Wasit Journal for Pure Science

Journal Homepage: https://wjps.uowasit.edu.iq/index.php/wjps/index
e-ISSN: 2790-5241 p-ISSN: 2790-5233

Finite Difference with Quintic B-Splines for Solving a system of nonlinear Volterra Integro-Differential Equations of integer order

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DOI: https://doi.org/ 10.31185/wjps.351
Received 10 March 2024; Accepted 28 April 2024; Available online 30 Jun 2024

ABSTRACT: This study presents a novel numerical approach for solving systems of nonlinear Volterra Integro-Differential Equations (VIDEs) of integer order by integrating the Finite Difference Method (FDM) with Quintic B-Splines. The proposed method aims to address the challenges posed by the nonlinearity and integral terms inherent in VIDEs while providing enhanced accuracy and stability in the numerical solution. An error estimate of the approximate solution is proved. The study focuses on developing an efficient computational method to approximate solutions for VIDEs, which frequently arise in various fields including physics, engineering, and biology. Through extensive numerical experiments and comparisons with existing methodologies, the effectiveness and accuracy of the proposed approach are demonstrated. The results highlight the applicability and reliability of the numerical method in solving systems of nonlinear VIDEs, thereby contributing to advancing computational methods for solving integral equations in mathematical modeling and scientific research. Finally, some examples are presented to illustrate the simplicity and the effectiveness of the propose method.

Keywords: Volterra Integro-Differential Equations, Finite Difference Method, Quintic B-Splines.

1. INTRODUCTION

Research efforts have focused on addressing the challenges posed by singularly perturbed VIDEs (SPVIDEs), which exhibit rapid oscillations or layers in their solutions. Various finite difference techniques, including upwind schemes, trapezoidal rule, Richardson extrapolation, and hybrid schemes, have been implemented to accurately solve SPVIDEs with first and second order uniform convergence.
Overall, the exploration of numerical solutions for VIDEs encompasses a wide range of methods that aim to provide accurate and efficient results for complex mathematical models in diverse scientific fields. See references: [1,2,3,4,5,6,7,8,9].

The Finite Difference Method is a widely-used numerical technique for solving differential equations by approximating derivatives with finite differences on a discretized grid. While conventional finite difference schemes provide satisfactory results for many problems, they may lack accuracy and efficiency, particularly in capturing complex behaviors associated with nonlinearities. To address this limitation, our approach incorporates Quintic B-Splines, which offer higher-order polynomial interpolation for the spatial discretization, leading to smoother and more accurate approximations of the solution.[23]

The integration of Quintic B-Splines with the Finite Difference Method enhances the overall accuracy and stability of the numerical solution, especially in systems with strong nonlinearity and integral terms. By leveraging the flexibility and precision of Quintic B-Splines, we aim to improve the fidelity of the numerical solution and expand the applicability of numerical methods to a wider range of problems.[27]

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Integral terms in the VIDEs are handled using numerical quadrature techniques, ensuring accurate approximation of the integral contributions to the equations. The resulting system of integro-differential equations is transformed into a system of algebraic equations, which is solved using numerical methods to obtain the discrete solution at each grid point.

he books edited by Wazwaz [10] and Linz [11] contain some different methods to solve the system of nonlinear Volterra integro-differential equations analytically.

There are some methods to solve System of Nonlinear Volterra Integro-Differential Equations of Integer order of the second kind, such as, Sumudu decomposition method [12], Modified Homotopy-perturbation Method [13], Differential transformation [14]. Mohamed [15] used the Taylor series expansion method to solve fractional singular integro-differential equations with the Cauchy Kernel. Jani et al. [16] presented the solutions of fractional integro-differential equations with Bernstein polynomials. Many authors investigated the solution of second order fractional differential equations using various methods, including Legendre collocation method [17], Adomian Decomposition method [18], the spline collocation approach [19]. Rahman and Islam [20] to solve the Volterra Integral equations using La-guerre polynomials as a trial function. and in [21] A.Necib and A.Merad are use Laplace transformation and Homotopy perturbation method for solving the pseudohyperbolic integrodifferential problems with purely integral conditions. [22] M A Shallal, A H Taqi, B F Jumaa, H Rezazadeh, and M Inc are using a cubic Hermite finite element method to find the numerical solutions to the 1-D Burgers' equation. [23] M A Shallal, B F Jumaa are use Finite Difference Techniques for Two-Dimensional Advection-Diffusion Equation.

Any Volterra integro-differential equation is characterized by the existence of one or more of the derivatives $u'(x), u''(x)$, outside the integral sign. The Volterra integro-differential equations may be observed when we convert an initial value problem to an integral equation by using Leibnitz rule.[24], [25].

