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Bernstein Operational Matrix for Solving Variable Order Fractional Differential Equations with Generalized Caputo-Type

Alaa Abbas Taha[1]*, Basim Nasih Abood[2]

[1,2]Department of Mathematics, College of Education for Pure Science, Wasit University, IRAQ

*Corresponding Author: Alaa Abbas Taha

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ABSTRACT: In this research, we present novel generalized derivative founded on the newly constructed novel generalized Caputo Fractional variable order derivatives (NGCFVODs). Utilizing these operators, a numerical methodology has been formulated to resolve the Fractional Variable Order Differential Equations (FVODEs). We estimate the solution for FVODEs by employing Bernstein polynomials as foundational vectors. We have further expanded the derivative operational matrix of Bernstein Polynomials (bPs) to generalized derivative an operational matrix in sense of NGCFVODs. The efficiency of developed numerical methodology is tested by a taking a various test example. We also a compare results of our suggested approach with the methodologies existing in academic papers. In this study the Fractional variable order differential operator of new generalized Caputo was described by three categories:
(i) various value in ρ and Fractional variable order a parameter, (ii) various value in a fractional parameters while Fractional variable order and ρ parameter is fixed, and (iii) various value in Fractional variable order parameter controlling fractional and ρ parameter. Numerical examples are given to demonstrate the theoretical analysis and verify the efficiency of the proposed method.

Keywords: Fractional-order differential equation, Generalized Caputo derivative, Operational matrix, Bernstein polynomials.

1. INTRODUCTION

The topic fractional calculus is afield involves studying of differential for an arbitrary order. Originally, Fractional Calculus (FC) was regarded solely as a theoretical mathematical concept with virtually no practical applications. The subject of fractional calculus is extremely challenging and has many illdefined concepts to deal with. In any case, the requirement for new methods of solving differential equations when fractional order is necessary in physical and engineering applications [1]. However, over the subsequent decades, there's been substantial progress in both the abstract and applied segments of mathematics regarding FC [2]. Various fractional operator is documented in academic literatures, but of these, significant use has been observed of the Caputo Differential Operator (CDO) which is extensively deployed to address Initial Value Problem of a fractional orders [3]. The application of mathematical modeling, specifically using Caputo Differential Equations, to an array of real-world phenomena has been thoroughly explored. For instance, researchers in Reference [4] utilized Convection Fractional Differential Equations (CFDEs) as a means to examine the dynamics of Casson fluid flow. In [5], The scholars meticulously examined the functioning of current within electrical circuits by employing CFDEs. The prevalent fractional integrated-operator is primarily influenced depending on how important the parameters ρ and α are in a fractional computation. The generalized-Caputo type fractional derivative [6] shares equivalent properties with the Caputo derivative. Additionally, it furnishes a practical technique for the creation and administration of mathematical models using fractional measurements. The most recent iteration of generalize Caputo fractional derivatives boasts a unique attribute, absent from other variants of fractional derivatives. These include versions like the traditional Caputo, Atanganas Baleanu and Caputo Fabrezio fractional derivative [7,18]. Except for fractional order parameters, the ρ parameters is also quite effective at creating graphs from the acquired data. The settings of this specific parameter can be changed to produce a wide variety of graphs. The majority of Functional Differential

