Comparison Convergent Numerical Result for Fractal Caputo and Fractal Caputo Fabrizo

https://doi.org/10.31185/wjcm.VolX.IssX.XX

Shaima K. ALaiwi (✉)
College of Education for Pure Sciences, Wasit University, Iraq

Basim N. Abood
College of Education for Pure Sciences, Wasit University, Iraq

Basim.nasih@yahoo.com

Abstract:—The new numerical technicality operational matrix based on shifted Legendre polynomials to solve the linear and nonlinear meaning of the fractal fractional of Caputo and Caputo-Fabrizio. By using this method, we get perfect and precise results. Studying the comparative of the convergence of numerical solutions with the accuracy of the fractal fractional Caputo and the fractal fractional Caputo-Fabrizio .

Keywords— Fractal fractional Caputo derivative, Fractal fractional Caputo-Fabrizio operator, the shifted Legendre polynomials, Spectral method.

1. Introduction

Through replacement of the single kernel in the classic derivative of fractal fractional Caputo with the ordinary kernel. Exponential kernel has used by the fractal fractional Caputo-Fabrizio (FFCF) operator, which is a non-single kernel FFCF have suggested the modern operator. It does not only have two various exemplifications for locative and temporal variables, but the entire impact of the memory can be pertraged else [1]. In heat convey model this modern operator has been successfully utilized [2], Freedman and nonlinear Baggs model [3], the equation of space time fractal fractional propagation [4], mathematical paradigms for an unstable Maxwell fluid flux and its thermic demeanor in a micro-pipe [5], mass-spring-damper system [6], and fractal fractional Maxwell liquid [7]. In this research, to the trouble determined in the sense of the fractal fractional Caputo (FFC) and FFCF operators some existent analytical and numeral methods to solve fractal fractional calculus trouble have been expanded, amongst them is the paper of Morales Delgado et.al. In [8], to detect the essential solution for the fractal fractional advection propagation equation with the exporter the authors employ integral converts where the derivative is considered in FFCF sense. Nevertheless, since FFCF operator is comparatively modernistic, there are still comparatively bounded works conducted to gain the authoritative, precision and simple solving for the fractional calculus trouble determined in FFC and FFCF operators. Moreover, in solving several fractal fractional calculus troubles operational matrix (OPM) method relied on perpendicular function was successfully utilized which are acquainted in classic sense of FFC. The method minimizes these troubles to solve a system of algebraic equations, thence extremely simplify the trouble. In this research field the major contribution starts with the seminal paper concerning Legendre wavelets OPM through Yousefi and Razzaghi [9] and OPM relied on Legendre polynomials in [10]. To solve changeable arrangement fractal fractional differential equations this OPM process has been expanded as in [11]. Nevertheless, there are still no OPM related processes to solve the troubles determined in FFC and FFCF operators. Thereafter, by pursuing the work of Dehghan and Saadatmandi [10], we derive the OPM relied on SLP to solve troubles in FFC and FFCF operators.

It is the first time that the OPM is used for solving the problem in FFCF sense. The goal from this paper is to compare the accuracy, strength, and convergence of solutions between the FFC and FFCF.

The article is arranged as the following: section 2 clarify fractal fractional of Caputo and fractal fractional of Caputo-Fabrizio derivative. In section 3 Fractal Fractional operational matrix for Caputo and Caputo-Fabrizio.

In section 4 the Legendre OPM of fractal fractional derivative is gained. In section 5 clarify Procedure of the operational matrix of fractal fractional derivative. In section 6 the suggested method is utilized to many examples.

2. Basic concept

2.1 Fractal Fractional operator

Definition 2.1: The FFC left-sided ${}^{FFC}\mathrm{D}^{\omega,\beta}$ of a function $y(z) \in Y^1(0,b)$ with $0 < \omega < 1$, is acquainted as:

$${}^{FFC}\mathrm{D}^{\omega,\beta} y(z) = \frac{1}{\Gamma(1-\omega)} \int_0^z \frac{dy(\tau)}{d\tau^\beta} (z-\tau)^{-\omega} d\tau$$

Definition 2.2: The definition of Caputo of the fractal fractional-order derivative is acquainted as:

$${}^{FFC}\mathrm{D}^{\omega,\beta} y(z) = \frac{1}{\Gamma(1-\omega)} \int_0^z \frac{dy(\tau)}{d\tau^\beta} (z-\tau)^{-\omega} d\tau$$

Where $\omega > 0$ is the order of the derivative.

In [1], FFCF submitted the novel operator through substituting the single Kernel

$(z-\tau)^{-\omega}$ with $e^{\left(\frac{-\omega(z-\tau)}{(1-\omega)}\right)}$ and $\frac{1}{\Gamma(1-\omega)}$ with $\frac{M(\omega)}{1-\omega}$ in Eq (1) to acquire:

For $0 < \omega < 1$, $a \in [-\infty, z)$ and $y(z) \in Y^1(a,b), b > a$ the FFCF operator or more accurately the left-sided FFCF operator of y (z) is acquainted as:

$${}^{FFCF}\mathrm{D}^{\omega,\beta}_{a+;z} y(z) = \frac{M(\omega)}{1-\omega} \int_a^z \frac{dy(\tau)}{d\tau^\beta}\ e^{\frac{-\omega(z-\tau)}{(1-\omega)}} d\tau$$

Where the normalization function is $M(\omega)$ for example $M(0) = M(1) = 1$.