Overall, the integration of Finite Difference Method with Quintic B-Splines offers a promising approach for tackling challenging problems in VIDEs, paving the way for advancements in numerical techniques for solving integro-differential equations with integer order

Accordingly, this work studies the system of first, second order Volterra integro-differential equations of the second kind. The following form formulates the unknown functions that appear inside and derivative of unknown function outside the integral sign.[26]

$$Z_i^{(n)}(x) + \sum_{k=1}^{n-1} \mathcal{F}_{ik}(x) Z_i^{(k)}(x) + \mathcal{F}_{i0}(x) Z_i(x) = \mathcal{G}_i(x) + \mathcal{P}Z_i(x), \quad i = 1,2,\ldots,m \tag{1.1}$$

Where $\mathcal{P}Z_i(x) = \sum_{j=1}^{m} \int_0^x \mathcal{K}_{ij}(x,y)\, \mathcal{Q}(y, Z_j(y))\,dy \;, i = 1,2,\ldots,m\,,\; x \in [0,1]$

With the initial conditions: $Z_i^{(k)}(x_0) = Z_{i0}^{k}, \quad k = 0,1,\ldots,n-1,\; i = 1,2,\ldots,m$ (1.2)

where $Z_i(y)$ are unknown functions, $Z^{(k)}{}_i(x)$ are the derivative of unknown function, Where $\mathcal{Q}(y, Z_j(y))$ is a nonlinear function of $Z_j(y)$, the functions $\mathcal{G}_i(x), \mathcal{F}_{ik}(x)$ $i = 1,\ldots,m$ and kernels $\mathcal{K}_{ij}(x,y), 1 \le i,j \le m$ are given real-valued functions on subsets of $\mathcal{R}^3$ and $\mathcal{R}^1$, respectively.

2. QUINTIC B-SPLINE COLLOCATION METHOD

Let's consider a uniform partition of the problem domain denoted as $\Delta_m = [a = x_0, x_m = b]$ ith knots x_i, here $i = 0,\ldots,m$. The mesh spacing is denoted as $h = \frac{b-a}{m}$, and $x_i = a + ih$. In addition, there are supplementary knots $x_{-2}, x_{-1}, x_{m+1}, x_{m+2}$ located outside the problem domain. The quintic B-spline basis functions, denoted as $\mathcal{T}Q_i(x)$,are formulated at mesh points as follows

$$\mathcal{TQ}_i(x) = \frac{1}{h^5}\begin{cases} (x-x_{i-3})^5 & x \in [x_{i-3}, x_{i-2}] \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 & x \in [x_{i-2}, x_{i-1}] \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 + 15(x-x_{i-1})^5 & x \in [x_{i-1}, x_i] \\ (x_{i+3}-x)^5 - 6(x_{i+2}-x)^5 + 15(x_{i+1}-x)^5 & x \in [x_i, x_{i+1}] \\ (x_{i+3}-x)^5 - 6(x_{i+2}-x)^5 & x \in [x_{i+1}, x_{i+2}] \\ (x_{i+3}-x)^5 & x \in [x_{i+2}, x_{i+3}] \\ 0 & otherwise \end{cases} \tag{2.1}$$

The set of splines, $\{\mathcal{TQ}_{-2}, \mathcal{TQ}_{-1}, \mathcal{TQ}_0, \mathcal{TQ}_1, \dots, \mathcal{TQ}_m, \mathcal{TQ}_{m+1}, , \mathcal{TQ}_{m+2}\}$ forms a basis for the function defined over the domain Δ_m.The values of $\mathcal{TQ}_i$ and it's derivatives are in Table(1)

Table (1):V alues of the quintic B-spline basis functions and it's derivatives at different nodes

x	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
$\mathcal{TQ}_i(x)$	0	1	26	66	26	1	0
$\mathcal{TQ}'_i(x)$	0	5/h	50/h	0	-50/h	-5/h	0
$\mathcal{TQ}''_i(x)$	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0
$\mathcal{TQ}'''_i(x)$	0	$60/h^3$	$-120/h^3$	0	$120/h^3$	$-60/h^3$	0

3: APPROXIMATION OF FUNCTIONS BY QUINTIC B-SPLINE COLLOCATION

Sometimes when a function $Z(x)$ is to be approximated by a polynomial of the form

$$Z(x) = \sum_n^\infty \alpha_n x^n + \mathbb{E}_\mathcal{N}(x) \qquad |x| \le 1 \tag{3.1}$$

where $|\mathbb{E}_n(x)|$ does not exceed an allowed limit, it is possible to reduce the degree of the polynomial by a process called economization of power series. The procedure is to convert the polynomial to a linear combination of Quintic B-Spline Collocation:

$$\sum_{i=1}^\mathcal{N} \alpha_i x^i = \sum_{i=-2}^{m+2} \beta_i \mathcal{TQ}_i(x) \qquad i = 0,1,2,\dots \tag{3.2}$$

It may be possible to drop some of the last terms without permitting the error to exceed the prescribed limit. Since $|\mathcal{TQ}_i(x)| \le 1$, the number of terms which can be omitted is determined by the magnitude of the coefficient β_i [27].

4: FINITE DIFFERENCE APPROXIMATION

We assume the underlying function $u : R \to R$ is smooth. Let us define the following Forward finite difference operators:

$$D_+u(x) := \frac{u(x+h)-u(x)}{h} \tag{4.1}$$

Here, h is called the mesh size. By Taylor expansion, we can get

$u'(x) = D_+u(x) + O(h),$

We can also approximate $u'(x)$ by other finite difference operators with higher order errors.

For example, $u'(x) = D_3u(x) + O(h\,3),$

where $D_3u(x) = \frac{1}{6h}(2u(x+h) + 3u(x) - 6u(x-h) + u(x-2h)).$

In general, we can derive finite difference approximation for $u\,(k)$ at specific point x by the values of u at some nearby stencil points $x_j, j = 0,\dots,n$ with $n \ge k$.

