell]] = [\alpha x_2 + \alpha x_3 + \alpha x_4, [\alpha x_1 + \alpha x_3 + \beta x_3 + \beta x_3 + \beta x_4 + \beta z]]\)
4) We claim that the 4-dimensional subspace \(V = \{x_1, x_2, x_3, x_4\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\). Then \(x = \alpha x_1 + \alpha x_3 + \alpha x_4, y = \alpha x_1 + \alpha x_3 + \alpha x_4\) for some \(\alpha, \alpha, \alpha, \alpha, \alpha \in R\). Since \([x, [y, \ell]] = [\alpha x_2 + \alpha x_3 + \alpha x_4, [\alpha x_1 + \alpha x_3 + \beta x_3 + \beta x_3 + \beta x_4 + \beta z]]\)
It remains to show that \( [x, y] \neq 0 \) since
\[
[x, y] = [a_1x_1 + ax_2 + ax_4, a_1x_1 + ax_2 + ax_4]
= (a_1d_0 - a_1a_0x_1, (a_1d_0 - a_1a_0)x_2) \neq 0
\]
Therefore, \( V \) is a non-commutative \( I \)-ideal of \( L \), as required.

5) We claim that the 4-dimensional subspace \( V = \text{span} \{x_1, x_2, x_3, z\} \) is an \( I \)-ideal of \( L \). We need to show that \( [V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then
\[
x = a_1x_1 + ax_2 + ax_3 + ax_4, y = a_1x_1 + ax_2 + ax_3 + ax_4 \text{ for some } a_1, a_2, a_3, a_4, a_5 \in R.
\]
Since
\[
[x, y] = [a_1x_1 + ax_2 + ax_3 + ax_4, a_1x_1 + ax_2 + ax_3 + ax_4]
= (a_1d_0 - a_1a_0x_1, (a_1d_0 - a_1a_0)x_2) \neq 0
\]
Therefore, \( V \) is a commutative \( I \)-ideal of \( L \), as required.

6) We claim that the 4-dimensional subspace \( V = \text{span} \{x_1, x_2, x_3, x_4\} \) is an \( I \)-ideal of \( L \). We need to show that \( [V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then
\[
x = a_1x_1 + ax_2 + ax_3 + ax_4, y = a_1x_1 + ax_2 + ax_3 + ax_4 \text{ for some } a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \in R.
\]
Since
\[
[x, y] = [a_1x_1 + ax_2 + ax_3 + ax_4, a_1x_1 + ax_2 + ax_3 + ax_4]
= (a_1d_0 - a_1a_0x_1, (a_1d_0 - a_1a_0)x_2) \neq 0
\]
Therefore, \( V \) is a non-commutative \( I \)-ideal of \( L \), as required.

\[\square\]

3.5 Remark: 3.4 Theorem is not true if we state that every 1, 2, 3 and 4-dimensional subspace is an \( I \)-ideal because \( L = L_1 \) contains a 1, 2, 3 and 4-dimensional subspace which is not an \( I \)-ideal. As one can see in the following examples.

3.6 Example: Recall that we fix a basis \( \{x_1, x_2, x_3, x_4, z\} \) of \( L \) with the Lie multiplication \( [x_1, x_2] = x_1, [x_2, x_3] = x_2 \) and otherwise is zero. Let \( \ell \in L \). Then \( \ell = [x_1, x_2] + [x_3, x_4] + [z, x] \) for some \( \ell_1, \ell_2, \ell_3, \ell_4, \ell_5 \in R \).

1) We claim that the 1-dimensional subspace \( V = \text{span} \{x_1\} \) is not an \( I \)-ideal of \( L \). Let \( x, y \in V \). Then \( x = a_1x_1, y = a_2x_4 \) for some \( a_1, a_2 \in R \).

\[
[x, y] = [a_1x_1, a_2x_4] = a_1a_2x_1 \in V
\]

2) We claim that the 2-dimensional subspace \( V = \text{span} \{x_1, x_4\} \) is not an \( I \)-ideal of \( L \). Let \( x, y \in V \). Then \( x = a_1x_1 + ax_4, y = a_2x_1 + ax_4 \) for some \( a_1, a_2, a_3, a_4, a_5 \in R \).

\[
[x, y] = [a_1x_1 + ax_4, a_2x_1 + ax_4] = a_1a_2x_1 \in V
\]

3) We claim that the 3-dimensional subspace \( V = \text{span} \{x_1, x_3, x_4\} \) is not an \( I \)-ideal of \( L \). Let \( x, y \in V \). Then \( x = a_1x_1 + ax_3 + ax_4, y = a_1x_1 + ax_3 + ax_4 \) for some \( a_1, a_2, a_3, a_4, a_5 \in R \).

\[
[x, y] = [a_1x_1 + ax_3 + ax_4, a_1x_1 + ax_3 + ax_4] = a_1a_2x_1 \in V
\]

