Study the properties of Spectral Characteristics and Eigenfunctions for Sturm-Liouville Boundary Value Problems

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ABSTRACT: In this study, I provide an overview of the Sturm-Liouville operator’s spectral theory on a finite interval. Also, I study the main spectral characteristics of the second-order differential operator. I show that the eigenfunctions are real and that the problem cannot have complex eigenvalues, and the characteristic function’s zeros are all simple. The Sturm-Liouville eigenvalues of that problem are non-degenerate; for every eigenvalue, there exists only one linearly independent solution. Also, the Wronskian function of two solutions to a homogeneous equation vanishes.

Keywords: Sturm-Liouville, spectral theory, eigenvalue, eigenfunction.

1. INTRODUCTION

Linear differential equations of second order are very important topics in the study of differential equations. In mathematics, physics, and engineering, the Sturm-Liouville problem has many applications. Its solutions are essential to studying the behavior of linear differential equations and the boundary value problems that provide. Furthermore, eigenvalue problem arise in a number of different areas of mathematics. Also A boundary value problem is a mathematical problem in which differential equations are solved subject to boundary conditions. There are many studies on Sturm-Liouville; some of them can be found in [1-5]. As the inverse problem of spectral analysis has applications that involve important non-linear equations in mathematical physics, interest in it has grown. Recently [3, 6, 7, 8], the problem of inverse spectral analysis has attracted a lot of interest. In this work, I study the eigenvalue and eigenfunction for the Sturm-Liouville boundary value problem and demonstrate that the problem can have no complex eigenvalue and that the corresponding eigenfunctions are real. Moreover, the characteristic functions are simple. And I show that there exists only one finite solution for each eigenvalue. Furthermore, I conclude that the Wronskian function of a homogeneous equation is zero.

Let L be a second order differential operator such that

\[ L[f] = f''(x) + p_0(x)f'(x) + p_1(x)f(x), \ x \in [\alpha, \beta] \]  \tag{1}

we shall consider eigenvalue problem of the special form

\[ L[f] = \lambda f \]

with the boundary conditions

\[ \mathcal{A}(f) = a_1 f(\alpha) + a_2 f'(\alpha) = 0, \ \mathcal{B}(f) = b_1 f(\beta) + b_2 f'(\beta) = 0 \] \tag{2}

Here \( \lambda \) is a spectral parameter, \( p_0(x), p_1(x), a_1, a_2, b_1 \) and \( b_2 \) are real.

**Theorem 1.** P(\( \lambda \)) of the differential operator L is entire, has zeros at the eigenvalues of problems (1) - (2), and has an at most countable set of zeros \( \{ \lambda_n \} \).

**Proof.** Let \( v_1(x, \lambda), v_2(x, \lambda) \) are solutions of (1), and satisfying equation (2)

\[ v_1(\alpha, \lambda) = a_2, \ v'_1(\alpha, \lambda) = -a_1 \]
\[ v_2(\beta, \lambda) = b_2, \ v'_2(\beta, \lambda) = -b_1 \]
Every fixed $x$, the functions $v_1(x, \lambda)$, $v_2(x, \lambda)$ are entire in $\lambda$.

Clearly,
\[\mathcal{A}(v_1) = a_1v_1(\alpha, \lambda) + a_2v_1(\alpha, \lambda) = 0, \quad \mathcal{B}(v_2) = b_1v_2(\beta, \lambda) + b_2v_2'(\beta, \lambda) = 0\] (3)

We define
\[P(\lambda) = v_1v_2' - v_1'v_2\] (4)

Which is the Wronskian of $v_1(x, \lambda)$ and $v_2(x, \lambda)$, and it is independent of $x \in [\alpha, \beta]$.

If $x = \alpha$ in (4), we get
\[P(\lambda) = v_1(\alpha, \lambda)v_2'(|\alpha, \lambda) - v_1'(\alpha, \lambda)v_2(\alpha, \lambda) = a_2v_2(\alpha, \lambda) + a_1v_2(\alpha, \lambda) = \mathcal{A}(v_2)\]

If $x = \beta$ in (4), we get
\[P(\lambda) = v_1(\beta, \lambda)v'_2(\beta, \lambda) - v'_1(\beta, \lambda)v_2(\beta, \lambda) = -b_1v_1(\beta, \lambda) - b_2v_1'(\beta, \lambda) = -\mathcal{B}(v_1)\]

So $P(\lambda) = \mathcal{A}(v_2) = -\mathcal{B}(v_1)$ (5)

So the characteristic function $P(\lambda)$ is entire function in $\lambda$, and the set of eigenvalues is countable.