That is, $u^{(k)}(x) = \sum_{j=0}^n c_j u(x_j) + O(h^{\,p-k+1})$

for some p as larger as possible. Here, the mesh size h denotes $max\{|x_i - x_j|\}$. As we shall see later that we can choose $p = n$. To find the coefficients c_j , $j = 0, \ldots, n$, we make Taylor expansion of $u(x_j)$ about the point x: $u(x_j) = \sum_{i=0}^{p} \frac{1}{i!} (x_j - x)^i u^{(i)}(x) + O(h^{p+1})$.
We plug this expansion formula of $u(x_j)$ into the finite difference approximation formula for $u^{(k)}(x)$:

$$u^{(k)}(x) = \sum_{j=0}^{n} c_j \sum_{i=0}^{p} \frac{1}{i!} (x_j - x)^i u^{(i)}(x) + O(h^{p-k+1}). \tag{4.2}$$

Comparing both sides, we obtain

$$\sum_{j=0}^{n} \frac{1}{i!} \frac{(x_j - x)^i}{h^i} c_j = \begin{cases} 1 & if\ i = k \\ 0 & otherwise \end{cases}, for\ i = 0, \ldots, p.$$

5: APPLICATION OF THE FINITE DIFFERENCE METHOD WITH QUINTIC B-SPLINES

In this section, we introduce an approximate method for solution of system of nonlinear Volterra integro-differential equations of integer order of the form (1.1).

$$Z_i^{(n)}(x) + \sum_{k=1}^{n-1} \mathcal{F}_{ik}(x) Z_i^{(k)}(x) + \mathcal{F}_{i0}(x) Z_i(x) = \mathcal{G}_i(x) + \mathcal{P}Z_i(x), \quad i = 1,2,\ldots,m, \tag{5.1}$$

where

$$\mathcal{P}Z_i(x) = \sum_{j=1}^{m} \int_0^x \mathcal{K}_{ij}(x,y)\, Q(y, Z_j(y))dy \ , i = 1,2,\ldots,m\ ,\ x \in [0,1]$$

For this suppose, first we approximate the derivative $Z_i^{(n)}(x)$ and $\sum_{k=1}^{n-1} Z_i^{(k)}(x)$ as Finite Difference equation (4.2)
And then we approximate $Z_i(x)$ as

$$Z_i(x) \approx \tilde{Z}_i(x) = \sum_{\ell=-2}^{m+1} \beta_{i\ell} \mathcal{T}Q_\ell(x) \qquad i = 1,2,\ldots,m \tag{5.2}$$

i.e. $\tilde{Z}_i(x) = \mathcal{B}_i{}^T \mathcal{T}Q(x)$ (5.3)

With

$$\mathcal{B}_i = [\beta_{i,-2}, \beta_{i,-1}, \ldots, \beta_{i,m+1}]^T \tag{5.4}$$

$$\mathcal{T}Q(x) = [\mathcal{T}Q_{-2}(x), \mathcal{T}Q_{-1}(x), \ldots, \mathcal{T}Q_{m+1}(x)] \tag{5.5}$$

where $\mathcal{T}Q_\ell(x)$ is Quintic B-Splines function of degree 5 and β_ℓ is unknown parameters, to be determined and n is number of piecewise polynomials. The approximate solution is not producing an identically zero function but a function is called residual function.
Then the absolute error of the formula can be defined as:

Absolute Error = $|Z_i(x) - \tilde{Z}_i(x)|$

The formulation for system of nonlinear Volterra integro-differential equation will be discussed by some numerical problems as below.

6. NUMERICAL EXAMPLES

we consider some nonlinear system of Volterra integro-differential equations with convolution kernel, as the exact solutions are available in the literature. For all the examples we used, solutions are obtained by the proposed method and thus compared with the exact solutions using Finite Difference Method with Quintic B-Splines.Thus,the convergences of each nonlinear system of Volterra integro-differential equations can be calculated by absolute error:

6.1 Example

Consider the following second kind system of two Nonlinear Volterra integro-differential equation of first order of convolution kernel:

$$\left.\begin{aligned}\Phi'(\xi) &= 1 + 3\xi^2 + \tfrac{1}{3}\xi^4 + \tfrac{3}{14}\xi^8 + \int_0^{\xi}(\xi - 2\tau)\left(\Phi^2(\tau) + \Psi^2(\tau)\right)d\tau \\ \Psi'(\xi) &= 1 - 3\xi^2 + \tfrac{8}{15}\xi^2 + \int_0^{\xi}(\xi - 2\tau)\left(\Phi^2(\tau) - \Psi^2(\tau)\right)d\tau\end{aligned}\right\}$$

where $\Phi(0) = 0, \Psi(0) = 0$

with the exact solutions are $\Phi(\xi) = \xi + \xi^3$, $\Psi(\xi) = \xi - \xi^3$

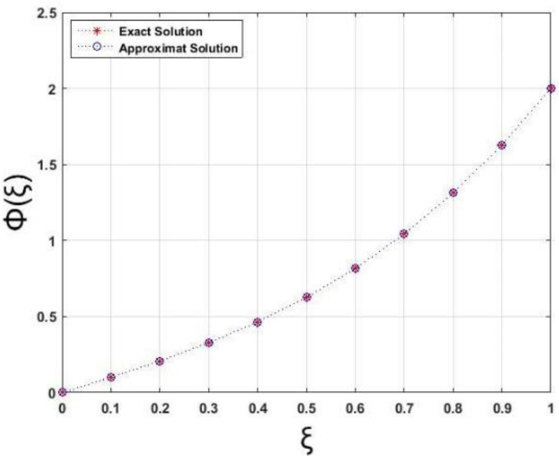
By using the proposed method, we solved this example. The numerical results are reported in Table (1), Table (2), Figure (1), and Figure (2).