4) We claim that the 4-dimensional subspace \( V = \text{span} \{x_2, x_3, x_4, z\} \) is not an \( I \)-ideal of \( L \). Let \( x, y \in V \). Then \( x = a_1x_2 + ax_3 + ax_4 + az, y = a_2x_2 + ax_3 + ax_4 + az \) for some \( a_1, a_2, a_3, a_4, a_5, a_6 \in R \).

\[
[x, y] = [a_1x_2 + ax_3 + ax_4 + az, a_2x_2 + ax_3 + ax_4 + az] = a_1a_2x_1 \in V
\]

3.7 Proposition: Suppose that \( L = L_2 \) and \( L' \not\subseteq Z \). Then the following is hold.

1. \( L \) contains a 1-dimensional \( I \)-ideal which is not ideal.
2. \( L \) contains a 2-dimensional commutative \( I \)-ideal which is not ideal.
3. \( L \) contains a 3-dimensional commutative \( I \)-ideal which is not ideal.
4. \( L \) contains a 3-dimensional non-commutative \( I \)-ideal.
5. \( L \) contains a 4-dimensional commutative \( I \)-ideal.
6. \( L \) contains a 4-dimensional non-commutative \( I \)-ideal.
Proof: By 2.6Theorems and 2.7, there is a basis \{x_1, x_2, x_3, x_4, z\} of \(L_2\) with the Lie multiplication \([x_1, x_4] = -x_1 + px_2, [x_2, x_4] = x_1\) and otherwise is zero, where \(p \in R\). Let \(L \subseteq L\). Then \(L = \beta x_1 + \beta x_2 + \beta_1 x_3 + \beta x_4 + \beta z\) for some \(\beta, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in R\).

1) We claim that the 1-dimensional subspace \(V = \text{span} \{x_2\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\) then \(x = \alpha x_2\) for some \(\alpha, \beta \in R\).

\([x, y] = [x, \alpha x_2] = \alpha [x, x_2] = \alpha \beta x_1 + \beta x_2 + \beta_1 x_3 + \beta x_4 + \beta z = \alpha \beta x_1 \in V\). Therefore, \(V\) is a commutative I-ideal of \(L\).

Now we need to show that \(V\) is not ideal. Since
\([x, y] = [\alpha x_2, \beta x_1] = \alpha [x, x_1] = 0 \in V\).

2) We claim that the 2-dimensional subspace \(V = \text{span} \{x_2, x_3\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\) then \(x = \alpha x_2 + \alpha x_3, y = \gamma x_2 + \gamma x_3\) for some \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in R\). Since
\([x, y] = [x, \alpha x_2 + \alpha x_3, \beta x_1 + \beta x_3 + \beta x_4 + \beta z] = [\alpha x_2, \alpha \beta x_1] = 0 \in V\).

3) We claim that the 3-dimensional subspace \(V = \text{span} \{x_1, x_2, x_3\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\) then \(x = \alpha x_1 + \alpha x_2 + \alpha x_3, y = \gamma x_1 + \gamma x_2 + \gamma x_3\) for some \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in R\). Since
\([x, y] = [x, \alpha x_1 + \alpha x_2 + \alpha x_3, \beta x_2 + \beta x_3 + \beta x_4 + \beta z] = [\alpha x_1, \alpha \beta x_2 + \alpha \beta x_3] = 0 \in V\).

4) We claim that the 3-dimensional subspace \(V = \text{span} \{x_1, x_2, x_4\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\) then \(x = \alpha x_1 + \alpha x_2 + \alpha x_3, y = \gamma x_1 + \gamma x_2 + \gamma x_4\) for some \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in R\). Since
\([x, y] = [x, \alpha x_1 + \alpha x_2 + \alpha x_3, \beta x_2 + \beta x_3 + \beta x_4 + \beta z] = [\alpha x_1, \alpha x_2 + \alpha x_3, \beta x_2 + \beta x_3 + \beta x_4 + \beta z] = 0 \in V\).

5) We claim that the 4-dimensional subspace \(V = \text{span} \{x_1, x_2, x_3, z\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\) then \(x = \alpha x_1 + \alpha x_2 + \alpha x_3, y = \gamma x_1 + \gamma x_2 + \gamma x_3 + \gamma x_4\) for some \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in R\). Since
\([x, y] = [x, \alpha x_1 + \alpha x_2 + \alpha x_3, \beta x_2 + \beta x_3 + \beta x_4 + \beta z] = [\alpha x_1, \alpha x_2 + \alpha x_3, \beta x_2 + \beta x_3 + \beta x_4 + \beta z] = 0 \in V\).