*Corresponding author: alaa.abbas@uowasit.edu.iq

Equations (FDEs) present significant analytical challenges. Consequently, there is an urgent necessity for numerical resolution methods. There are several numerical methods for solving functional differential equations in the literature, but the operational matrices method, when combined with the collocation method and the Tau method, is one that is frequently used to solve both partial and ordinary functional differential equations [8]. The new numerical technicality operational matrix based on shifted Legendre polynomials to solve the linear and nonlinear fractional differential equations[9].This method's foundation is the transforming functional differential equations into an algebraic system of equations that computer softwares can readily solve. Last but not least, this technique estimates, the solution is represented by the orthogonal polynomial basis vectors. Atangana in [10] established three new derivatives of a fractional parameter: the exponential, Mittag-Leffler, and power laws. many mathematicians developed this a new field and used this a novel derivative to a various fractional issues including the real-life problems and system of the FDEs [11], physiology, diffusion equations, economy, physics, electrochemical processes, physiology and fluids [12,19]. Motivated by above-mentioned studies, we present new operators for integration and differentiation, a novel generalized Caputo derivative for a fractional variable order, and we create a numerical method using a recently discovered derivative, the operation matrix of the Bernstein polynomial, in the context of the novel generalized Caputo derivatives for a solving fractional functional differential equation (FFDEs). Furthermore, we assess the practicability of our advanced numerical approach by juxtaposing the results derived from this proposed technique with those found in scholarly literatures. Our analysis revealed that the numerical results derived from our established method align proficiently with those obtained through alternative techniques.

2. PRELIMINARIES

Let's elucidate the generalized Caputo fractional derivative and provide illustrative examples. Our subsequent endeavor involves examining its properties.

Definition 2.1 [13]. Let $f:\mathbb{R}\to\mathbb{R}$ be a function in a state of continuity,the Caputo fractional order derivatives of order α of function $f(x)$ is provided by:

$$\mathbf{cD_X^{\alpha}f(x)}=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^x\frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}}\mathbf{d(\tau)}, & \mathbf{m-1<\alpha\le m}\\ \frac{d^m}{dx^m}\mathbf{f(x)}, & \boldsymbol{\alpha=m}\end{cases}$$

Definition 2.2 [14]. Novel generalized Caputo-fractional derivative We've got:

$cD^{\gamma,\rho}C=0$, C is a constant.

Furthermore, if $m-1<\gamma<m, k>m-1$ and $k\notin\mathbb{N}$

$$cD^{\gamma,\rho}(t^{\rho}-a^{\rho})^{k}=\begin{cases}\rho^{\gamma}\frac{\Gamma(k+1)}{\Gamma(k-\gamma+1)}(t^{\rho}-a^{\rho})^{k-\gamma}, & k\in\mathbb{N}_0 \text{ and } k\ge\lceil\gamma\rceil \text{ or } k\in\mathbb{N} \text{ and } >\lfloor\gamma\rfloor\\ 0\ ,k\in\mathbb{N}_0 \text{ and } k<\lceil\gamma\rceil\end{cases}$$

Theorem 2.3. The Caputo derivative is the linear operators, i.e. for any $a,b\in R$

$CD^{\alpha}\big(af(t)+bg(t)\big)=aCD^{\alpha}f(t)+b\,CD^{\alpha}g(t)$.

Proof: Let $CD^{\alpha}f(t)$, $CD^{\alpha}g(t)$ be the Caputo derivative of the function f and g, respectively. Then

$$CD^{\alpha}\big(af(t)+bg(t)\big)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{(af(t)+bg(t))^{(m)}}{(t-\tau)^{\alpha-m+1}}d\tau,$$

$$=\frac{1}{\Gamma(m-\alpha)}\left(a\int_0^t\frac{f^m(t)}{(t-\tau)^{\alpha-m+1}}d\tau+b\int_0^t\frac{g^m(t)}{(t-\tau)^{\alpha-m+1}}d\tau\right),$$

$$=\frac{1}{\Gamma(m-\alpha)}a\int_0^t\frac{f^m(t)}{(t-\tau)^{\alpha-m+1}}d\tau+\frac{1}{\Gamma(m-\alpha)}b\int_0^t\frac{g^m(t)}{(t-\tau)^{\alpha-m+1}}d\tau.$$

$$=aCD^{\alpha}f(t)+b\,CD^{\alpha}g(t).$$

Theorem 2.4. The Caputo fractional derivative it holds

$$D^{\alpha}C=0,\quad C\text{ is a constant.}$$

Proof: As common $0<m-1<\alpha<m, m\in\mathbb{N}$,This implies $m\ge1$. Implementing the Concept of Caputo's Fractional Derivative (2.1) and meanwhile the n-th derivative $c^{(m)}$ $(m\in\mathbb{N}, m\ge1)$ of a constant equal 0, it follows