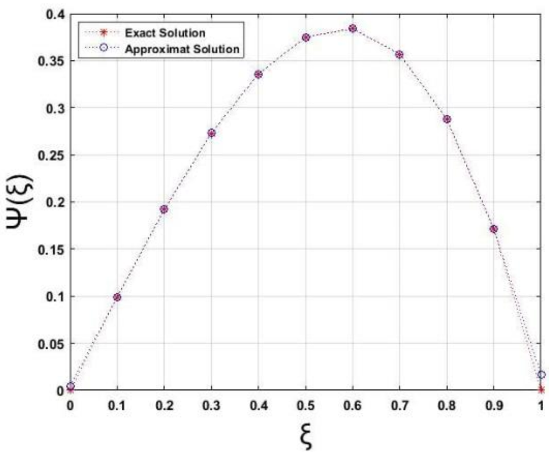
Table (1): show a comparison between the exact and approximate solutions using Finite Difference with Quintic B-Splines Method for $\Phi(x)$ for example1, depending on absolute error with h=0.1.

ξ	Exact Solution	Approx. Solution	Error
0.0	0.000000000000000	0.000087532467627	8.753246762700003e-05
0.1	0.101000000000000	0.101025891496443	2.589149644299793e-05
0.2	0.208000000000000	0.208031836451356	3.183645135601543e-05
0.3	0.327000000000000	0.327029940004515	2.994000451500556e-05
0.4	0.464000000000000	0.463957971585619	4.202841438100124e-05
0.5	0.625000000000000	0.625048426669897	4.842666989701705e-05
0.6	0.816000000000000	0.816009264318741	9.264318741042388e-06
0.7	1.043000000000000	1.043096810305853	9.681030585317885e-05
0.8	1.312000000000000	1.312079421415326	7.942141532590874e-05
0.9	1.629000000000000	1.628910789038737	8.921096126290884e-05
1.0	2.000000000000000	2.000099242761913	9.924276191286552e-05

Table (2): show a comparison between the exact and approximate solutions using Finite Difference with Quintic B-Splines Method for $\Psi(x)$ for example1, depending on absolute error with h=0.1.

ξ	Exact Solution	Approx. Solution	Error
0.0	0.000000000000000	0.004726635185887	2.663518588699992e-05
0.1	0.099000000000000	0.099025691728700	2.569172869999747e-05
0.2	0.192000000000000	0.191922287175521	7.771282447899264e-05
0.3	0.273000000000000	0.273093135561288	9.313556128798251e-05
0.4	0.336000000000000	0.335972593939962	2.740606003803769e-05
0.5	0.375000000000000	0.375096815820661	9.681582066101147e-05
0.6	0.384000000000000	0.383967752770107	3.224722989303430e-05
0.7	0.357000000000000	0.357004093250636	4.093250635994750e-06
0.8	0.288000000000000	0.287900868385442	9.913161455799013e-05
0.9	0.171000000000000	0.171093508881047	9.350888104697752e-05
1.0	0.000000000000000	0.016406241236227	6.241236226998612e-06





. **Figure (1**): show a comparison between the exact and approximate solution using Finite Difference with Quintic B-Splines Method for $\Phi(x)$ and $\Psi(x)$ for example1, depending on absolute error with h=0.1.