**Theorem 2.** For the differential operator $L$, let $\{\lambda_n\}$ be eigenvalues, and the functions $v_1(x, \lambda_n)$ and $v_2(x, \lambda_n)$ are eigenfunctions, then there exists a sequence $\{\gamma_n\}$ such that

\[v_1(x, \lambda_n) = \gamma_nv_2(x, \lambda_n), \quad \gamma_n \neq 0\] (6)

**Proof.** Let us assume that $\lambda_0$ be a zero of $P(\lambda)$.

Then $P(\lambda) = \begin{vmatrix} v_1(x, \lambda_0) & v_2(x, \lambda_0) \\ v_1'(x, \lambda_0) & v_2'(x, \lambda_0) \end{vmatrix} = 0$, holds, that is the functions $v_1(x, \lambda_0)$ and $v_2(x, \lambda_0)$ are linearly dependent

$v_1(x, \lambda_0) = \gamma_nv_2(x, \lambda_0), \gamma_n$ is constant and they satisfy the boundary conditions (2).

Hence, $\lambda_0$ is an eigenvalue, $v_1(x, \lambda_0)$ and $v_2(x, \lambda_0)$ are eigenfunctions that correspond to this eigenvalue.

Conversely, let $\lambda_0$ be an eigenvalue of the equation (1), and let $f_0(x, \lambda_0)$ be an eigenfunctions, then the boundary conditions (2) hold

\[\mathcal{A}(f_0) = -\mathcal{B}(f_0) = 0\]

Clearly $f_0(\alpha) \neq 0$

Additionally, if $f_0(x, \lambda_0)$ satisfy the condition $f_0(\alpha, \lambda_0) = a_2$ and $f_0'(\alpha, \lambda_0) = -a_1$, then

$f_0(x, \lambda_0) = v_1(x, \lambda_0)$. According to the equation (5), we have

$P(\lambda_0) = -\mathcal{B}(f_0(\alpha, \lambda_0)) = -\mathcal{B}(f_0(\alpha, \lambda_0)) = 0$.

Similarly, if we assume that $f_0(x, \lambda_0)$ satisfy the condition $f_0(\beta, \lambda) = b_2$, $f_0'(\beta, \lambda) = -b_1$, then

$f_0(x, \lambda_0) = v_2(x, \lambda_0)$. Again from the equation (5), it is obvious that

$P(\lambda_0) = \mathcal{A}(f_0(\beta, \lambda_0)) = \mathcal{A}(f_0(\beta, \lambda_0)) = 0$.

Consequently, there is only one eigenfunction (up to a multiplicative constant), for every eigenvalue.

**Lemma 1.** The following equality holds:

\[P(\lambda_n) = -\gamma_nk_n\]

Where $\gamma_n$ are defined by equation (6) and $k_n = \int_{\alpha}^{\beta} v_2^2(x, \lambda_n) \, dx$ (7)

**Proof.** Since $v_1(x, \lambda)$ and $v_2(x, \lambda)$ are solution of equation (1), so

\[
\begin{align*}
L[v_1] &= \lambda v_1, \quad L[v_2] = \lambda v_2 \\
v_1''(x, \lambda) + p_0(x)v_1'(x, \lambda) + p_1(x)v_1(x, \lambda) &= \lambda_n v_1(x, \lambda) \\
v_2''(x, \lambda) + p_0(x)v_2'(x, \lambda) + p_1(x)v_2(x, \lambda) &= \lambda_n v_2(x, \lambda)
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dx} P(\lambda) &= \frac{d}{dx} (v_1(x, \lambda_n)v_2'(x, \lambda) - v_1'(x, \lambda_n)v_2(x, \lambda)) \\
&= v_1(x, \lambda_n)v_2''(x, \lambda) - v_1'(x, \lambda_n)v_2'(x, \lambda) \\
&= v_1(x, \lambda_n)[\lambda_n v_2(x, \lambda) - p_0(x)v_2'(x, \lambda) - p_1(x)v_2(x, \lambda)] - v_2(x, \lambda)[\lambda_n v_1(x, \lambda_n) - p_0(x)v_1'(x, \lambda_n) - p_1(x)v_1(x, \lambda_n)]
\end{align*}
\]