$$D^{\alpha}c=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d(\tau)=0.$$

2.5. Bernstein polynomials

In the mathematical field for numerical analysis ‘a Bernstein polynomial’ named after Sergei Natanovich Bernstein, is a polynomial in the Bernstein form, that is a linear combination of Bernstein basis polynomials. Polynomials in Bernstein form was first used by Bernstein in a constructive proof for the Stone-Weierstrass approximation theorem.

Definition 2.6. Then $n+1$ Bernstein basis polynomials of degree n are defined as:

$$B_{i,n}(x)=\binom{n}{i}x^i(1-x)^{n-i},\ i=0,1,2,\dots,n$$

Where $\binom{n}{i}$ is a binomial coefficient.The Bernstein polynomials of degree n form a basis at the vector space Π_n of polynomials of degree at most n.A linear combination of Bernstein basis polynomials:

$$B_n(x) = \sum_{i=0}^{n} c_i B_{i,n}(x)$$

Is called a Bernstein polynomial or polynomial in Bernstein form of degree n.The coefficients c_i are called Bernstein coefficients.

3. PROPERTIES OF BERNSTEIN POLYNOMIALS

The Bernstein basis polynomials have the following properties:

1) $B_{i,n}(x) = 0$, if $i < 0$ or $i > n$.

2) $B_{i,n}(0) = \delta_{i,0}$ and $B_{i,n}(1) = \delta_{i,n}$, where δ is the Kronecker delta function given by:

$$\delta_{i,n} = \begin{cases} 1, & \text{if } i = n \\ 0, & \text{if } i \neq n \end{cases}$$

3) $B_{i,n}$ has a root with multiplicity i at a point $x = 0$.

4) $B_{i,n}$ has a root with multiplicity (n-i) at a point $x = 1$.

5) $B_{i,n} \geq 0$ for $x \in [0,1]$.

6) $B_{i,n}(1 - x) = B_{n-i,n}(x)$.

7) The derivative can be written as a combination of two polynomials of lower degree:

$$B'_{i,n}(x) = n(B_{i-1,n-1}(x) - B_{i,n-1}(x))$$

8) The integral is constant for a given n

$$\int_0^1 B_{i,n}(x)dx = \frac{1}{n+1}, \forall i = 0,1,\dots,n$$

9) If $n \neq 0$ then $B_{i,n}(x)$ has a unique local maximum of the interval [0,1] at $x = \frac{i}{n}$ This maximum takes the value:

$$i^i n^{-n}(n-i)\binom{n}{i}.$$

10) The Bernstein basis polynomials of degree n form a partition of unity:

$$\sum_{i=0}^{n} B_{i,n}(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} = (x + (1-x))^n = 1.$$

11) By taking the first derivative of $(x + y)^n$, where$y = 1 - x$, it can show that:$\sum_{i=0}^{n} iB_{i,n}(x) = nx$

12) The second derivative of $(x + y)^n$ where$y = 1 - x$ can be used to show

$$\sum_{i=0}^{n} i(i-1)B_{i,n}(x) = n(n-1)x^2.$$

13) A Bernstein polynomial can always be written as a linear combination for polynomials of higher degree:

$$B_{i,n-1}(x) = \frac{n-i}{n} B_{i,n}(x) + \frac{i+1}{n} B_{i+1,n}(x).$$

4. IMPLEMENTATIONS OF OPERATIONAL MATRIX FRACTIONAL DERIVATIVE

We underscore the significance of the operational matrix linked to a fractional derivative by implementing it to resolve order fractional-differential equations. The subsistence, uniqueness and continuous reliance of solutions pertaining to this specific matter are deliberated comprehensively in [15]

$$D_t^{q(t)} u(t) + \gamma_1 u'(t) + \gamma_2 u(t) = f(t), \quad u(0) = u_0 \qquad \dots(1)$$

where $f(t) \in L^2[0,1]$ is known, $u(t) \in L^2[0,1]$ is the unidentified function that we aim to estimate with substantial precision, γ_1, γ_2 and u_0 are all constants.