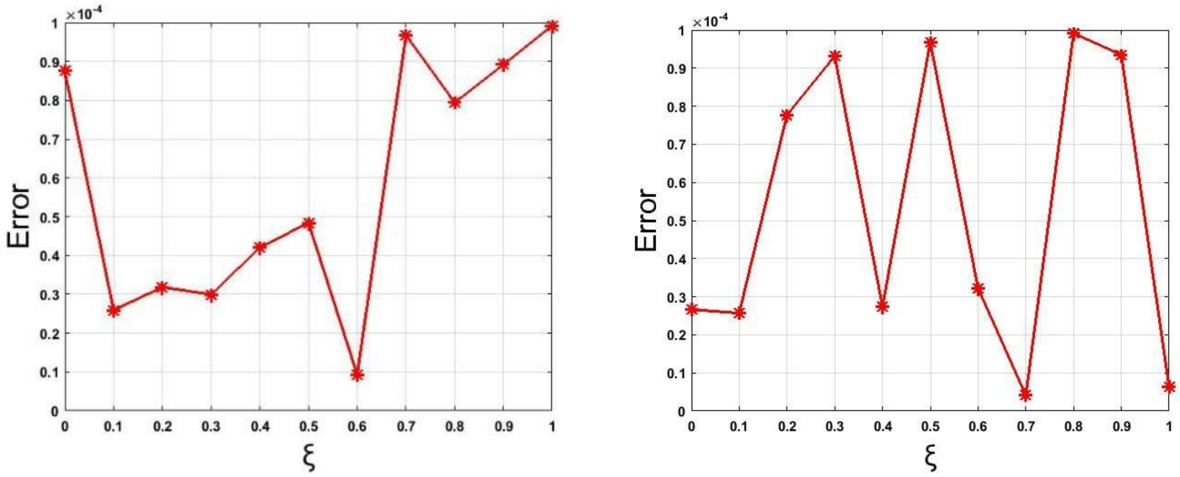


Figure (2): show an Error for the solution using Finite Difference with Quintic B-Splines Method for $\Phi(x)$ and $\Psi(x)$ for example1, depending on absolute error with h=0.1

6.2 Example

Consider the following second kind system of two Nonlinear Volterra integro-differential equation of second order of convolution kernel:

$$\left.\begin{aligned} \Phi''(\xi) + 2\Phi'(\xi) - \Phi(\xi) &= \frac{2}{3}\sin(\xi) + \frac{2}{3}sin(2\xi) + 2cos(\xi) - 5\xi + 2 + \int_0^{\xi} \cos(\xi - \tau)\left(\Phi^2(\tau) + \Psi^2(\tau)\right)d\tau \\ \Psi''(\xi) - \Psi'(\xi) + 3\Psi(\xi) &= -(2+\xi)\sin(\xi) + (1+\xi^2)\cos(\xi) + 3\xi - 1 + \int_0^{\xi} \sin(\xi - \tau)\left(\Phi^2(\tau) - \Psi^2(\tau)\right)d\tau \end{aligned}\right\}$$

Where $\Phi(0) = 0, \Phi'(0) = 2, \Psi(0) = 0, \Psi'(0) = 0$

The exact solutions are $\Phi(\xi) = \xi + \sin(\xi)$, $\Psi(\xi) = \xi - \sin(\xi)$

By using the proposed method, we solved this example. The numerical results are reported in Table (3), Table (4), Figure (3), and Figure (4).

Table (3): show a comparison between the exact and approximate solutions using Finite Difference with Quintic B-Splines Method for $\Phi(x)$ for example2, depending on absolute error with h=0.1.

ξ	Exact Solution	Approx. Solution	Error
0.0	0.000000000000000	0.000034561743718	3.456174371800017e-05
0.1	0.199833416646828	0.199757832987410	7.558365941798262e-05
0.2	0.398669330795061	0.398616556932597	5.277386246399507e-05
0.3	0.595520206661340	0.595516919646175	3.287015164943341e-06
0.4	0.789418342308650	0.789340459900048	7.788240860207019e-05
0.5	0.979425538604203	0.979487221740683	6.168313648002854e-05
0.6	1.164642473395035	1.164739222060555	9.674866552011885e-05
0.7	1.344217687237691	1.344118790412166	9.889682552488566e-05

0.8	1.517356090899523	1.517297633199841	5.845769968204628e-05
0.9	1.683326909627483	1.683244930128200	8.197949928301362e-05
1.0	1.841470984807897	1.841531714050749	6.072924285205339e-05

Table (4): show a comparison between the exact and approximate solutions using Finite Difference with Quintic B-Splines Method for $\Psi(x)$ for example2, depending on absolute error with h=0.1.

ξ	Exact Solution	Approx. Solution	Error
0.0	0.000000000000000	-0.000007461738257	7.461738257000000e-06
0.1	0.000166583353172	0.000239562969783	7.297961661100001e-05
0.2	0.001330669204939	0.001297189599968	3.347960497100005e-05
0.3	0.004479793338660	0.004478587481861	1.205856799000454e-06
0.4	0.010581657691350	0.010616573438534	3.491574718400038e-05
0.5	0.020574461395797	0.020634117935710	5.965653991299794e-05
0.6	0.035357526604965	0.035376989351838	1.946274687299882e-05
0.7	0.055782312762309	0.055785518251219	3.205488910001941e-06
0.8	0.082643909100477	0.082625707182610	1.820191786700010e-05
0.9	0.116673090372517	0.116759224486823	8.613411430599682e-05
1.0	0.158529015192104	0.158529237401358	2.222092539971854e-07