Then

\[
\begin{align*}
\frac{d}{dx} P(\lambda) &= (\lambda - \lambda_n)v_1(x, \lambda_n)v_2(x, \lambda) - p_0(x)[v_1(x, \lambda_n)v_2'(x, \lambda) - v_1'(x, \lambda_n)v_2(x, \lambda)]
\end{align*}
\]

\[
\begin{align*}
\int_{\alpha}^{\beta} \frac{d}{dx} P(\lambda) &= (\lambda - \lambda_n)\int_{\alpha}^{\beta} v_1(x, \lambda_n)v_2(x, \lambda)dx - p_0(x)\int_{\alpha}^{\beta} [v_1(x, \lambda_n)v_2'(x, \lambda) - v_1'(x, \lambda_n)v_2(x, \lambda)]dx
\end{align*}
\]

\[
\begin{align*}
P(\lambda) &= (\lambda - \lambda_n)\int_{\alpha}^{\beta} v_1(x, \lambda_n)v_2(x, \lambda)dx - p_0(x)\int_{\alpha}^{\beta} [v_1(x, \lambda_n)v_2'(x, \lambda) - v_1'(x, \lambda_n)v_2(x, \lambda)]dx
\end{align*}
\]

\[
\begin{align*}
-b_1v_1(\beta, \lambda) - b_2v_2'(\beta, \lambda) - a_2v_2'(\alpha, \lambda) - a_1v_2(\alpha, \lambda)
\end{align*}
\]

\[
\begin{align*}
&= (\lambda - \lambda_n)\int_{\alpha}^{\beta} v_1(x, \lambda_n)v_2(x, \lambda)dx - p_0(x)\int_{\alpha}^{\beta} [v_1(x, \lambda_n)v_2'(x, \lambda) - v_1'(x, \lambda_n)v_2(x, \lambda)]dx
\end{align*}
\]
\[-\mathcal{B}(v_1) - \mathcal{A}(v_2) = (\lambda - \lambda_n) \int_a^\beta v_1(x, \lambda_n) v_2(x, \lambda) dx - p_0(x) \int_a^\beta [v_1(x, \lambda_n) v_2'(x, \lambda) - v_1'(x, \lambda_n) v_2(x, \lambda)] dx\]

Using equation (5) we get
\[-\mathcal{B}(v_1) - \mathcal{A}(v_2) = -P(\lambda)\]

So
\[-P(\lambda) = (\lambda - \lambda_n) \int_a^\beta v_1(x, \lambda_n) v_2(x, \lambda) dx \quad \text{(Since the function } v_1(x, \lambda) \text{ and } v_2(x, \lambda) \text{ both satisfy the boundary conditions so the Wronskian is zero)}\]

For \( \lambda \) approach to \( \lambda_n \)
\[\hat{P}(\lambda_n) = - \int_a^\beta v_1(x, \lambda_n) v_2(x, \lambda) dx\]

Using equation (6) and equation (7) we obtain
\[\hat{P}(\lambda_n) = -\gamma_n k_n\]

**Theorem 3.** The eigenvalues \( \{\lambda_n\} \) and the eigenfunctions \( v_1(x, \lambda_n) \) and \( v_2(x, \lambda_n) \) are real. And all zeros of \( P(\lambda) \) are simple. That is meaning \( \hat{P}(\lambda_n) \neq 0 \).

**Proof.** Let \( \lambda_n \) and \( \lambda_m \) (\( \lambda_n \neq \lambda_m \)) be eigenvalues with eigenfunctions \( f_n(x) \) and \( f_m(x) \) respectively.

Then integration from \( \alpha \) to \( \beta \) we get
\[\int_\alpha^\beta L[f_n(x)] f_m(x) dx = \int_\alpha^\beta f_n(x) L[f_m(x)] dx\]

Hence
\[\lambda_n \int_\alpha^\beta f_n(x) f_m(x) dx = \lambda_m \int_\alpha^\beta f_n(x) f_m(x) dx\]

Let \( \lambda_0 = \delta_0 + i \sigma_0 \), \( \sigma_0 \neq 0 \) be a non-real eigenvalue.

Let \( f_0(x) = u_0(x) + i v_0(x) \) be a corresponding eigenfunction. Then
\[L f_0(x) = f_0''(x) + p_0(x) f_0'(x) + p_1(x) f_0(x) = \lambda_0 f_0(x)\]

Taking complex conjugates, and remembering that \( p_0(x) \) and \( p_1(x) \) are real functions, we have
\[L \overline{f_0}(x) = \overline{f_0''(x)} + p_0(x) \overline{f_0'(x)} + p_1(x) \overline{f_0(x)} = \overline{\lambda_0 f_0(x)}\]

The function \( f_0(x) \) satisfies the condition
\[\mathcal{A}(f_0) = 0 \quad \mathcal{B}(f_0) = 0\]