In order to solve eq. (1), we set the approximate solution of equation (1) to be:

$$u(t) \approx \sum_{i=0}^{n} c_i P_{i,n}(t) = c^T \Phi(t) \qquad \dots(2)$$

c=[c0,c1,…,cn]T and $\Phi(t) = [P0(t), P1(t), \dots, Pn(t)]^T$

.

Operating $D_t^{q(t)}$ to the both sides of eq.(2) and by using equation:

$$D_t^{q(t)} u(t) = c^T D_t^{q(t)} \Phi(t) = c^T A G A^{-1} \Phi(t) \qquad \dots(3)$$

$$G(t)=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \rho^q\frac{\Gamma(2)}{\Gamma(2-\gamma)}(t^\rho-a^\rho)^{-q(t)} & 0 & \dots & 0 \\ 0 & 0 & \rho^q\frac{\Gamma(3)}{\Gamma(3-\gamma)}(t^\rho-a^\rho)^{-q(t)} & \dots & 0 \\ 0 & 0 & 0 & \dots & \rho^q\frac{\Gamma(n+1)}{\Gamma(n-\gamma+1)}(t^\rho-a^\rho)^{-q(t)} \end{bmatrix} \quad \dots(4)$$

$$A=\begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \dots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n}{1}\binom{n-1}{0} & \dots & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & (-1)^0\binom{n}{n} \end{bmatrix} \dots(5)$$

replace equations (2)-(5) into equation (1), we can express it in the following manner:

$c^T AGA^{-1}\Phi(t) + \gamma_1 c^T AD^{(1)}A^{-1}\Phi(t) + \gamma_2 c^T\Phi(t) = f(t)$ …(6)

$c^T\Phi(0) = u_0$

As a result, upon the computation of the values of Φ and f on [0,1], using $t_i = \frac{2_i+1}{2(n+1)}$ for i=0,1,…,n,

Therefore, we get following system of algebraic equations:

$c^T AGA^{-1}\Phi(t_i) + \gamma_1 c^T AD^{(1)}A^{-1}\Phi(t_i) + \gamma_2 c^T\Phi(t_i) = f(t_i)$ …(7)

$c^T\Phi(0) = u_0$

The unknown variable 'c' can be accurately determined by solving a system of algebraic equations provided by (7), and upon substituting into equation (2), the desired solution is attained.

5. ILLUSTRATIVE EXAMPLES

This section illustrates the solution of three separate problems involving Fractional Variable Order Differential Equations (FVODEs) using the sophisticated operational matrix approach based on Bernstein polynomials. To provide a comprehensive understanding of our research, graphs have been meticulously generated, and the resulting data has been systematically organized into tables for clarity. We validate our numerical results by comparing them with exact solutions and other approaches found in scholarly literature. All computations and programming were performed using Matlab software.

Example 5.1

Consider the variable order $\alpha(t)$ for a linear Fractional variable order [16].

$D^{\alpha,\rho}u(t) = f(t),\ \ 0 \le \alpha(t) \le 1\ \ t \in [0,1],$ (8)

with $u(0) = 0$ and $\alpha(t) = \sin(t)$, $f(t) = \frac{\Gamma(3)t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + 3\frac{\Gamma(2)t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))}$. The exact solution of Eq. (8) is $u(t) = t^2 + 3t$. We apply the present method with

$\rho=1, N = 3$, $\alpha(t_1)=0.1247$, $\alpha(t_2)=0.3663$, $\alpha(t_3)=0.5851$, $\alpha(t_4)=0.7675$