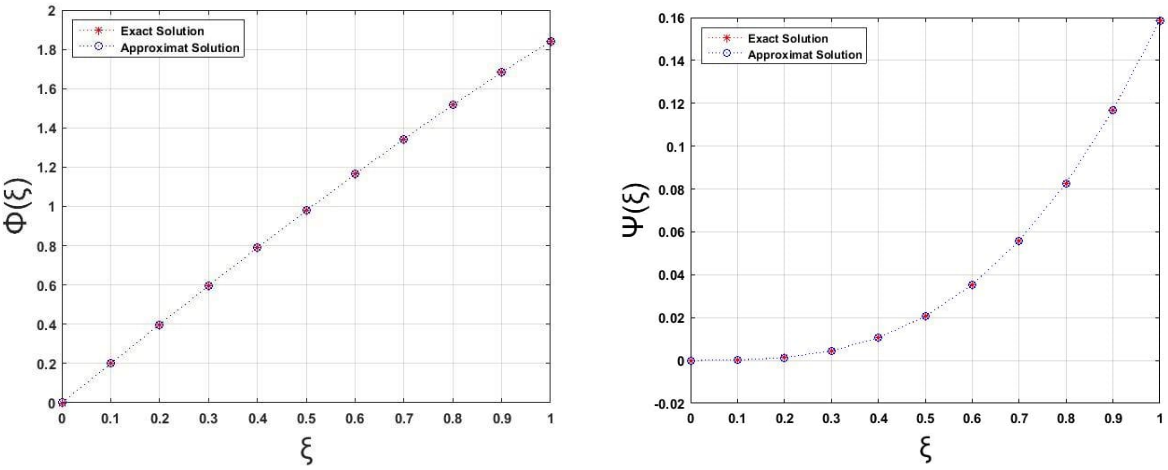


Figure (3): show a comparison between the exact and approximate solutions using Finite Difference with Quintic B-Splines Method for $\Phi(x)$ and $\Psi(x)$ for example2, depending on absolute error with h=0.1.

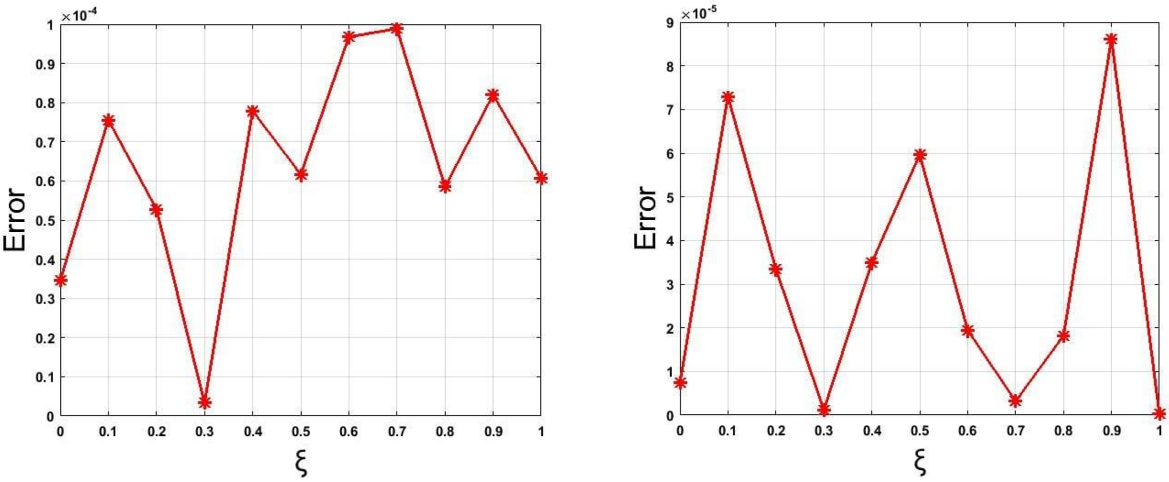


Figure (4): show an Error for the solution using Finite Difference with Quintic B-Splines Method for $\Phi(x)$ and $\Psi(x)$ for example2, depending on absolute error with h=0.1.

6.3 Example

Consider the following second kind system of three Nonlinear Volterra integro-differential equation of first order:

$$\left.\begin{aligned} \Phi''(\xi) + \xi\Phi'(\xi) + \Phi(\xi) &= (2-\xi)e^{\xi} + \int_0^{\xi} e^{(\xi-2\tau)}\left(\Phi^2(\tau) + \Psi(\tau)\right)d\tau \\ \Psi''(\xi) - 3\Psi'(\xi) &= -2\xi e^{\xi} - 2e^{2\xi} + \int_0^{\xi}(e^{(\xi-4\tau)}\,\Psi^2(\tau) + e^{(\xi-3\tau)}\Theta(\tau))d\tau \\ \Theta''(\xi) &= -2\xi e^{\xi} + 9e^{3\xi} + \int_0^{\xi}(e^{(\xi-6\tau)}\,\Theta^2(\tau) + e^{(\xi-\tau)}\Phi(\xi))d\tau \end{aligned}\right\}$$

where $\Phi(0) = 1, \Phi'(0) = 1, \Psi(0) = 1, \Psi'(0) = 2, \Theta(0) = 1, \Theta'(0) = 3$

with the exact solutions are $\Phi(\xi) = e^{\xi}$, $\Psi(\xi) = e^{2\xi}$, $\Theta(\xi) = e^{3\xi}$

By using the proposed method, we solved this example. The numerical results are reported in Table (5), Table (6), Table (7), Figure (5), and Figure (6).

Table (5): show a comparison between the exact and approximate solutions using Finite Difference with Quintic B-Splines Method $\Phi(x)$ for example3, depending on absolute error with h=0.1.