6) We claim that the 4-dimensional subspace \(V = \text{span} \{x_1, x_2, x_3, x_4\}\) is an I-ideal of \(L\). We need to show that \([V, [V, L]] \subseteq V\). Let \(x, y \in V\) then \(x = \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha x_4, y = \gamma x_1 + \gamma x_2 + \gamma x_3 + \gamma x_4\) for some \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in R\). Since
\([x, y] = [x, \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha x_4, \beta x_2 + \beta x_3 + \beta x_4 + \beta z] = [\alpha x_1, \alpha x_2 + \alpha x_3 + \alpha x_4, \beta x_2 + \beta x_3 + \beta x_4 + \beta z] = 0 \in V\).

It remains to show that \([x, y] \neq 0\) Since
\([x, y] = [x, \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha x_4, \beta x_2 + \beta x_3 + \beta x_4 + \beta z] = 0 \in V\).

Therefore, \(V\) is a non-commutative I-ideal of \(L\), as required.
\[ [x, y] = [a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4, a_5x_1 + a_6x_2 + a_7x_3 + a_8x_4] \]
\[ = (-αa_8 + αa_9 + αa_{10} - αa_{11})x_1 + p(αa_{12} - αa_{13})x_2 = 0 \]
Therefore, \( V \) is a non-commutative I-ideal of \( L \), as required.

3.8 Remark: 3.7 Theorem is not true if we state that every 1, 2, 3, and 4-dimensional subspace is an I-ideal because \( L = L_2 \) contains a 1, 2, 3, and 4-dimensional subspace which is not an I-ideal. As one can see in the following examples.

3.9 Example: Recall that we fix a basis \( \{x_1, x_2, x_3, x_4\} \) of \( L_2 \) with the Lie multiplication \( [x_1, x_1] = (-x_1 + px_2), [x_2, x_4] = x_1 \) and otherwise is zero, where \( p \in R. \) Let \( t \in L. \) Then \( t = β_1x_1 + β_2x_2 + β_3x_3 + β_4x_4 + β_5 \) for some \( β_1, β_2, β_3, β_4, β_5 \in R. \)

1) We claim that the 1-dimensional subspace \( V = \langle x_1 \rangle \) is not an I-ideal of \( L. \) Let \( x, y \in V. \) Then \( x = α_{11}x_1, y = α_{12}x_4 \) for some \( α_{11}, α_{12} \in R. \) Since
\[ [x, y] = [x, α_{12}x_4] = [α_{11}x_1 + α_{12}x_4, β_1x_1 + β_2x_2 + β_3x_3 + β_4x_4 + β_5] \]
\[ = [α_{11}x_1, β_1x_1] \neq \beta_2, β_3, β_4, β_5 \in R. \]
Therefore, \( V \) is not an I-ideal of \( L. \)

2) We claim that the 2-dimensional subspace \( V = \langle x_1, x_4 \rangle \) is not an I-ideal of \( L. \) Let \( x, y \in V. \) Then \( x = α_{13}x_1 + α_{24}x_4 \) and \( y = α_{31}x_1 + α_{42}x_4 \) for some \( α_{13}, α_{24}, α_{31}, α_{42} \in R. \) Since
\[ [x, y] = [α_{13}x_1 + α_{24}x_4, α_{31}x_1 + α_{42}x_4] = [α_{13}x_1, α_{31}x_1] + [α_{24}x_4, α_{42}x_4] \neq \beta_2, β_3, β_4, β_5 \in R. \]
Therefore, \( V \) is not an I-ideal of \( L. \)

3) We claim that the 3-dimensional subspace \( V = \langle x_1, x_3, x_4 \rangle \) is not an I-ideal of \( L. \) Let \( x, y \in V. \) Then \( x = α_{13}x_1 + α_{31}x_3 + α_{42}x_4 \) and \( y = α_{13}x_1 + α_{24}x_4 \) for some \( α_{13}, α_{31}, α_{42} \in R. \) Since
\[ [x, y] = [α_{13}x_1, α_{31}x_3 + α_{42}x_4] = [α_{13}x_1, α_{31}x_1] + [α_{42}x_4, α_{42}x_4] \neq \beta_2, β_3, β_4, β_5 \in R. \]
Therefore, \( V \) is not an I-ideal of \( L. \)

4) We claim that the 4-dimensional subspace \( V = \langle x_2, x_3, x_4, x_5 \rangle \) is not an I-ideal of \( L. \) Let \( x, y \in V. \) Then \( x = α_{13}x_1 + α_{24}x_4 + α_{31}x_3 + α_{42}x_4 + α_{51}x_5 \) and \( y = α_{13}x_1 + α_{24}x_4 + α_{13}x_1 + α_{42}x_4 + α_{51}x_5 \) for some \( α_{13}, α_{24}, α_{31}, α_{42}, α_{51} \in R. \) Since
\[ [x, y] = [α_{13}x_1, α_{13}x_1 + α_{24}x_4 + α_{31}x_3 + α_{42}x_4 + α_{51}x_5] = [α_{13}x_1, α_{13}x_1] + [α_{24}x_4, α_{24}x_4] + [α_{31}x_3, α_{31}x_3] + [α_{42}x_4, α_{42}x_4] + [α_{51}x_5, α_{51}x_5] \neq \beta_2, β_3, β_4, β_5 \in R. \]
Therefore, \( V \) is not an I-ideal of \( L. \)