Here ω denoted as the fractal fractional order, β denotes the fractal order and the integral has power law kernel and,

$$\frac{dy(\tau)}{dz^\beta} = \lim_{z\to\tau} \frac{y(z)-y(\tau)}{z^\beta - \tau^\beta}$$

$$= \frac{1}{\beta\ \tau^{\beta-1}} \frac{d}{d\tau}\ y(\tau)$$

2.2 Fractal Fractional Derivative

For the Caputo derivative, we have [15]

$$\mathfrak{D}^{\omega} C = 0\,, \qquad (C \; is \; a \; constant),$$

$$\mathfrak{D}^{\omega} z^{\beta} = \begin{cases} 0 & for\ \beta \in \mathbb{N}_0 \ and\ \beta < \lceil \omega \rceil \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\omega)} z^{\beta-\omega}, & for \beta \in \mathbb{N}_0 \ and\ \beta \geq \lceil \omega \rceil \ or\ \beta \notin \mathbb{N}_0 \ and\ \beta > \lfloor \omega \rfloor \end{cases}$$

Theorem 1: Let $n < \omega < n + 1$, for a presented integer $\beta \geq \lceil \omega \rceil$, the FFCF operator of order $\omega = \lceil \omega \rceil$ is of $(\tau - a)\beta$ presented as [13]:

$$^{FFCF}\mathfrak{D}_{a+;\tau}^{\omega,\beta}(\tau-a)^{\beta} = \frac{M(\upsilon)\Gamma(\beta+1)}{1-\upsilon}\left[\left(\sum_{\delta=0}^{\beta-n-1}\frac{(-1)^{\delta}(\tau-a)^{\beta-n-1-\delta}}{\Gamma(\beta-n-\delta)(\frac{\upsilon}{1-\upsilon})^{\delta+1}}\right) + \frac{(-1)^{\beta-n}}{(\frac{\upsilon}{1-\upsilon})^{\beta-n}}e^{\frac{-\upsilon(\tau-a)}{1-\upsilon}}\right]$$

Proof: in [13]

2.3 Some Properties of the Shifted Legendre Polynomials

The notable Legendre polynomials are acquainted with this interval [-1, 1] and can be resolved with the guide of the accompanying repeat formulation [14]:

$$L_{\delta+1}(t) = \frac{2\delta+1}{\delta+1} t L_{\delta}(t) - \frac{\delta}{\delta+1} L_{\delta-1}(t), \quad \delta = 1,2,\ldots$$

Where $L_0(t) = 1$ and $L_1(t) = t$. For utilizing these polynomials on the interval $z \in [0,1]$ we limit which is named SLP through presenting the alteration of variable $= 2z - 1$.
Let the SLP $L_{\delta}(2z - 1)$ be indicated through $P_{\delta}(z)$. Then $P_{\delta}(z)$ can be acquired as the following:

$$P_{\delta+1}(z) = \frac{(2\delta+1)(2z-1)}{(\delta+1)} P_\delta(z) - \frac{\delta}{\delta+1} P_{\delta-1}(z), \quad \delta = 1,2,\dots$$

Where $P_0(z) = 1$ and $P_1(z) = 2z - 1$. The analytic form of the SLP $P_\delta(z)$of degree $\boldsymbol{\delta}$ is given by:

$$P_\delta(z) = \sum_{s=0}^{\delta} (-1)^{\delta+s} \frac{(\delta+s)!}{(\delta-s)!} \frac{z^s}{(s!)^2}$$

Notice that $P_\delta(0) = (-1)^\delta$ *and* $P_\delta(1) = 1$.
The orthogonality condition is

$$\int_0^1 P_\delta(z) P_\eta(z) dz = \begin{cases} \frac{1}{2\delta+1}, & \delta = \eta \\ 0 , & \delta \neq \eta \end{cases}$$

A function $g(z)$ square-integrable in [0, 1] may be expressed in terms of SLP as:

$$g(z) = \sum_{\eta=0}^{\infty} c_\eta P_\eta(z)$$

Where the coefficients c_η are presented through

$$c_\eta = (2\eta+1) \int_0^1 g(z) P_\eta(z) dz, \qquad \eta = 1,2,\dots$$

Practically speaking, only the first $(N+1)$-terms SLP are consider.
So we have

$$g(z) = \sum_{\eta=0}^{N} c_\eta P_\eta(z) = C^T \emptyset(z)$$

Where the shifted Legendre vector $\phi(z)$ and the shifted Legendre coefficient vector C are presented by

$$C^T = [c_0 , \dots , c_N]$$

$$\phi(z) = [P_0(z) , P_1(z), \dots , P_N(z)]^T$$

The derivative of the vector $\phi(z)$ can be expressed through

$$\frac{d\phi(z)}{dz} = \mathrm{D}^{(1)}\phi(z),$$

Where $\mathrm{D}^{(1)}$is the $(N + 1) \times (N + 1)$ OPM of derivative presented through

$$\mathrm{D}^{(1)} = (d_{\delta\eta}) = \begin{cases} 2(2\eta+1), & for\ \eta = \delta - s, \begin{cases} s = 1,3,\dots,N, & if\ N\ odd, \\ s = 1,3,\dots,N-1, & if\ N\ even, \end{cases} \\ 0 \quad , & otherwise \end{cases}$$

