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Study the properties of Spectral Characteristics and Eigenfunctions for Sturm-Liouville Boundary Value Problems

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ABSTRACT: In this study, I provide an overview of the Sturm-Liouville operator's spectral theory on a finite interval. Also, I study the main spectral characteristics of the second-order differential operator. I show that the eigenfunctions are real and that the problem cannot have complex eigenvalues, and the characteristic function's zeros are all simple. The Sturm-Liouville eigenvalues of that problem are non-degenerate; for every eigenvalue, there exists only one linearly independent solution. Also, the Wronskian function of two solutions to a homogeneous equation vanishes.

Keywords: Sturm-Liouville, spectral theory, eigenvalue, eigenfunction.

1. INTRODUCTION

Linear differential equations of second order are very important topics in the study of differential equations. In mathematics, physics, and engineering, the Sturm-Liouville problem has many applications. Its solutions are essential to studying the behavior of linear differential equations and the boundary value problems that provide. Furthermore, eigenvalue problem arise in a number of different areas of mathematics. Also A boundary value problem is a mathematical problem in which differential equations are solved subject to boundary conditions. There are many studies on Sturm-Liouville; some of them can be found in [1-5]. As the inverse problem of spectral analysis has applications that involve important non-linear equations in mathematical physics, interest in it has grown. Recently [3, 6, 7, 8], the problem of inverse spectral analysis has attracted a lot of interest. In this work, I study the eigenvalue and eigenfunction for the Sturm-Liouville boundary value problem and demonstrate that the problem can have no complex eigenvalue and that the corresponding eigenfunctions are real. Moreover, the characteristic functions are simple. And I show that there exists only one finite solution for each eigenvalue. Furthermore, I conclude that the Wronskian function of a homogeneous equation is zero.

Let L be a second order differential operator such that

$$L[f] = f''(x) + p_0(x)f'(x) + p_1(x)f(x)\ ,\ x \in [\alpha, \beta] \tag{1}$$

we shall consider eigenvalue problem of the special form

$$L[f] = \lambda f$$

with the boundary conditions

$$\mathcal{A}(f) = a_1 f(\alpha) + a_2 f'(\alpha) = 0\ ,\ \mathcal{B}(f) = b_1 f(\beta) + b_2 f'(\beta) = 0 \tag{2}$$

Here λ is a spectral parameter, $p_0(x), p_1(x), a_1, a_2, b_1$ and b_2 are real.

Theorem 1. P(λ) of the differential operator L is entire, has zeros at the eigenvalues of problems (1) - (2), and has an at most countable set of zeros $\{\lambda_n\}$.

Proof. Let $v_1(x,\lambda), v_2(x,\lambda)$ are solutions of (1), and satisfying equation (2)

$v_1(\alpha,\lambda) = a_2\ ,\ \ v_1'(\alpha,\lambda) = -a_1$
$v_2(\beta,\lambda) = b_2\ ,\ \ v_2'(\beta,\lambda) = -b_1$

Every fixed x, the functions $v_1(x,\lambda), v_2(x,\lambda)$ are entire in λ.
Clearly,

$$\mathcal{A}(v_1) = a_1 v_1(\alpha,\lambda) + a_2 v_1'(\alpha,\lambda) = 0, \mathcal{B}(v_2) = \beta_1 v_2(\beta,\lambda) + \beta_2 v_2'(\beta,\lambda) = 0 \quad (3)$$

We define

$$\mathrm{P}(\lambda) = v_1 v_2' - v_1' v_2 \quad (4)$$

Which is the Wronskian of $v_1(x,\lambda)$ and $v_2(x,\lambda)$, and it is independent of $x \in [\alpha,\beta]$.
If $x = \alpha$ in to (4), we get

$$\mathrm{P}(\lambda) = v_1(\alpha,\lambda)v_2'(\alpha,\lambda) - v_1'(\alpha,\lambda)v_2(\alpha,\lambda)$$
$$= a_2 v_2'(\alpha,\lambda) + a_1 v_2(\alpha,\lambda) = \mathcal{A}(v_2)$$

If $x = \beta$ in to (4), we get

$$\mathrm{P}(\lambda) = v_1(\beta,\lambda)v_2'(\beta,\lambda) - v_1'(\beta,\lambda)v_2(\beta,\lambda)$$
$$= -b_1 v_1(\beta,\lambda) - b_2 v_1'(\beta,\lambda) = -\mathcal{B}(v_1)$$

So $\mathrm{P}(\lambda) = \mathcal{A}(v_2) = -\mathcal{B}(v_1)$ (5)
So the characteristic function $\mathrm{P}(\lambda)$ is entire function in λ, and the set of eigenvalues is countable.