3.10 Proposition: Suppose that \( L = L_3 \) and \( L \not\subseteq \mathbb{Z}. \) Then the following is hold.
1. \( L \) contains a 1-dimensional I-ideal.
2. \( L \) contains a 2-dimensional commutative I-ideal.
3. \( L \) contains a 2-dimensional non-commutative I-ideal.
4. \( L \) contains a 3-dimensional commutative I-ideal.
5. \( L \) contains a 3-dimensional non-commutative I-ideal.
6. \( L \) contains a 4-dimensional non-commutative I-ideal.

Proof: By 2.6 Theorems and 2.7, there is a basis \( \{x_1, x_2, x_3, x_4, x_5\} \) of \( L_3 \) with the Lie multiplication \( [x_1, x_1] = x_1, [x_2, x_4] = x_3 \) and otherwise is zero. Let \( t \in L. \) Then \( t = β_1x_1 + β_2x_2 + β_3x_3 + β_4x_4 + β_5 \) for some \( β_1, β_2, β_3, β_4, β_5 \in R. \)

1) We claim that the 1-dimensional subspace \( V = \langle x_2 \rangle \) is an I-ideal of \( L. \) We need to show that \( [V, [V, L]] \subseteq V. \) Let \( x, y \in V. \) Then \( x = α_{13}x_1 \) for some \( α_{13} \in R. \) Since
\[ [x, y] = [x, α_{13}x_1] = [α_{13}x_1, α_{13}x_1] = 0 \in V, \]
\[ [V, V, L] \subseteq V \Rightarrow V \text{ is an I-ideal of } L. \]

2) We claim that the 2-dimensional subspace \( V = \langle x_2, z \rangle \) is an I-ideal of \( L. \) We need to show that \( [V, [V, L]] \subseteq V. \) Let \( x, y \in V. \) Then \( x = α_{13}x_1 + α_{14}x_2 + α_{15}x_5, y = α_{13}x_1 + α_{14}x_2 + α_{15}x_5 \) for some \( α_{13}, α_{14}, α_{15} \in R. \) Since
\[ [x, y] = [x, α_{13}x_1 + α_{14}x_2 + α_{15}x_5] = [α_{13}x_1, α_{13}x_1] + [α_{14}x_2, α_{14}x_2] + [α_{15}x_5, α_{15}x_5] = 0 \in V, \]
\[ [V, [V, L]] \subseteq V \Rightarrow V \text{ is an I-ideal of } L. \]

Now we need to show that \( V \) is commutative. Since
\[ [x, y] = [α_{13}x_1 + α_{14}x_2 + α_{15}x_5] = 0. \] Therefore, \( V \) is a commutative I-ideal of \( L. \)

3) We claim that the 2-dimensional subspace \( V = \langle x_1, x_3 \rangle \) is an I-ideal of \( L. \) We need to show that \( [V, [V, L]] \subseteq V. \) Let \( x, y \in V. \) Then \( x = α_{13}x_1 + α_{14}x_3, y = α_{13}x_1 + α_{14}x_3 \) for some \( α_{13}, α_{14} \in R. \) Since
\[ [x, y] = [x, α_{13}x_1 + α_{14}x_3] = [α_{13}x_1, α_{13}x_1] + [α_{14}x_3, α_{14}x_3] = 0 \in V, \]
\[ [V, [V, L]] \subseteq V \Rightarrow V \text{ is an I-ideal of } L. \]
It remains to show that \([x, y] \neq 0\) Since
\[\begin{align*}
[x, y] &= [\alpha x_1 + a_3 x_3, \alpha x_1 + a_3 x_3] = (\alpha x_1 - a_3 x_3)x_1 \neq 0
\end{align*}\]
Therefore, \(V\) is a non-commutative \(I\)-ideal of \(L\).

4. We claim that the 3-dimensional subspace \(V = \text{span} \{x_1, x_2, z\}\) is an \(I\)-ideal of \(L\). We need to show that \([V, \{V, L\}] \subseteq V\). Let \(x, y \in V\). Then \(x = ax_1 + ax_2 + ax_3, y = ax_1 + ax_2 + az, z = ax_1 + ax_2 + z\) for some \(a_1, a_2, a_3, a_4, a_5, a_6 \in R\). Since
\[\begin{align*}
[x, y] &= [x, y] = [x_1 + x_2 + ax_3, x_1 + x_2 + az, x_1 + x_2 + ax_3 + az] = 0 \in V
\end{align*}\]
Therefore, \(V\) is an \(I\)-ideal of \(L\).