ξ	Exact Solution	Approx. Solution	Error
0.0	1.000000000000000	0.999991620097705	8.379902294963770e-06
0.1	1.105170918075648	1.105085783077343	8.513499830486992e-05
0.2	1.221402758160170	1.221419296641356	1.653848118587931e-05
0.3	1.349858807576003	1.349821220298748	3.758727725489131e-05
0.4	1.491824697641270	1.491845551443130	2.085380186001018e-05
0.5	1.648721270700128	1.648747799562429	2.652886230092300e-05
0.6	1.822118800390509	1.822188379372198	6.957898168891141e-05
0.7	2.013752707470477	2.013818235264831	6.552779435375911e-05
0.8	2.225540928492468	2.225517510529226	2.341796324190071e-05
0.9	2.459603111156950	2.459538504061257	6.460709569289236e-05
1.0	2.718281828459046	2.718325950551392	4.412209234594400e-05

Table (6): show a comparison between the exact and approximate solutions using Finite Difference with Quintic B-Splines Method for $\Psi(x)$ for example3, depending on absolute error with h=0.1.

ξ	Exact Solution	Approx. Solution	Error
0.0	1.000000000000000	1.000015001590479	1.500159047895799e-05
0.1	1.221402758160170	1.221492274923094	8.951676292401345e-05
0.2	1.491824697641270	1.491863322865272	3.862522400210899e-05
0.3	1.822118800390509	1.822149115575026	3.031518451690474e-05
0.4	2.225540928492468	2.225587805397848	4.687690538007772e-05
0.5	2.718281828459046	2.718349879803561	6.805134451504102e-05
0.6	3.320116922736547	3.320166006223929	4.908348738208446e-05
0.7	4.055199966844675	4.055262766305873	6.279946119747137e-05
0.8	4.953032424395115	4.953124405998664	9.198160354895180e-05
0.9	6.049647464412947	6.049708109190452	6.064477750467034e-05
1.0	7.389056098930650	7.389028140473412	2.795845723824186e-05

Table (7): show a comparison between the exact approximate solutions using Finite Difference with Quintic B-Splines Method for $\Theta(x)$ for example3, depending on absolute error with h=0.1.

ξ	Exact Solution	Approx. Solution	Error
0.0	1.000000000000000	0.999919486168310	8.051383169005444e-05
0.1	1.349858807576003	1.349880855753847	2.204817784412505e-05
0.2	1.822118800390509	1.822074058079877	4.474231063200662e-05
0.3	2.459603111156949	2.459529883477434	7.322767951478681e-05
0.4	3.320116922736548	3.320126190294342	9.267557794068182e-06
0.5	4.481689070338065	4.481726006862082	3.693652401626224e-05
0.6	6.049647464412945	6.049559337404816	8.812700812832475e-05
0.7	8.166169912567646	8.166159375384122	1.053718352395094e-05
0.8	11.02317638064160	11.02309644232725	7. 993831434838228e-05
0.9	14.87973172487283	14.87976579874256	3.407386972398285e-05
1.0	20.08553692318766	20.08551482134658	2.210184107909186e-05

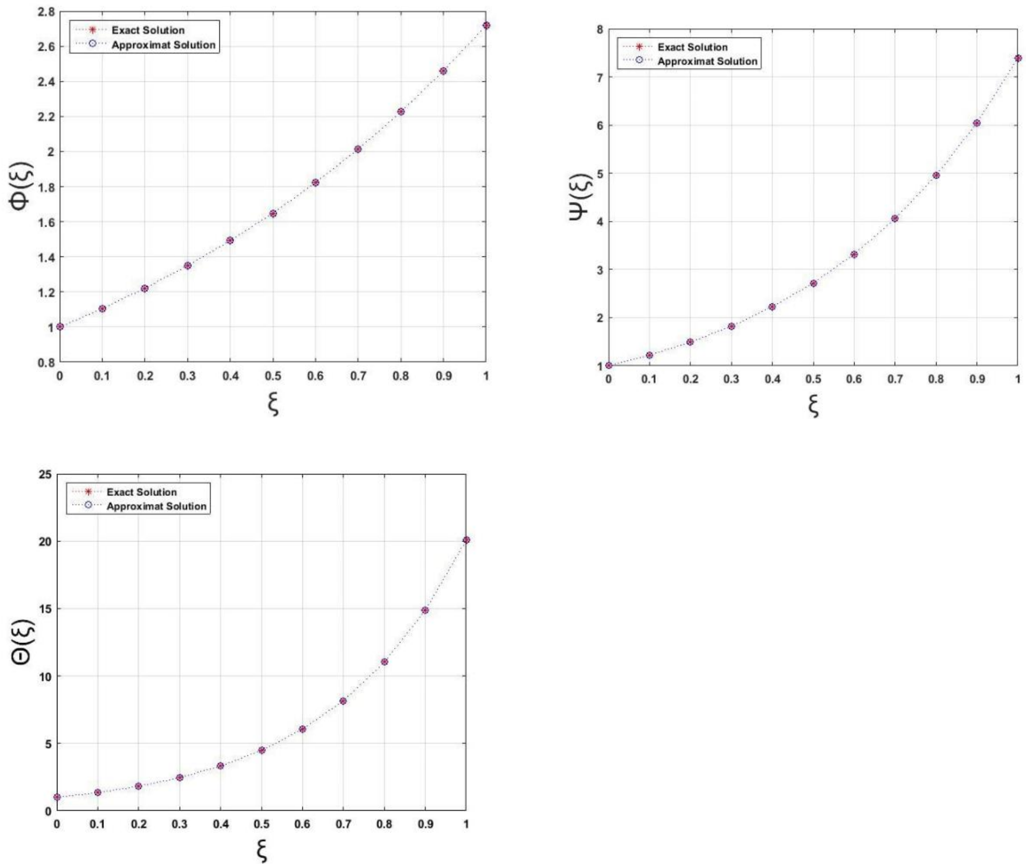


Figure (5): show a comparison between the exact and approximate solutions using Finite Difference with Quintic B-Splines Method for $\Phi(x)$, $\Psi(x)$ and $\Theta(x)$ for example3, depending on absolute error with h=0.1.