For instance, for even N we have

$$\mathrm{D}^{(1)} = 2\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \cdots & 2N-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 0 & 2N-1 & 0 \end{pmatrix}$$

3. Fractal Fractional OPM

$$\mathfrak{D}^{(\omega,\beta)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{\lceil\omega\rceil,0,s} & \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{\lceil\omega\rceil,1,s} & \cdots & \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{\lceil\omega\rceil,N,s} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{s=\lceil\omega\rceil}^{\delta} \theta_{\delta,0,s} & \sum_{s=\lceil\omega\rceil}^{\delta} \theta_{\delta,1,s} & \cdots & \sum_{s=\lceil\omega\rceil}^{\delta} \theta_{\delta,N,s} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{s=\lceil\omega\rceil}^{N} \theta_{N,0,s} & \sum_{s=\lceil\omega\rceil}^{N} \theta_{N,1,s} & \cdots & \sum_{s=\lceil\omega\rceil}^{N} \theta_{N,N,s} \end{pmatrix}$$

Where $\theta_{\delta,\eta,s}$ of the FFC is presented through

$$\theta_{\delta,\eta,s} = (2\eta+1)\sum_{i=0}^{\eta} \frac{(-1)^{\delta+\eta+s+i}(\delta+s)!\,(i+\eta)!}{(\delta-s)!\,s!\,\Gamma(s-\omega+1)(\eta-i)!\,(i)^2(s+i-\omega+1)}$$

Notice that in $\mathfrak{D}^{(\omega,\beta)}$, the premier $\lceil\omega\rceil$ rows, are all zero.

Where $\theta_{\delta,\eta,s}$ of the FFCF is presented through

$$\theta_{\delta,\eta,s} = \frac{(2\eta+1)M(\upsilon)}{(1-\upsilon)}\sum_{i=0}^{\eta} \frac{(-1)^{\delta+\eta+i}(\delta+s)!\,(\eta+i)!}{(\delta-s)!\,s!\,(\eta-i)!\,i!}\left[\frac{(-1)^{1-\lceil\omega\rceil}}{\gamma^{s-\lceil\omega\rceil+i+2}} + \sum_{r=0}^{i} \frac{(-1)^{\lceil\omega\rceil}e^{-\gamma}}{(i-r)!\,\gamma^{s-\lceil\omega\rceil+r+2}} + \sum_{r=0}^{s-\lceil\omega\rceil} \frac{(-1)^{s+r}}{(i)!\,\Gamma(s-\lceil\omega\rceil-r+1)(\gamma)^{r+1}(s-\lceil\omega\rceil-r+i+1)}\right]$$

Where

$$\gamma = \frac{\upsilon}{1-\upsilon}$$

$$\delta = \lceil\omega\rceil, \dots, N, \quad \eta = 0, 1, 2, \dots, N-1$$

4. OPM for fractal fractional order differential equation

By utilizing (12), can be written the higher derivative as follows [14]:

$$\frac{d^n\phi(z)}{dz^n} = (\mathrm{D}^{(1)})^n\phi(z)$$

Where $n \in N$ and the superscript symbol, in $\mathrm{D}^{(1)}$, indicate matrix powers. Thus

$$\mathrm{D}^{(n)} = (\mathrm{D}^{(1)})^n \;, \quad n = 1,2,\dots$$

5. Procedure of the OPM of fractal fractional derivative

In this section, we apply OPM of fractal fractional derivative for solving multi-order fractal fractional differentiation equation for showing the high significance of it. The continuous dependence and existence of the solving to this problem are debated in [16].

5.1 Linear multi-order fractal fractional differential equation

Consider the linear multi-order fractal fractional differentiation equation

$$\mathrm{D}^{\omega,\beta}g(z) = a_1\mathrm{D}^{\beta_1}g(z) + \cdots + a_s\mathrm{D}^{\beta_s}g(z) + a_{s+1}\, g(z) + a_{s+2}h(z),$$

With initial conditions

$$g^{(\delta)}(0) = d_\delta, \;\; \delta = 0, \dots, n,$$

where a_η , for $\eta = 1, \dots, s + 2$ are real constant coefficients and also $n < \omega \le n + 1$, $0 < \beta_1 < \beta_2 < \dots < \beta_s < \omega$, and $\mathrm{D}^{\omega,\beta}$ indicates the fractal fractional derivative of Caputo of order ω.