$C^T D^{\alpha,\rho}\phi(t) - f(t) = 0$

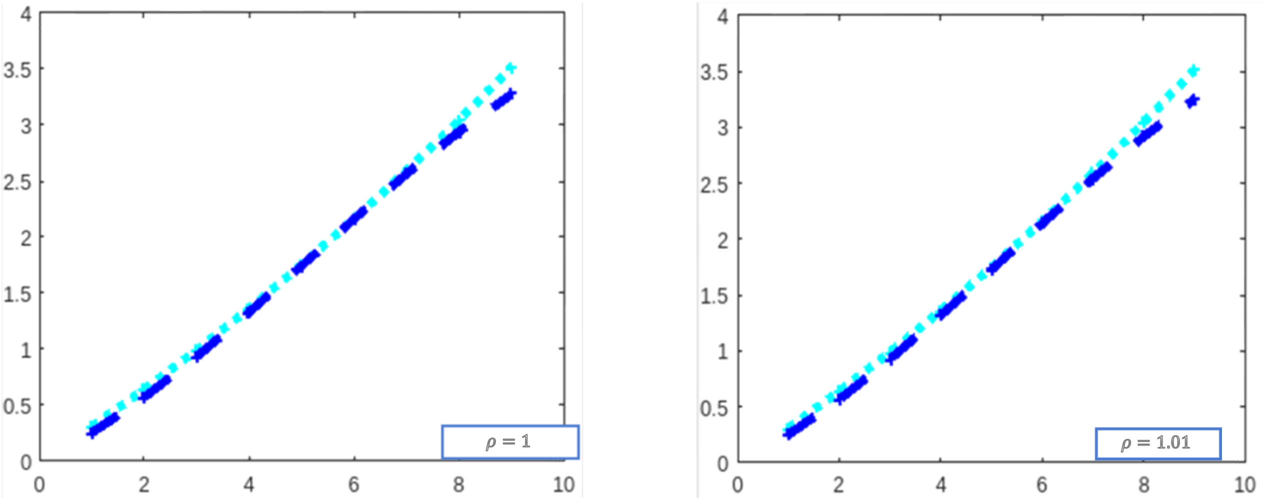
Table 1. The Approximate solution when**(ρ=1, N=3)** for example (5.1)

t	Exact Solution	Approximate Solution N=3
0.1	0.3100	0.2527
0.2	0.6400	0.5684
0.3	0.9900	0.9319
0.4	1.3600	1.3282
0.5	1.7500	1.7422
0.6	2.1600	2.1587
0.7	2.5900	2.5628
0.8	3.0400	2.9392
0.9	3.5100	3.2730

Table 2. The Approximate solution when**(ρ=1.01, N=3)** for example (5.1)

t	**Exact Solution**	**Approximate Solution N=3**
0.1	0.3100	0.2521
0.2	0.6400	0.5659
0.3	0.9900	0.9266
0.4	1.3600	1.3194
0.5	1.7500	1.7295
0.6	2.1600	2.1421
0.7	2.5900	2.5426
0.8	3.0400	2.9160
0.9	3.5100	3.2477

Figure 1. The Approximate solution of ρ=1 & 1.01, for example (5.1)



Example 5.2

Consider the variable order $\alpha(t)$ for a linear Fractional variable order [16].

$$D^{\alpha,\rho}u(t) = f(t), \quad 0 \leq \alpha(t) \leq 1 \quad t \in [0,1], \qquad (9)$$

With $u(0) = 0$ and $\alpha(t) = t/2$, $f(t) = \frac{\Gamma(3)t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + 3\frac{\Gamma(2)t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))}$. The exact solution of Eq.(9) is $u(t) = t^2 + 3t$. We apply the present method with