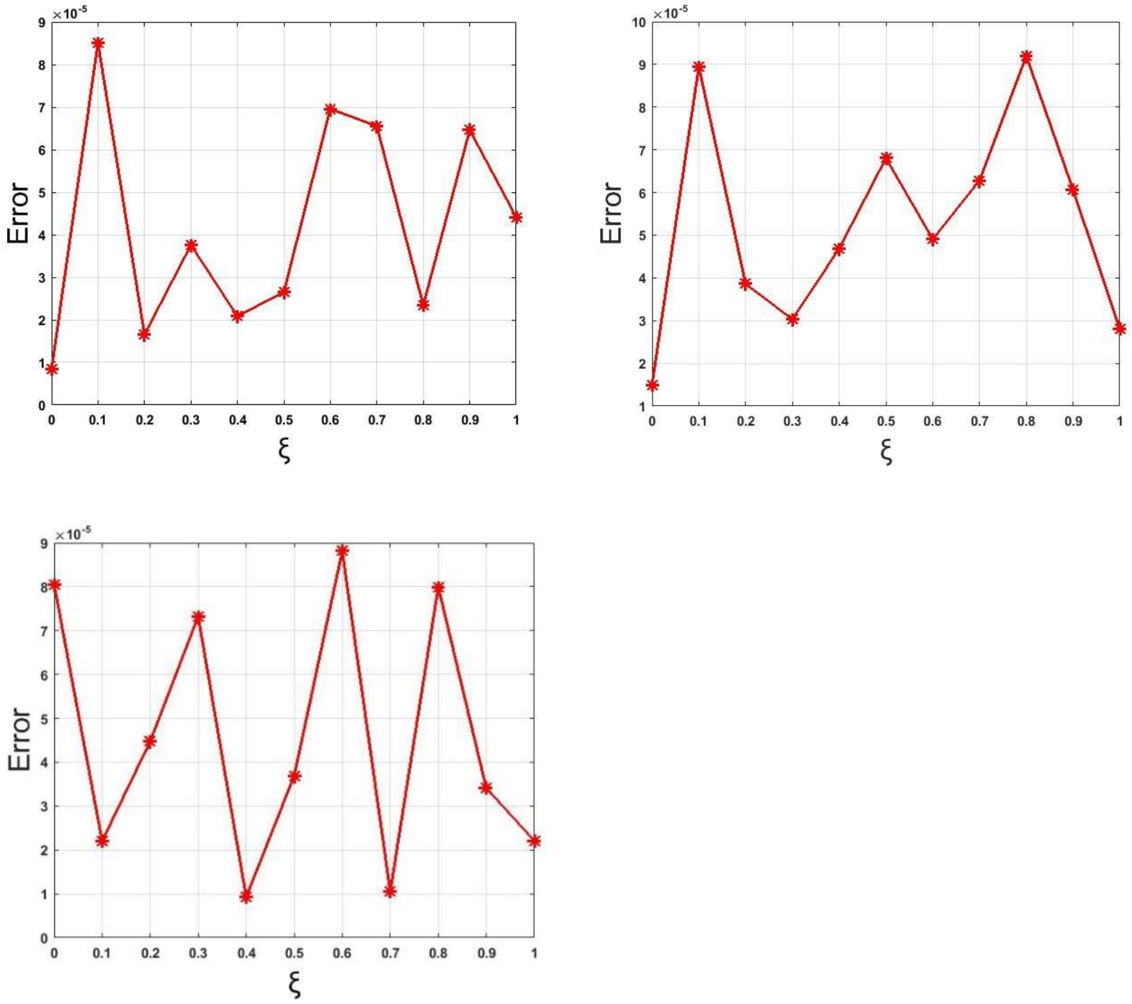


Figure (6): show an Error for the solution using Finite Difference with Quintic B-Splines Method for $\Phi(x)$, $\Psi(x)$ and $\Theta(x)$for example3, depending on absolute error with h=0.1

7. CONCLUSION

The application of the Finite Difference Method with Quintic B-Splines for solving systems of nonlinear Volterra Integro-Differential Equations (VIDEs) of integer order offers a promising avenue for obtaining accurate and stable numerical solutions to complex mathematical models. Through the systematic integration of finite difference methods and quintic B-spline interpolation techniques, this approach addresses the challenges associated with nonlinearities and integral terms inherent in VIDEs, providing a versatile and reliable tool for scientific and engineering applications. The combined approach of finite difference methods and quintic B-spline interpolation enhances the accuracy and stability of numerical solutions for VIDEs. Quintic B-splines offer higher-order polynomial interpolation, leading to smoother and more accurate representations of the solution compared to lower-order interpolations. This results in improved fidelity in capturing the dynamic behavior of the system.

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