By solving problem (19) and (20) we approximate $g(z)$ and $h(z)$ by the SLP as:

$$g(z) \simeq \sum_{\delta=0}^{N} c_\delta\, P_\delta(z) = C^T\phi(z)$$

$$h(z) \simeq \sum_{\delta=0}^{N} h_\delta\, P_\delta(z) = H^T\phi(z)$$

Where vector $H = [h_0, \dots, h_N]^T$ is known but $C = [c_0, \dots, c_N]^{\mathrm{T}}$ is an unbeknown vector.

By utilizing Eq (21) we have

$$\mathrm{D}^{\omega,\beta} g(z) \simeq C^T \mathrm{D}^{\omega,\beta} \phi(z) \simeq C^T \mathrm{D}^{(\omega,\beta)} \phi(z),$$

$$\mathrm{D}^{\beta\eta} g(z) \simeq C^T \mathrm{D}^{\beta\eta} \phi(z) \simeq C^T \mathrm{D}^{(\beta\eta)} \phi(z), \quad \eta = 1,2, \dots, s.$$

By utilizing Eqs. (21)- (24) The residue $R_N(z)$ for Eq. (19) can be written as:

$$R_N(z) \simeq \left(C^T \mathrm{D}^{\omega,\beta} - C^T \sum_{\eta=1}^{s} a_\eta \mathrm{D}^{(\beta\eta)} - a_{s+1} C^T - a_{s+2} H^T \right) \phi(z)$$

As in a model tau method [17], we generate $N - n$ linear equations by applying

$$\langle R_N(z) P_\eta(z) \rangle = \int_0^1 R_N(z) P_\eta(z) dz = 0 \ , \eta = 0,1, \dots, N - n - 1 \, .$$

Also, through replacing Eqs. (18) and (21) in Eq. (20) we get

$$g(0) = C^T \phi(0) = d_0,$$

$$g^{(1)}(0) = C^T \mathrm{D}^{(1)} \phi(0) = d_1,$$

$$\vdots$$

$$g^{(n)}(0) = C^T \mathrm{D}^{(n)} \phi(0) = d_n,$$

Eqs. (26) and (27) generate $(N-n)$ and $(n+1)$ set of linear equations, respectively. These linear equations can be solved for unbeknown coefficients of the vector C. Subsequently, $g(z)$ which is presented in Eq. (21) can be calculated.

5.2 Nonlinear multi-order fractal fractional differential equation

Consider the nonlinear multi-order fractal fractional differential equation

$$\mathcal{D}^{\omega,\beta} g(z) = Y\left(z\,, g(z)\,, \mathcal{D}^{\beta_1} g(z)\,, \dots, \mathcal{D}^{\beta_s} g(z)\right)$$

With initial conditions

$$g^{(\delta)}(0) = d_\delta, \qquad \delta = 0, \dots, n,$$

where $n < \omega \leq n + 1$, $0 < \beta_1 < \beta_2 < \ldots < \beta_s < \omega$, and $\mathcal{D}^{\omega,\beta}$ indicates the fractal fractional derivative of Caputo of order ω . It ought to be noticed that Y *can* be nonlinear in generic.

To utilizing SLP for this problem, we firstly approximate $g(z), \mathcal{D}^{\omega,\beta} g(z)$ *and* $\mathcal{D}^{\beta\eta} g(z)$, for $\eta = 0, \ldots, s$ as Eqs. (21), (23) and (24) respectively. Through replacing these equations in Eq. (28) we obtain

$$C^T \mathcal{D}^{(\omega,\beta)} \phi(z) \simeq Y\left(z\,, C^T\phi(z)\,, C^T\mathcal{D}^{(\beta_1)}\phi(z)\,, \dots, C^T\mathcal{D}^{(\beta_s)}\phi(z)\right)$$

Also, through replacing Eqs. (18) and (21) in Eq. (29) we get

$$g(0) = C^T\phi(0) = d_0,$$

$$g^{(\lambda)}(0) = C^T\mathcal{D}^{(\delta)}\phi(0) = d_\delta, \qquad \delta = 1, 2, \ldots, n.$$

We firstly calculate Eq. (30) at $(N- n)$ points, to find the solution $g(z)$. we utilize the first $(N-n)$ roots of shifted Legendre of $P_{N+1}(z)$ for suitable collocation points.

Together these equations with Eq. (31) generate $(N + 1)$ nonlinear equations which disbanded by utilizing the iterative method of Newton. Subsequently, $g(z)$ presented in Eq. (21) can be calculated.

6. Numerical Examples

In part, of linear and nonlinear fractal fractional differential equations with the left-sided FFC and FFCF operators, some numerical examples are solved by utilizing the enforcement of the recently derived OPM for left-sided FFCF and FFC operators.

6.1 Examples of linear Fractal Fractional Differential Equations

Example 1. Consider the following linear fractal fractional differential equation

$$^{FFC}\mathrm{D}^{\omega,\beta} g(z) = z, \qquad g(0) = 0.$$

The exact solution $0.5206\, z^{1.94}$. By enforcing the technicality depicted in section (5.1) with $N = 9$,

$$C^T \mathrm{D}^{\omega,\beta} \Phi(z) - G^T\, \Phi(z) = 0$$

Table1: The Absolute errors for different value of ω = 0.95, for example (1).