$N = 2, \rho = 1, \alpha(t_1) = 0.125, \alpha(t_2) = 0.375, \alpha(t_3) = 0.625$

$$C^T D^{\alpha,\rho}\phi(t) - f(t) = 0$$

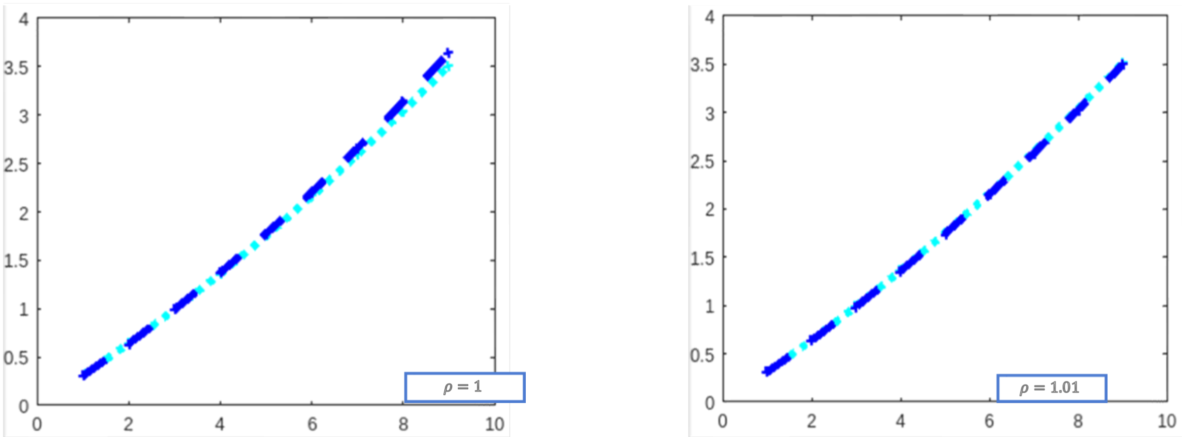
Table 3. The Approximate solution when **(ρ=1, N=2)** for example (5.2)

t	**Exact Solution**	**Approximate Solution N=2**
0.1	0.3100	0.3063
0.2	0.6400	0.6370
0.3	0.9900	0.9922
0.4	1.3600	1.3719
0.5	1.7500	1.7761
0.6	2.1600	2.2047
0.7	2.5900	2.6579
0.8	3.0400	3.1355
0.9	3.5100	3.6375

Table 4. The Approximate solution when **(ρ=1.01, N=3)** for example (5.2)

t	**Exact Solution**	**Approximate Solution N=3**
0.1	-0.0900	-0.0887
0.2	-0.1600	-0.1580
0.3	-0.2100	-0.2076
0.4	-0.2400	-0.2376
0.5	-0.2500	-0.2478
0.6	-0.2400	-0.2381
0.7	-0.2100	-0.2085
0.8	-0.1600	-0.1588
0.9	-0.0900	-0.0889

Figure 2. The Approximate solution when ρ=1 & 1.01, for example (5.2)



Example 5. 3

Consider the variable order $\gamma(t)$ for a nonlinear Fractional variable order [17].

$$D^{\gamma,\rho}u(t) + u(t) = f(t), \quad 0 < \gamma(t) \leq 1 \;\; t \in [0,1], \qquad (10)$$

With $u(0) = 0$ and $\gamma(t) = 1 - 0.5e^{-t}$, $f(t) = 2\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} - \frac{t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))} - t^2 + t$. The exact solution of Eq. (10) is $u(t) = t^2 - t$. We apply the present method with $N = 3$

$$C^T D^{\gamma,\rho}\phi(t) + C^T\phi(t) = f(t)$$

Table 5. The Approximate solution when**($N = 3$, ρ=1)** for example (5.3)

t	**Exact solution**	**Approximate solution N=3**
0.1	-0.0900	-0.0945
0.2	-0.1600	-0.1600
0.3	-0.2100	-0.2153
0.4	-0.2400	-0.2436
0.5	-0.2500	-0.2519
0.6	-0.2400	-0.2410
0.7	-0.2100	-0.2121
0.8	-0.1600	-0.1661
0.9	-0.0900	-0.1040