Now we need to show that \(V\) is commutative. Since
\[\begin{align*}
[x, y] &= [x_1 + x_2 + az, x_1 + x_2 + az, x_1 + x_2 + az] = 0 \in V
\end{align*}\]
Therefore, \(V\) is a commutative \(I\)-ideal of \(L\).

5. We claim that the 3-dimensional subspace \(V = \text{span} \{x_1, x_2, x_3\}\) is an \(I\)-ideal of \(L\). We need to show that \([V, \{V, L\}] \subseteq V\). Let \(x, y \in V\). Then \(x = ax_1 + ax_2 + ax_3, y = ax_1 + ax_2 + ax_3 + ax_4\) for some \(a_1, a_2, a_3, a_4, a_5, a_6 \in R\). Since
\[\begin{align*}
[x, y] &= [x, y] = [x_1 + x_2 + a_4 x_4, b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 + b_5 z] = 0 \in V
\end{align*}\]
Therefore, \(V\) is an \(I\)-ideal of \(L\).

6. We claim that the 4-dimensional subspace \(V = \text{span} \{x_1, x_2, x_3, x_4\}\) is an \(I\)-ideal of \(L\). We need to show that \([V, \{V, L\}] \subseteq V\). Let \(x, y \in V\). Then \(x = ax_1 + ax_2 + ax_3 + ax_4, y = ax_1 + ax_2 + ax_3 + ax_4\) for some \(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \in R\). Since
\[\begin{align*}
[x, y] &= [x, y] = [a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4, b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 + b_5 z] = 0 \in V
\end{align*}\]
Therefore, \(V\) is a non-commutative \(I\)-ideal of \(L\), as required.

3.11 Remark: 3.10Theorem is not true if we state that every \(1, 2, 3, 4\)-dimensional subspace is an \(I\)-ideal because \(L = L_1\) contains a 1, 2, 3, 4-dimensional subspace which is not an \(I\)-ideal. As one can see in the following examples.

3.12 Example: Recall that we fix a basis \(\{x_1, x_2, x_3, x_4, z\}\) of \(L_4\) with the Lie mul- tiplication \([x_1, x_2] = x_1 x_2 - x_2 x_1\) and otherwise is zero. Let \(t \in L\). Then \(t = b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 + b_5 z\) for some \(b_1, b_2, b_3, b_4, b_5 \in R\).

1. We claim that the 1-dimensional subspace \(V = \text{span} \{x_1\}\) is not an \(I\)-ideal of \(L\). Let \(x, y \in V\). Then \(x = a_1 x_1, y = a_2 x_1\) for some \(a_1, a_2 \in R\). Since
\[\begin{align*}
[x, y] &= [a_1 x_1, a_2 x_1] = a_1 a_2 x_1 \in V
\end{align*}\]Therefore, \(V\) is not an \(I\)-ideal of \(L\).

2. We claim that the 2-dimensional subspace \(V = \text{span} \{x_1, x_2\}\) is not an \(I\)-ideal of \(L\). Let \(x, y \in V\). Then \(x = ax_1 + ax_2, y = ax_3 + ax_4\) for some \(a_1, a_2, a_3, a_4 \in R\). Since
\[\begin{align*}
[x, y] &= [ax_1 + ax_2, ax_3 + ax_4] = (a_1 a_2 - a_3 a_4)x_1 x_2 \not\in V
\end{align*}\]Therefore, \(V\) is not an \(I\)-ideal of \(L\).

3. We claim that the 3-dimensional subspace \(V = \text{span} \{x_1, x_2, x_3\}\) is not an \(I\)-ideal of \(L\). Let \(x, y \in V\). Then \(x = ax_1 + ax_2 + ax_3, y = ax_4 + ax_5\) for some \(a_1, a_2, a_3, a_4, a_5 \in R\). Since
\[\begin{align*}
[x, y] &= [ax_1 + ax_2 + ax_3, ax_4 + ax_5] = (a_1 a_2 - a_3 a_4 + a_5)x_1 x_2 x_3 \not\in V
\end{align*}\]Therefore, \(V\) is not an \(I\)-ideal of \(L\).

4. We claim that the 4-dimensional subspace \(V = \text{span} \{x_2, x_3, x_4, z\}\) is not an \(I\)-ideal of \(L\). Let \(x, y \in V\). Then \(x = ax_3 + ax_4 + ax_2 + ax_1, z = ax_3 + ax_4 + ax_2 + ax_1\) for some \(a_1, a_2, a_3, a_4, a_5 \in R\). Since
\[\begin{align*}
[x, y] &= [ax_3 + ax_4 + ax_2 + ax_1, ax_3 + ax_4 + ax_2 + ax_1] = (a_1 a_2 + a_3 a_4 + a_5)x_1 x_2 x_3 x_4 z \not\in V
\end{align*}\]Therefore, \(V\) is not an \(I\)-ideal of \(L\).