Abs. Error of β		
z	0.98	0.99
0.1	6.33272497 e^{-4}	5.35556634 e^{-4}
0.2	1.20901104 e^{-3}	9.75384431 e^{-4}
0.3	1.08106160 e^{-3}	7.59135314 e^{-4}
0.4	5.096925597 e^{-4}	1.86648489e^{-4}
0.5	1.36126394 e^{-4}	6.45023111 e^{-5}
0.6	3.76981795 e^{-4}	3.14033053 e^{-4}
0.7	1.3287005 e^{-3}	8.48336444 e^{-4}
0.8	2.03742244 e^{-3}	9.62804444 e^{-4}
0.9	2.229664746 e^{-3}	4.39371269 e^{-4}

Table1: The Absolute errors for different value of ω = 0.95, for example (1).

Abs. Error of β		
z	0.98	0.99

0.1	4.57683093 e^{-4}	5.53517948 e^{-4}
0.2	1.44165637 e^{-3}	1.67287727 e^{-3}
0.3	2.77814482 e^{-3}	3.09572247 e^{-3}
0.4	4.36502560 e^{-3}	4.68337397 e^{-3}
0.5	6.14471036 e^{-3}	6.35067164 e^{-3}
0.6	8.07349162 e^{-3}	8.03146772 e^{-3}
0.7	1.01116439 e^{-2}	9.66841268 e^{-3}
0.8	1.22303121 e^{-2}	1.12181727 e^{-2}
0.9	1.44040145 e^{-2}	1.26456882 e^{-2}

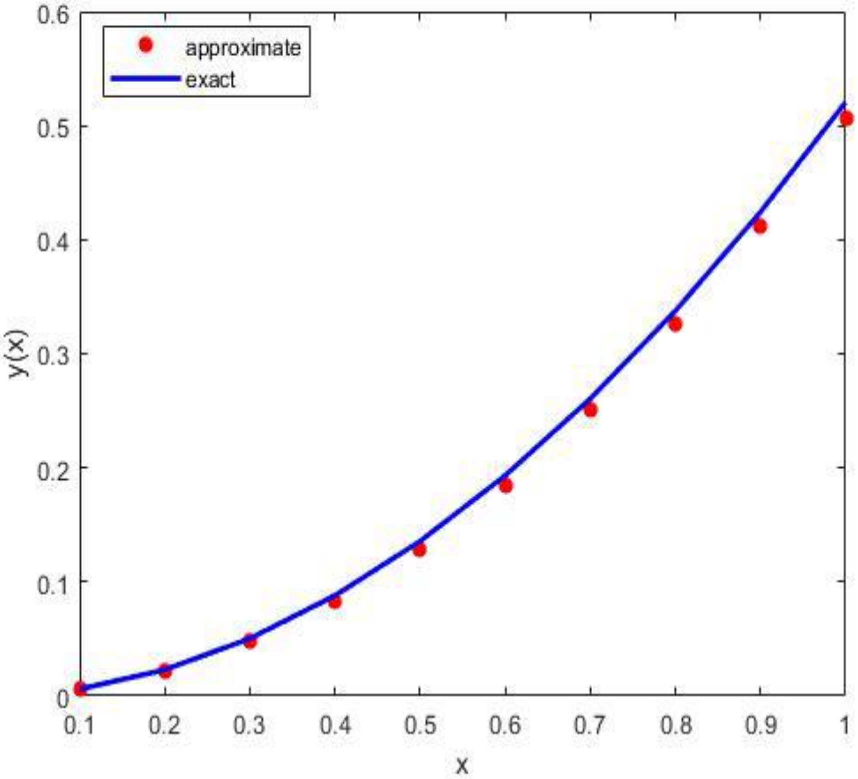

Figure 1: The exact solution and approximate solution of ω = 0.98, for example (1).

Example 2. Consider the following linear fractal fractional differential equation

$$^{FFCF}\mathrm{D}^{\omega,\beta} g(z) = z, \qquad g(0) = 0.$$

The exact solution $0.5206\, z^{1.94}$. By enforcing the technicality depicted in section (5.1) with $N = 9$,

$$C^T \mathrm{D}^{\omega,\beta} \Phi(z) - G^T\, \Phi(z) = 0$$

Table 3: The Absolute errors for different value of ω = 0.95, for example (2).

$\boldsymbol{Abs. Error\ of\ \beta}$		
z	0.98	0.99
0.1	4.07026907 e^{-3}	3.91809610 e^{-3}
0.2	6.59733458 e^{-3}	6.32871425 e^{-3}
0.3	8.02828359 e^{-3}	7.70952792 e^{-3}
0.4	8.56190922 e^{-3}	8.28724902 e^{-3}
0.5	8.33333735 e^{-3}	8.21791034 e^{-3}
0.6	7.43803558 e^{-3}	7.61683703 e^{-3}
0.7	5.94941171 e^{-3}	6.57482872 e^{-3}
0.8	3.93506356 e^{-3}	5.17010931 e^{-3}
0.9	1.45445011 e^{-3}	3.46855073 e^{-3}

Table4: The Absolute errors for different value of ω = 0.98, for example (2).