Table 6. The Approximate solution when **(ρ=1.01, N=3)** for example (5.3)

t	Exact solution	Approximate solution N=3
0.1	-0.0900	-0.0933
0.2	-0.1600	-0.1640
0.3	-0.2100	-0.2131
0.4	-0.2400	-0.2414
0.5	-0.2500	-0.2500
0.6	-0.2400	-0.2395
0.7	-0.2100	-0.2111
0.8	-0.1600	-0.1656
0.9	-0.0900	-0.1038

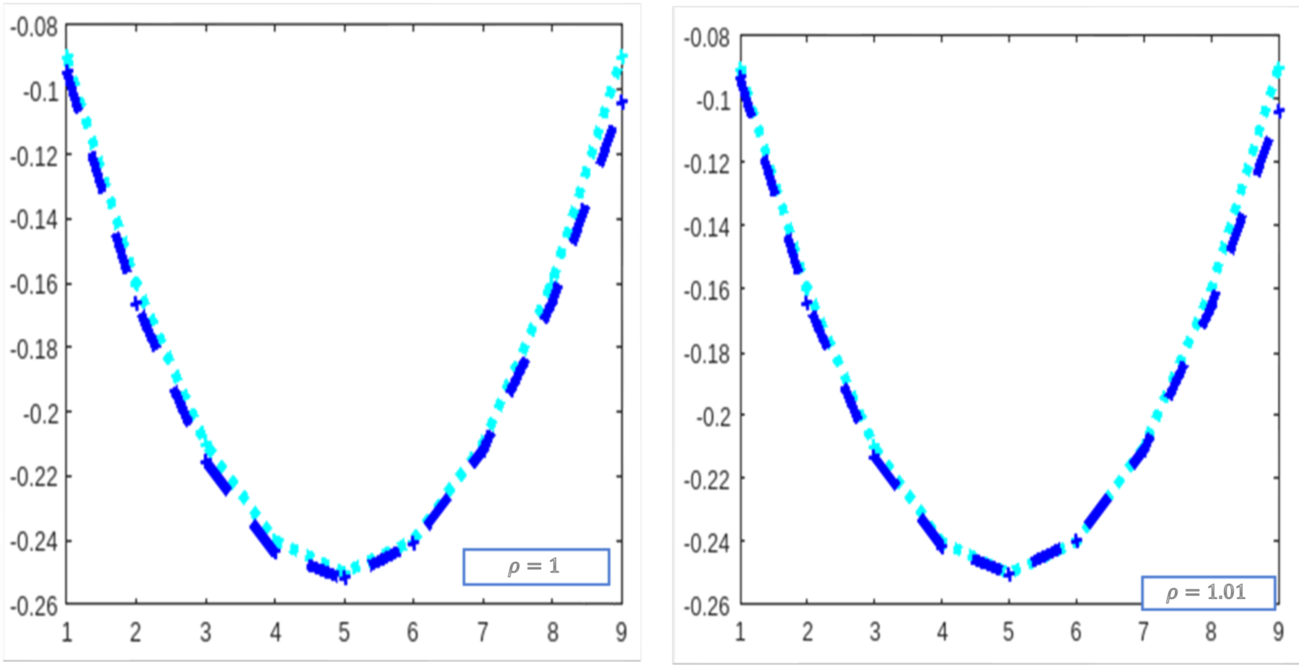


Figure 3. The Approximate solution when ρ=1 & 1.01, for example (5.3)

6. CONCLUSION

The recent numerical approximation has successfully addressed both linear and non-linear time-variable order fractional problems. This approach utilizes an operational matrix based on Bernstein polynomials in the new generalized Caputo fractional derivative sense, which reduces the problems to a linear (non-linear) set of algebraic equations. New characteristics are presented, and novel theorems are developed.

The plot of the solution illustrates the outcome and nature of this approach and its generalization in the fractional domain. Our achievement is robust and highly reliable in seeking solutions to variable order fractional problems. This approach reveals the outcomes of selecting different values for fractional order, parameters, and variable order, providing advantages for a more precise analysis of the problems. This latest differential would open new avenues for inquiry into variable order fractional differentiations.

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