3.13 Proposition: Suppose that \(L = L_1\) and \(L \not\in \mathcal{Z}\). Then the following is hold.
1. \(L\) contains a 1-dimensional \(I\)-ideal which is not ideal.
2. \(L\) contains a 2-dimensional commutative \(I\)-ideal which is not ideal.
3. \(L\) contains a 3-dimensional commutative \(I\)-ideal.
4. L contains a 3-dimensional non-commutative I-ideal.

5. L contains a 4-dimensional non-commutative I-ideal.

\textbf{Proof:} By 2.6Theorems and 2.7, there is a basis \{x_1, x_2, x_3, x_4, z\} of \(L_4\) with the Lie multiplication \([x_1, x_3] = -x_2, [x_1, x_4] = x_1, [x_2, x_3] = x_1, [x_2, x_4] = x_2\) and otherwise is zero. Let \( \ell \in L_4 \). Then \( \ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta z \) for some \( \beta_1, \beta_2, \beta_3, \beta_4, \beta \in R \).

1) We claim that the 1-dimensional subspace \( V = \text{span} \{x_1\} \) is an I-ideal of \( L_4 \). We need to show that \([V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha_1 x_1, y = \alpha_2 x_1 \) for some \( \alpha_1, \alpha_2 \in R \). Since

\[
[x, [y, \ell]] = 0.
\]

2) We claim that the 2-dimensional subspace \( V = \text{span} \{x_2, x_3\} \) is an I-ideal of \( L_4 \). We need to show that \([V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha_1 x_2 + \alpha_2 x_3, y = \alpha_3 x_2 + \alpha_4 x_3 + \alpha_5 x_4 + \alpha_6 z \) for some \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R \). Since

\[
[x, [y, \ell]] = 0.
\]

3) We claim that the 3-dimensional subspace \( V = \text{span} \{x_1, x_2, x_3, x_4\} \) is an I-ideal of \( L_4 \). We need to show that \([V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_4 + \alpha_6 z \) for some \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R \). Since

\[
[x, [y, \ell]] = 0.
\]

4) We claim that the 4-dimensional subspace \( V = \text{span} \{x_1, x_2, x_3, x_4\} \) is an I-ideal of \( L_4 \). We need to show that \([V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_4 + \alpha_6 z \) for some \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R \). Since

\[
[x, [y, \ell]] = 0.
\]

5) We claim that the 4-dimensional subspace \( V = \text{span} \{x_1, x_2, x_3, x_4\} \) is an I-ideal of \( L_4 \). We need to show that \([V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_4 + \alpha_6 z \) for some \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in R \). Since

\[
[x, [y, \ell]] = 0.
\]

\textbf{3.14 Remark:} 3.13Theorem is not true if we state that every 1, 2, 3, and 4-dimensional subspace is an I-ideal because \( L_4 \) contains a 1, 2, 3, and 4-dimensional subspace which is not I-Ideal. As one can see in the following examples.

\textbf{3.15 Example:} Recall that we fix a basis \( \{x_1, x_2, x_3, x_4, z\} \) of \( L_4 \) with the Lie multiplication \([x_1, x_3] = -x_2, [x_1, x_4] = x_1, [x_2, x_3] = x_1, [x_2, x_4] = x_2 \) and otherwise is zero. Let \( \ell \in L_4 \). Then \( \ell = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta z \) for some \( \beta_1, \beta_2, \beta_3, \beta_4, \beta \in R \).

1) We claim that the 1-dimensional subspace \( V = \text{span} \{x_4\} \) is not an I-Ideal of \( L_4 \). Let \( x, y \in V \). Then \( x = \alpha_1 x_4 + \alpha_2 x_4 \) for some \( \alpha_1, \alpha_2 \in R \). Since

\[
[x, [y, \ell]] = 0.
\]
\[ [\alpha x, \beta] = x, \beta = \alpha x + \beta x, y = \alpha x + \alpha z + \beta x + \beta z \in V \cap [V, [V, L]] \subseteq V. \]

Therefore, \( V \) is an L-ideal. 

2) We claim that the 2-dimensional subspace \( V = \text{span} \{ x_1, x_4 \} \) is not an ideal of \( L \). Let \( x, y \in V \). Then \( x = \alpha x + y, y = \alpha x + \alpha z + \beta x + \beta z \in V \). Since

\[ x, \alpha \] = \[ x_1, \alpha x + \alpha z + \beta x + \beta z \subseteq V \cap [V, [V, L]] \subseteq V. \]

Therefore, \( V \) is an L-ideal of \( L \).

3) We claim that the 3-dimensional subspace \( V = \text{span} \{ x_1, x_4, x_5 \} \) is not an L-ideal of \( L \). Let \( x, y \in V \). Then \( x = \alpha x + \alpha x + \alpha z, y = \alpha x + \alpha z + \beta x + \beta z \in V \). Since

\[ x, \alpha \] = \[ x_1, \alpha x + \alpha z + \beta x + \beta z \subseteq V \cap [V, [V, L]] \subseteq V. \]

Therefore, \( V \) is an L-ideal of \( L \).