$\boldsymbol{Abs. Error\ of\ \beta}$		
Z	0.98	0.99
0.1	1.36709723 e^{-3}	1.25309054 e^{-3}
0.2	1.54662335 e^{-3}	1.30828539 e^{-3}
0.3	9.65775982 e^{-4}	6.56285183 e^{-4}
0.4	2.19266765 e^{-4}	5.135029698 e^{-4}

0.5	1.84555682 e^{-3}	2.01279799 e^{-3}
0.6	3.82538987 e^{-3}	3.73245085 e^{-3}
0.7	6.11202200 e^{-3}	5.60798464 e^{-3}
0.8	8.62478202 e^{-3}	7.54516309 e^{-3}
0.9	1.129634595 e^{-2}	9.46808480 e^{-3}

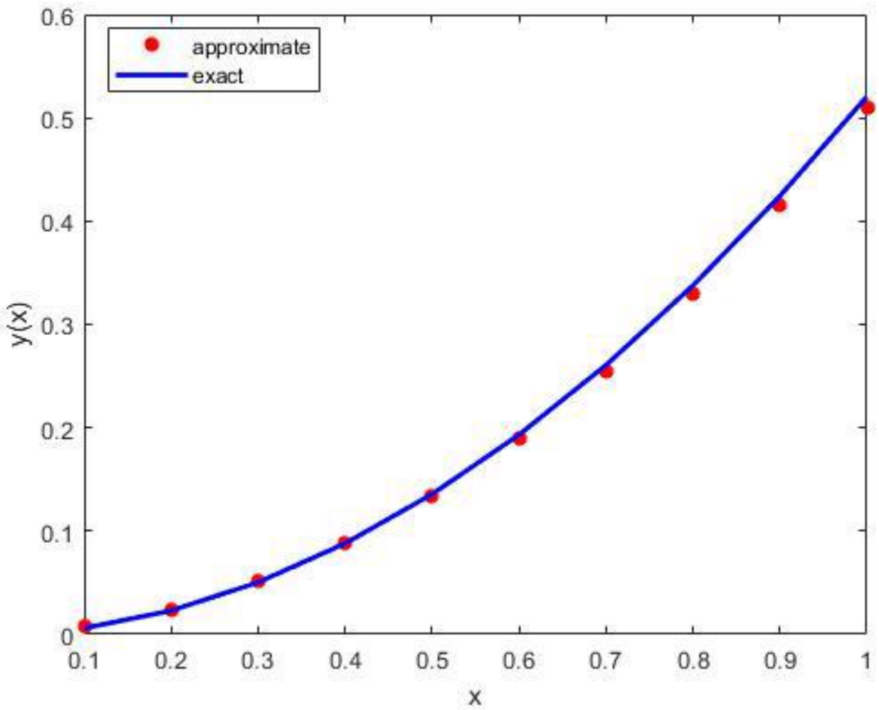

Figure 2: The exact solution and approximate solution of ω = 0.98, for example (2).

6.2 Examples of Nonlinear Fractal Fractional Differential Equations

Example 3. Consider the following nonlinear fractal fractional differential equation

$$\mathrm{D}^3 g(z) + {}^{FFC}\mathrm{D}^{\omega,\beta} g(z) + g^2(z) = z^4 \, , g(0) = \acute{g}(0) = 0 \, , \acute{\acute{g}}(0) = 2$$

$y(z) = z^2$ is the exact solution of this problem and N = 3.

We solved the above problem

$$C^T \mathrm{D}^3 \Phi(z) + C^T\, \mathrm{D}^{\omega,\beta} \Phi(z) + [C^T \Phi(z)]^2 - z^4 = 0$$

Table5: The Absolute errors for different value of ω = 2.5, for example (3).

Abs. Error of β			
z	2.95	2.98	2.99
0.1	4.39218151 e^{-10}	5.22071608 e^{-11}	6.94919216 e^{-11}
0.2	0.2351374537 e^{-9}	4.17657252 e^{-10}	5.55935438 e^{-10}
0.3	0.3118588907 e^{-8}	1.40959319 e^{-9}	1.87628213 e^{-9}
0.4	0.4281099631 e^{-8}	3.34125789 e^{-9}	4.44748358 e^{-9}
0.5	0.5549022716 e^{-8}	6.52589428 e^{-9}	8.68649139 e^{-9}
0.6	0.6948711254 e^{-8}	1.12767453 e^{-8}	1.50102571 e^{-8}
0.7	0.7150651833 e^{-7}	1.79070538 e^{-8}	2.38357324 e^{-8}
0.8	0.8224879705 e^{-7}	2.67300628 e^{-8}	3.55798688 e^{-8}
0.9	0.9320190048 e^{-7}	3.80590152 e^{-8}	5.06596179 e^{-8}