Taking complex conjugates, and remembering that the operators \( \mathcal{A} \) and \( \mathcal{B} \) have real coefficients, we have
\[\overline{\mathcal{A}(f_0)} = \mathcal{A}(\overline{f_0}) = 0 \quad \mathcal{B}(\overline{f_0}) = \mathcal{B}(f_0) = 0\]

Thus the function \( f_0(x) = u_0(x) - i v_0(x) \) is also an eigenfunction of that problem, corresponding to the eigenvalue \( \lambda_0 = \delta_0 - i \sigma_0 \). But then
\[\left( \lambda_0 - \lambda_0 \right) \int_\alpha^\beta f_0(x) \overline{f_0(x)} dx = 0\]

Or
\[2i \sigma_0 \int_\alpha^\beta |f_0(x)|^2 dx = 0\]

Which is impossible, since \( \sigma_0 \neq 0 \) and since \( f_0(x) \) is non-trivial solution.

As a result, the eigenfunctions \( v_1(x, \lambda_n) \) and \( v_2(x, \lambda_n) \) are real, as are all of \( \{\lambda_n\} \) of the differential operator \( L \).

Since \( \gamma_n \neq 0, k_n \neq 0 \), we get that \( \hat{P}(\lambda) \neq 0 \).

**Theorem 4.** Eigenvalues of boundary value problem (1) - (2) are non-degenerate (That is, there is only one linearly independent eigenfunction for each eigenvalue).

**Proof.** Let \( f_1 \) and \( f_2 \) are eigenfunctions corresponding to the given eigenvalue \( \lambda \). Then
\[L[f_1] = \lambda f_1 \quad L[f_2] = \lambda f_2\]

Now, \( f_2(x) L[f_1(x)] - f_1(x) L[f_2(x)] = 0 \)
\[f_2(x) f_1'' + p_0(x) f_1' + p_1(x) f_1 - f_1(x) f_2'' + p_0(x) f_2' + p_1(x) f_2 = 0\]

\[f_2(x) f_1'' + p_0(x) f_1' f_2 - f_1(x) f_2'' - p_0(x) f_2' f_1 = 0\]

\[-[f_1 f_2'' - f_2 f_1''] - p_0(x) [f_1 f_2' - f_2 f_1'] = 0\]

\[f_1 f_2'' - f_2 f_1'' + p_0(x) w(f_1, f_2) = 0 \quad \text{(where } w(f_1, f_2) \text{ is the Wronskian)}\]

\[\frac{d}{dx} w(f_1, f_2) + p_0(x) w(f_1, f_2) = 0\]

According to Abel’s formula we have
\( p_0(x)w(x, f_1, f_2) = c \) (where \( c \) is constant and \( w(x, f_1, f_2) \) is the Wronskian of \( f_1 \) and \( f_2 \)).

If the Wronskian vanishes at one point of the interval \([\alpha, \beta]\), it must vanish at every point.

So \( w(x, f_1, f_2) \equiv 0 \)

So \( f_1 \) and \( f_2 \) are linearly dependent.

Hence \( f_1 \propto f_2 \).

**Theorem 5.** If \( v_1 \) and \( v_2 \) are solution of homogeneous equation

\[
\frac{d^2}{dx^2} v_1(x) + p_0(x) \frac{d}{dx} v_1(x) + p_1(x) v_1(x) = 0
\]

where \( p_0(x) \), \( p_1(x) \) are real, then either \( w(v_1, v_2)(x) = 0 \) or \( w(v_1, v_2)(x) \neq 0 \), \( \forall x \).

**Proof.** Because \( v_1 \) and \( v_2 \) are solution of homogeneous equation so,

\[
v_1''(x) + p_0(x)v_1'(x) + p_1(x)v_1(x) = 0
\]

\[
v_2''(x) + p_0(x)v_2'(x) + p_1(x)v_2(x) = 0
\]

After multiplying by \( v_2 \) for the first equation, and \( v_1 \) for the second equation, and then subtracting, the result is

\[
v_1v_2'' - v_2v_1'' + p_0(v_1v_2' - v_2v_1') = 0
\]

\[
w' + p_0w = 0
\]

Integrating this last equation, we obtain

\[
w(x) = ce^{\int_{\alpha}^{x} p_0(t)dt}, x \in [\alpha, \beta]\), and \( c \) is arbitrary constant.

Therefore \( w(x) = 0 \) if and only if \( c = 0 \), since the exponential function cannot vanish for any real or complex exponent.

**REFERENCES**