4) We claim that the 4-dimensional subspace \( V = \text{span} \{ x_2, x_3, x_4, x_5 \} \) is not an L-ideal of \( L \). Let \( x, y \in V \). Then \( x = \alpha x + \alpha x + \alpha z, y = \alpha x + \alpha z + \beta x + \beta z \in V \). Since

\[ x, \alpha \] = \[ x_1, \alpha x + \alpha z + \beta x + \beta z \subseteq V \cap [V, [V, L]] \subseteq V. \]

Therefore, \( V \) is an L-ideal of \( L \).

3.16 Proposition: Suppose that \( L = L_4 \) and \( L \leq Z \). Then the following is hold.

1. \( L \) contains a 1-dimensional L-ideal which is not ideal.
2. \( L \) contains a 2-dimensional commutative L-ideal which is not ideal.
3. \( L \) contains a 3-dimensional non-commutative L-ideal.
4. \( L \) contains a 4-dimensional non-commutative L-ideal.

Proof: By 2.6 Theorems and 2.7, there is a basis \( \{ x_1, x_2, x_3, x_4, x_5 \} \) of \( L_4 \) with the Lie multiplication \( [x_1, x_4] = x_1, [x_2, x_3] = x_2 \) and otherwise is zero. Let \( \ell \in L \). Then \( \ell = \beta x_1 + \beta x_2 + \beta x_3 + \beta x_4 + \beta z \) for some \( \beta \in \mathbb{Z} \). We need to show that \( \ell \) is an ideal of \( L \).

1) We claim that the 1-dimensional subspace \( V = \text{span} \{ x_1 \} \) of \( L \) is an ideal of \( L \). We need to show that \( [V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha x \) for some \( \alpha, \beta \in \mathbb{R} \). Since

\[ x, \alpha \] = \[ x_1, \alpha x \in V \cap [V, [V, L]] \subseteq V. \]

Therefore, \( V \) is an L-ideal of \( L \).

2) We claim that the 2-dimensional subspace \( V = \text{span} \{ x_2, x_3 \} \) of \( L \) is an ideal of \( L \). We need to show that \( [V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha x + \alpha z, y = \alpha x + \alpha z \) for some \( \alpha, \beta, \gamma \in \mathbb{R} \). Since

\[ x, \alpha \] = \[ x_2, x_3, x_4, x_5 \in V \cap [V, [V, L]] \subseteq V. \]

Therefore, \( V \) is an L-ideal of \( L \).

It remains to us to show that \( V \) is an L-ideal of \( L \). Therefore, \( V \) is a commutative L-ideal of \( L \).

3) We claim that the 3-dimensional subspace \( V = \text{span} \{ x_1, x_2, x_3 \} \) of \( L \) is an ideal of \( L \). We need to show that \( [V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha x + \alpha z, y = \alpha x + \alpha z \) for some \( \alpha, \beta, \gamma \in \mathbb{R} \). Since

\[ x, \alpha \] = \[ x_1, x_2, x_3, x_4, x_5 \in V \cap [V, [V, L]] \subseteq V. \]

Therefore, \( V \) is an L-ideal of \( L \).

4) We claim that the 4-dimensional subspace \( V = \text{span} \{ x_1, x_2, x_3 \} \) of \( L \) is an ideal of \( L \). We need to show that \( [V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha x + \alpha z, y = \alpha x + \alpha z \) for some \( \alpha, \beta, \gamma \in \mathbb{R} \). Since

\[ x, \alpha \] = \[ x_1, x_2, x_3, x_4, x_5 \in V \cap [V, [V, L]] \subseteq V. \]

Therefore, \( V \) is a commutative L-ideal of \( L \).
In this paper, we proved that where \( L \) is a non-commutative 4-dimensional subspace of \([V, [V, L]]\).

Therefore, \( V \) is a non-commutative I-ideal of \( L \), as required.

5) We claim that the 4-dimensional subspace \( V = \text{span} \{x_1, x_2, x_3, z\} \) is an I-ideal of \( L \). We need to show that \([V, [V, L]] \subseteq V \). Let \( x, y \in V \). Then \( x = \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha z \), \( y = \beta x_1 + \beta x_2 + \beta x_3 + \beta z \) for some \( \alpha, \beta, \gamma, \delta \subseteq R \).

\[
\begin{align*}
\{x, y, z\} &= \{ \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha z, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \} \\
&= \{ \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha z \}
\end{align*}
\]

Therefore, \( V \) is a non-commutative I-ideal of \( L \), as required.

\[\square\]

3.17 Remark: 3.16 Theorem is not true if we state that every 1, 2, 3 and 4-dimensional subspace is an I-ideal because \( L = L_5 \) contains a 1, 2, 3 and 4-dimensional subspace which is not an I-ideal. As one can see in the following examples.