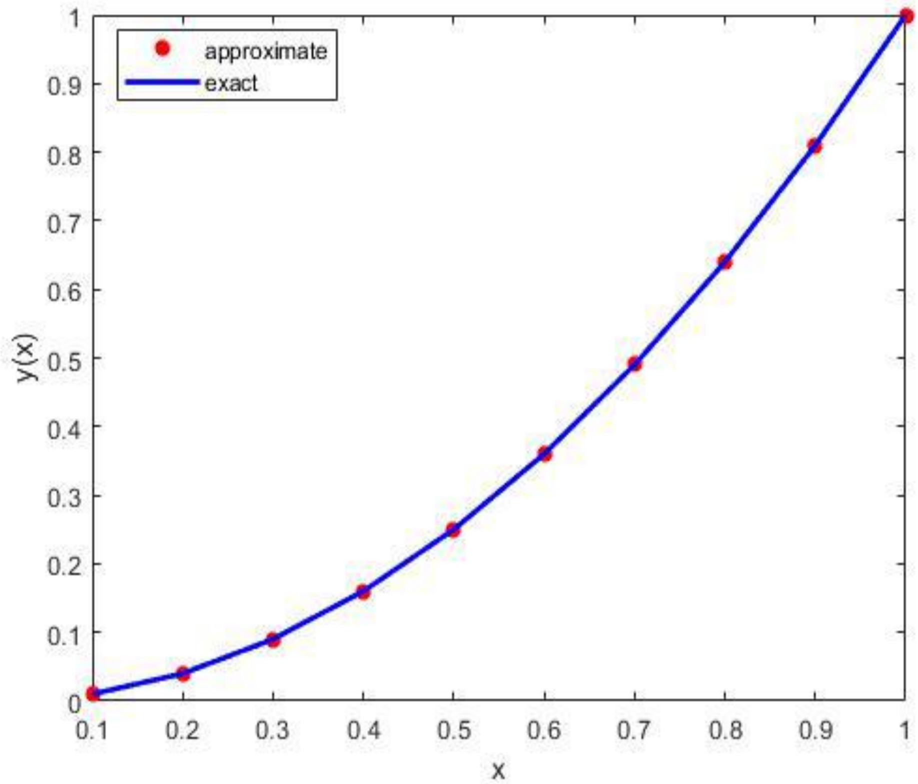

Figure 3: The exact solution and approximate solution of ω = 2.5, for example (3).

Example 4. Consider the following nonlinear fractal fractional differential equation

$$\mathrm{D}^3 g(z) + {}^{FFCF}\mathrm{D}^{\omega,\beta} g(z) + g^2(z) = z^4 , g(0) = \acute{g}(0) = 0 , \acute{\acute{g}}(0) = 2$$

$y(z) = z^2$ is the exact solution of this problem and N= 3.

We solved the above problem

$$C^T \mathrm{D}^3 \Phi(z) + C^T \, \mathrm{D}^{\omega,\beta} \Phi(z) + [C^T \Phi(z)]^2 - z^4 = 0$$

Table6: The Absolute errors for different value of ω= 2.5, for example (4).

$Abs. Error\ of\ \beta$			
z	2.95	2.98	2.99
0.1	4.27066859 e^{-10}	5.07711699 e^{-11}	6.75841409 e^{-11}
0.2	0.2341653491 e^{-9}	4.06169291 e^{-10}	5.40673194 e^{-10}
0.3	0.3115308053 e^{-8}	1.37082135 e^{-9}	1.82477205 e^{-9}
0.4	0.4273322793 e^{-8}	3.24935432 e^{-9}	4.32538563 e^{-9}
0.5	0.55338335797 e^{-8}	6.34639516 e^{-9}	8.44801881 e^{-9}
0.6	0.6922464426 e^{-8}	1.09665708 e^{-8}	1.45981765 e^{-8}
0.7	0.7146483934 e^{-7}	1.74145083 e^{-8}	2.31813636 e^{-8}
0.8	0.8218658234 e^{-7}	2.59948346 e^{-8}	3.46030851 e^{-8}
0.9	0.9311331743 e^{-7}	3.70121766 e^{-8}	4.92688458 e^{-8}