Theorem 2. For the differential operator L, let $\{\lambda_n\}$ be eigenvalues, and the functions $v_1(x,\lambda_n)$ and $v_2(x,\lambda_n)$ are eigenfunctions, then there exists a sequence $\{\gamma_n\}$ such that

$$v_1(x,\lambda_n) = \gamma_n v_2(x,\lambda_n),\ \gamma_n \neq 0 \quad (6)$$

Proof. Let us assume that λ_0 be a zero of $\mathrm{P}(\lambda)$.
Then $\mathrm{P}(\lambda) = \begin{vmatrix} v_1(x,\lambda_0) & v_2(x,\lambda_0) \\ v_1'(x,\lambda_0) & v_2'(x,\lambda_0) \end{vmatrix} = 0$, holds, that is the functions $v_1(x,\lambda_0)$ and $v_2(x,\lambda_0)$ are linearly dependent $v_1(x,\lambda_0) = \gamma_n v_2(x,\lambda_0)$, γ_n is constant and they satisfy the boundary conditions (2).
Hence, λ_0 is an eigenvalue, $v_1(x,\lambda_0)$ and $v_2(x,\lambda_0)$ are eigenfunctions that correspond to this eigenvalue.
Conversely, let λ_0 be an eigenvalue of the equation (1), and let $f_0(x,\lambda_0)$ be an eigenfunctions, then the boundary conditions (2) hold

$$\mathcal{A}(f_0) = \mathcal{B}(f_0) = 0.$$

Clearly $f_0(\alpha) \neq 0$
Additionally, if $f_0(x,\lambda_0)$ satisfy the condition $f_0(\alpha,\lambda) = a_2$ and $f_0'(\alpha,\lambda) = -a_1$, then $f_0(x,\lambda_0) = v_1(x,\lambda_0)$. According to the equation (5), we have

$$\mathrm{P}(\lambda_0) = -\mathcal{B}\big(v_1(\alpha,\lambda_0)\big) = -\mathcal{B}\big(f_0(\alpha,\lambda_0)\big) = 0.$$

Similarly, if we assume that $f_0(x,\lambda_0)$ satisfy the condition $f_0(\beta,\lambda) = b_2$, $f_0'(\beta,\lambda) = -b_1$, then $f_0(x,\lambda_0) = v_2(x,\lambda_0)$. Again from the equation (5), it is obvious that

$$\mathrm{P}(\lambda_0) = \mathcal{A}\big(v_1(\beta,\lambda_0)\big) = \mathcal{A}\big(f_0(\beta,\lambda_0)\big) = 0.$$

Consequently, There is only one eigenfunction (up to a multiplicative constant), for every eigenvalue.

Lemma 1. The following equality holds:

$$\dot{\mathrm{P}}(\lambda_n) = -\gamma_n k_n$$

Where γ_n are defined by equation (6) and $k_n = \int_\alpha^\beta {v_2}^2(x,\lambda_n)\,dx$ (7)

Proof. Since $v_1(x,\lambda)$ and $v_2(x,\lambda)$ are solution of equation (1), so

$$L[v_1] = \lambda v_1\ ,\ L[v_2] = \lambda v_2$$
$$v_1''(x,\lambda_n) + p_0(x)v_1'(x,\lambda_n) + p_1(x)v_1(x,\lambda_n) = \lambda_n v_1(x,\lambda_n)$$
$$v_2''(x,\lambda) + p_0(x)v_2'(x,\lambda) + p_1(x)v_2(x,\lambda) = \lambda v_2(x,\lambda)$$
$$\frac{d}{dx}P(\lambda) = \frac{d}{dx}\big(v_1(x,\lambda_n)v_2'(x,\lambda) - v_1'(x,\lambda_n)v_2(x,\lambda)\big) = v_1(x,\lambda_n)v_2''(x,\lambda) - v_1''(x,\lambda_n)v_2(x,\lambda)$$
$$= v_1(x,\lambda_n)[\lambda v_2(x,\lambda) - p_0(x)v_2'(x,\lambda) - p_1(x)v_2(x,\lambda)] - v_2(x,\lambda)[\lambda_n v_1(x,\lambda_n) - p_0(x)v_1'(x,\lambda_n) - p_1(x)v_1(x,\lambda_n)]$$
$$= (\lambda - \lambda_n)v_1(x,\lambda_n)v_2(x,\lambda) - p_0(x)[v_1(x,\lambda_n)v_2'(x,\lambda) - v_1'(x,\lambda_n)v_2(x,\lambda)]$$