3.18 Example: Recall that we fix a basis \( \{x_1, x_2, x_3, x_4, z\} \) of \( L_5 \) with the Lie multiplication \([x_1, x_4] = x_1, [x_2, x_3] = x_1, [x_2, x_4] = x_2 \) and otherwise is zero. Let \( \ell \in L \). Then \( \ell = \beta x_1 + \beta x_2 + \beta x_3 + \beta x_4 + \beta z \) for some \( \beta_1, \beta_2, \beta_3, \beta_4 \subseteq R \).

1) We claim that the 1-dimensional subspace \( V = \text{span} \{x_1\} \) is not an I-ideal of \( L \). Let \( x, y \in V \). Then \( x = \alpha x_1 \), \( y = \gamma x_4 \) for some \( \alpha, \gamma \subseteq R \).

\[
\begin{align*}
\{x, y, z\} &= \{ \alpha x_1, \gamma x_4, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \} \\
&= \{ \alpha x_1, \gamma x_4, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \}
\end{align*}
\]

Therefore, \( V \) is not an I-ideal of \( L \).

2) We claim that the 2-dimensional subspace \( V = \text{span} \{x_1, x_4\} \) is not an I-ideal of \( L \). Let \( x, y \in V \). Then \( x = \alpha x_1 + \alpha x_4 \), \( y = \gamma x_1 + \gamma x_4 \) for some \( \alpha, \gamma \subseteq R \).

\[
\begin{align*}
\{x, y, z\} &= \{ \alpha x_1 + \alpha x_4, \gamma x_1 + \gamma x_4, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \} \\
&= \{ \alpha x_1 + \alpha x_4, \gamma x_1 + \gamma x_4, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \}
\end{align*}
\]

Therefore, \( V \) is not an I-ideal of \( L \).

3) We claim that the 3-dimensional subspace \( V = \text{span} \{x_1, x_4, z\} \) is not an I-ideal of \( L \). Let \( x, y \in V \). Then \( x = \alpha x_1 + \alpha x_4 + \alpha z \), \( y = \gamma x_1 + \gamma x_4 + \gamma z \) for some \( \alpha, \gamma \subseteq R \).

\[
\begin{align*}
\{x, y, z\} &= \{ \alpha x_1 + \alpha x_4 + \alpha z, \gamma x_1 + \gamma x_4 + \gamma z, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \} \\
&= \{ \alpha x_1 + \alpha x_4 + \alpha z, \gamma x_1 + \gamma x_4 + \gamma z, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \}
\end{align*}
\]

Therefore, \( V \) is not an I-ideal of \( L \).

4) We claim that the 4-dimensional subspace \( V = \text{span} \{x_1, x_4, z\} \) is not an I-ideal of \( L \). Let \( x, y \in V \). Then \( x = \alpha x_1 + \alpha x_4 + \alpha z \), \( y = \gamma x_1 + \gamma x_4 + \gamma z \) for some \( \alpha, \gamma \subseteq R \).

\[
\begin{align*}
\{x, y, z\} &= \{ \alpha x_1 + \alpha x_4 + \alpha z, \gamma x_1 + \gamma x_4 + \gamma z, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \} \\
&= \{ \alpha x_1 + \alpha x_4 + \alpha z, \gamma x_1 + \gamma x_4 + \gamma z, \beta x_1 + \beta x_2 + \beta x_3 + \beta z \}
\end{align*}
\]

Therefore, \( V \) is not an I-ideal of \( L \).

Now we are ready to prove 3.17 Theorem. Recall that \( L \) is either \( L_n \) or \( L_1 \) or \( L_2 \) or \( L_3 \) or \( L_4 \) or \( L_5 \) or \( L_6 \) or \( L_7 \). We need to show that \( L \) contains a commutative and non-commutative I-ideal.

Proof: for 3.1 Proposition \( L \) contains a commutative and non-commutative I-ideal.

If \( L = L_n \), by the 3.4 Proposition \( L \) contains a commutative and non-commutative I-ideal.

If \( L = L_1 \), by the 3.7 Proposition \( L \) contains a commutative and non-commutative I-ideal.

If \( L = L_2 \), by the 3.10 Proposition \( L \) contains a commutative and non-commutative I-ideal.

If \( L = L_3 \), by the 3.13 Proposition \( L \) contains a commutative and non-commutative I-ideal.

If \( L = L_4 \), by the 3.16 Proposition \( L \) contains a commutative and non-commutative I-ideal.

4. CONCLUSION

In this paper, we proved that where \( L \) is a 4-dimensional real Lie algebras with 2-dimensional derived. Then \( L \) contains a commutative and non-commutative I-ideal.
REFERENCES