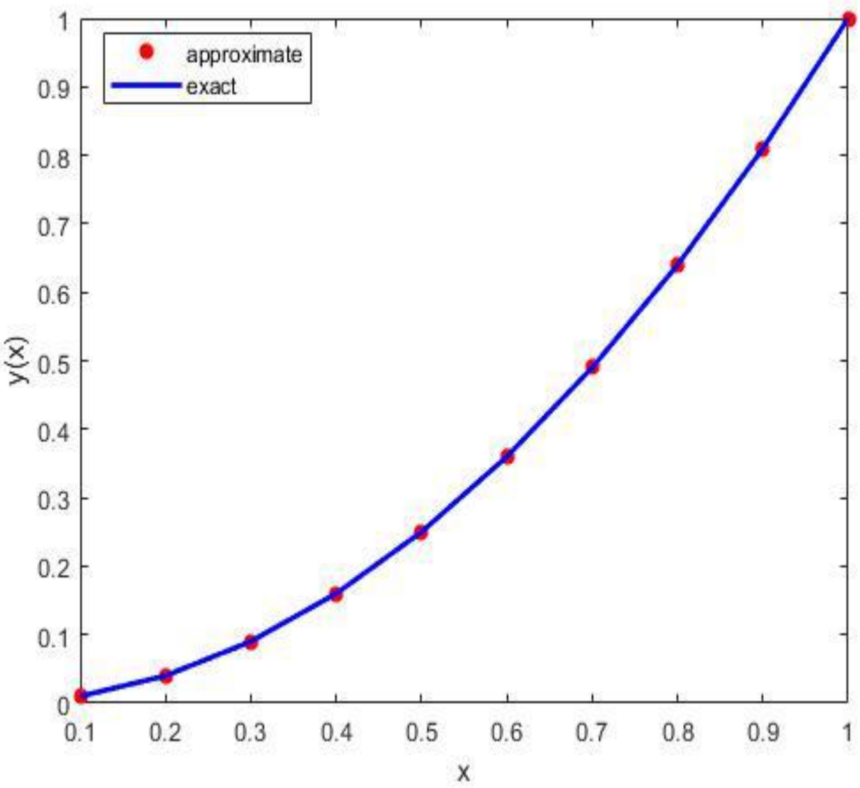

Figure 4: The exact solution and approximate solution of $\omega = 2.5$, for example (4).

7.Conclusion

To solve linear and non-linear of the fractal fractional Caputo and Caputo-Farbizo by utilizing operational matrix. Some numerical examples appear that the method is soft to employ and giving aloft fineness. The numerical results of fractal Caputo-Fabizo is more accuracy and convergence when compare with fractal Caputo. The new operational matrix of the operator of fractal fractional Caputo-Fabrizio inherits the gorgeous advantage from the well-known operational matrix of the fractional derivative of fractal fractional Caputo.

The method reduces the problem in the fractal fractional of Caputo and fractal fractional Caputo-Fabrizio for solving a system of algebraic equations, hence greatly simplifying the problem.

8.References

[1] M.Caputo,M.Fabrizio,A new definition of fractional derivative without singular kernel,Progr.Fract.Differ.Appl 1 (2) (2015) 1-13 .

[2] M. Abdulhameed, D. Vieru, R. Roslan, Modeling electro-magnetohydrodynamic thermo-fluidic transport of biofluids with new trend of fractional derivative without singular kernel,Physica A Stat.Mech.Appl.484(2017)233-252.

[3] A. Atangana, I. Koca, on the new fractional derivative and application to nonlinear Baggs and Freedman model, J. Nonlinear Sci. Appl. 9(2016)2467-2480.

[4] J.Gómez – Aguilar, Space-time fractional diffusion equation using a derivative with nonsingular and regular kernel, Physica. A Stat. Mech. Appl. 465(2017)562-572.

[5] M. Abdulhameed, D. Vieru, R. Roslan, Magnetohydrodynamic electroosmotic flow of Maxwell fluids with Caputo-Fabrizio derivatives through circular tubes, Comput. Math. Appl. (2017) DOI: 10.1016/j.camwa.2017.07.040.

[6] N. Al-Salti, E.karimov, K.sadarangani, on a differential equation with Caputo-Fabrizio fractional derivative of order $1 < \beta \leq 2$ and application to mass-springdamper system, Progr. Fract. Differ. Appl. 2(4) (2015)257-263.

[7] G. Feng, Y. Xiao-Jun, Fractional Maxwell fluid with fractional derivative without singular kernel, Thermal Science 20(2016)871-876.

[8] I. A. Mirza, D. Vieru, Fundamental solution to advection-diffusion equation with time-fractional Caputo-Fabrizio derivative, Comput. Math. Appl. 73(1) (2017)1-10.

[9] M. Razzaghi, S. Yousefi, the Legendre wavelets operational matrix of integration, Int. J. Syst. Sci. 32(4) (2001)495-502.

[10] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractionalorder differential equations, Comput. Math. Appl. 59(3) (2010)1326-1336.

[11] A. Bhrawy, M. Zaky, Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation, Nonlinear Dynam. 80(1-2) (2015)101-116.

[12] Saadatmandi, Abbas, and Mehdi Dehghan." A new operational matrix for solving fractional-order differential equations". Computers and mathematics with applications. 59.3(2010):1326-1336.

[13] Loh, Jian Rong, et al." On the new properties of Caputo-Fabrizio operator and its application in deriving shifted Legendre operational matrix." Applied Numerical Mathematics 132(2018):138-153.

[14] Deng J. and Ma L.,"Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations", Appl. Math. Lett. 23(2010)676-680.

[15] Caputo M.,"Linear models of dissipation whose Q is almost frequency independent part II", J. Roy. Astral. Soc., 13:529-539.1967.

[16] Bhrawy, A.H., Alofi, A.S. and Ezz-Eldien, S.S.," A quadrature tau method for variable coefficients fractional differential equations", Appl. Math Lett. 24, 2146-2152(2011). doi: 10.1016/j.aml.2011.06.016.

[17] C.Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamic, Prentice-Hall, Englewood Cliffs, NJ, 1988.

[18] Atangana,Abdon,and Ali Akgul."On solutions of fractal fractional differential equations."DiscreteContinuous Dynamical System-S14.10 (2021):3441.

Article submitted 16 October 2021. Published as resubmitted by the authors 29 November 2021.