Then

$$\frac{d}{dx}P(\lambda) = (\lambda - \lambda_n)v_1(x,\lambda_n)v_2(x,\lambda) - p_0(x)[v_1(x,\lambda_n)v_2'(x,\lambda) - v_1'(x,\lambda_n)v_2(x,\lambda)]$$
$$\int_\alpha^\beta \frac{d}{dx}P(\lambda) = (\lambda - \lambda_n)\int_\alpha^\beta v_1(x,\lambda_n)v_2(x,\lambda)dx - p_0(x)\int_\alpha^\beta [v_1(x,\lambda_n)v_2'(x,\lambda) - v_1'(x,\lambda_n)v_2(x,\lambda)]dx$$
$$\mathrm{P}(\lambda)\Big|_\alpha^\beta = (\lambda - \lambda_n)\int_\alpha^\beta v_1(x,\lambda_n)v_2(x,\lambda)dx - p_0(x)\int_\alpha^\beta [v_1(x,\lambda_n)v_2'(x,\lambda) - v_1'(x,\lambda_n)v_2(x,\lambda)]dx$$
$$-b_1 v_1(\beta,\lambda) - b_2 v_1'(\beta,\lambda) - a_2 v_2'(\alpha,\lambda) - a_1 v_2(\alpha,\lambda)$$
$$= (\lambda - \lambda_n)\int_\alpha^\beta v_1(x,\lambda_n)v_2(x,\lambda)dx - p_0(x)\int_\alpha^\beta [v_1(x,\lambda_n)v_2'(x,\lambda) - v_1'(x,\lambda_n)v_2(x,\lambda)]dx$$

$$-\mathcal{B}(v_1)-\mathcal{A}(v_2)=(\lambda-\lambda_n)\int_\alpha^\beta v_1(x,\lambda_n)v_2(x,\lambda)dx-p_0(x)\int_\alpha^\beta [v_1(x,\lambda_n)v_2{}'(x,\lambda)-v_1{}'(x,\lambda_n)v_2(x,\lambda)]dx$$

Using equation (5) we get

$-\mathcal{B}(v_1)-\mathcal{A}(v_2)=-\mathrm{P}(\lambda)$

So

$-\mathrm{P}(\lambda)=(\lambda-\lambda_n)\int_\alpha^\beta v_1(x,\lambda_n)v_2(x,\lambda)dx$ (Since the function $v_1(x,\lambda)$ and $v_2(x,\lambda)$ both satisfy the boundary conditions so the Wronskian is zero)

For λ approach to λ_n

$$\dot{\mathrm{P}}(\lambda_n)=-\int_\alpha^\beta v_1(x,\lambda_n)v_2(x,\lambda)dx$$

Using equation (6) and equation (7) we obtain

$\dot{\mathrm{P}}(\lambda_n)=-\gamma_n k_n$

Theorem 3. The eigenvalues $\{\lambda_n\}$ and the eigenfunctions $v_1(x,\lambda_n)$ and $v_2(x,\lambda_n)$ are real. And all zeros of $\mathrm{P}(\lambda)$ are simple. That is meaning $\dot{\mathrm{P}}(\lambda_n)\neq 0$.

Proof. Let λ_n and λ_m ($\lambda_n\neq\lambda_m$) be eigenvalues with eigenfunctions $f_n(x)$ and $f_m(x)$ respectively.
Then integration from α to β we get

$$\int_\alpha^\beta L[f_n(x)]f_m(x)dx=\int_\alpha^\beta f_n(x)L[f_m(x)]dx$$

Hence

$$\lambda_n\int_\alpha^\beta f_n(x)f_m(x)dx=\lambda_m\int_\alpha^\beta f_n(x)f_m(x)dx$$

Let $\lambda_0=\delta_0+i\sigma_0,\ \sigma_0\neq 0$ be a non-real eigenvalue.
Let $f_0(x)=u_0(x)+iv_0(x)$ be a corresponding eigenfunction. Then

$$Lf_0(x)=f_0{}''(x)+p_0(x)f_0{}'(x)+p_1(x)f_0(x)=\lambda_0 f_0(x)$$

Taking complex conjugates, and remembering that $p_0(x)$ and $p_1(x)$ are real functions, we have

$$L\overline{f_0}(x)=\overline{f_0}''(x)+p_0(x)\overline{f_0}'(x)+p_1(x)\overline{f_0}(x)=\overline{\lambda_0}\overline{f_0}(x)$$

The function $f_0(x)$ satisfies the condition

$\mathcal{A}(f_0)=0$, $\mathcal{B}(f_0)=0$

Taking complex conjugates, and remembering that the operators $\mathcal{A}$ and $\mathcal{B}$ have real coefficients, we have

$\overline{\mathcal{A}(f_0)}=\mathcal{A}(\overline{f_0})=0$, $\overline{\mathcal{B}(f_0)}=\mathcal{B}(\overline{f_0})=0$

Thus the function $\overline{f_0}(x)=u_0(x)-iv_0(x)$ is also an eigenfunction of that problem, corresponding to the eigenvalue $\overline{\lambda_0}=\delta_0-i\sigma_0$. But then

$$(\lambda_0-\overline{\lambda_0})\int_\alpha^\beta f_0(x)\overline{f_0}(x)dx=0$$

Or

$$2i\sigma_0\int_\alpha^\beta |f_0(x)|^2dx=0$$

Which is impossible, since $\sigma_0\neq 0$ and since $f_0(x)$ is non-trivial solution.
As a result, the eigenfunctions $v_1(x,\lambda_n)$ and $v_2(x,\lambda_n)$ are real, as are all of $\{\lambda_n\}$ of the differential operator L.
Since $\gamma_n\neq 0, k_n\neq 0$, we get that $\dot{\mathrm{P}}(\lambda)\neq 0$.

Theorem 4. Eigenvalues of boundary value problem (1) - (2) are non-degenerate (That is, there is only one linearly independent eigenfunction for each eigenvalue).

Proof. Let f_1 and f_2 are eigenfunctions corresponding to the given eigenvalue λ. Then

$$L[f_1]=\lambda f_1$$
$$L[f_2]=\lambda f_2$$

Now, $f_2(x)L[f_1(x)]-f_1(x)L[f_2(x)]=0$

$$f_2(x)[f_1{}''+p_0(x)f_1{}'+p_1(x)f_1]-f_1(x)[f_2{}''+p_0(x)f_2{}'+p_1(x)f_2]=0$$
$$f_2(x)f_1{}''+p_0(x)f_2(x)f_1{}'-f_1(x)f_2{}''-p_0(x)f_1(x)f_2{}'=0$$
$$-[f_1f_2{}''-f_2f_1{}'']-p_0(x)[f_1f_2{}'-f_2f_1{}']=0$$

$f_1f_2{}''-f_2f_1{}''+p_0(x)w(f_1,f_2)=0$ (where $w(f_1,f_2)$ is the Wronskian)

$$\frac{d}{dx}w(f_1,f_2)+p_0(x)w(f_1,f_2)=0$$

According to Abel's formula we have

$p_0(x)w(x, f_1, f_2) = c$ (where c is constant and $w(x, f_1, f_2)$ is the Wronskian of f_1 and f_2).
If the Wronskian vanishes at one point of the interval $[\alpha, \beta]$, it must vanishe at every point
So $w(x, f_1, f_2) \equiv 0$
So f_1 and f_2 are linearly dependent.
Hence $f_1 \propto f_2$.

Theorem 5. If v_1 and v_2 are solution of homogeneous equation
$f''(x) + p_0(x)f'(x) + p_1(x)f(x) = 0$ where $p_0(x), p_1(x)$ are real, then either $w(v_1, v_2)_{(x)} = 0$ or $w(v_1, v_2)_{(x)} \neq 0, \forall x$.

Proof. Because v_1 and v_2 are solution of homogeneous equation so,

$$v_1''(x) + p_0(x)v_1'(x) + p_1(x)v_1(x) = 0$$
$$v_2''(x) + p_0(x)v_2'(x) + p_1(x)v_2(x) = 0$$

After multiplying by v_2 for the first equation, and v_1 for the second equation, and then subtracting, the result is

$$v_1v_2'' - v_2v_1'' + p_0(v_1v_2' - v_2v_1') = 0$$
$$w' + p_0w = 0$$

Integrating this last equation, we obtain $w(x) = ce^{\int_\alpha^x p_0(t)dt}$, $x \in [\alpha, \beta]$, and c is arbitrary constant.
Therefore $w(x) = 0$ if and only if $c = 0$, since the exponential function cannot vanish for any real or complex exponent.

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