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Results for Multivalent Functions Third-Order Differential Subordination and Superordination by Using New Differential Operator

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ABSTRACT: Presenting a study of the differential subordination and superordination difficulties or obtaining the results of the third-order differential subordination and superordination is the goal of this research. These outcomes were attained by using a new differential operator $J_{\beta,\lambda}^{m}(f * g)(z)$. which also produced Third-Order sandwich outcomes. Numerous works address the issues of first- and second-order differential subordination and superordination in the unit disk, but only a small number address the issues of third-order differential subordination and superordination. For example, suppose p(z) analytic functions in the open unit disk: $\Omega = \{z \in C : |z| < 1\}$, as well suppose $¥ : C^4 \times \Omega \rightarrow C$. In this work, we look at the issue of identifying the characteristics of functions p(z) that meet the following differential superordination of third order:

$$U \subset \{¥(p(z), z p'(z), z^2 p''(z), z^3 p'''(z) ; z \in \Omega)\} .$$

Keywords: Analytic Function, Multivalent Function, Sandwich Results, Admissible Function.

1. INTRODUCTION

Let $L(\Omega)$ be the class of analytic functions in the open unit disk: $\Omega = \{z \in C : |z| < 1\}$.
For $n \in$ N= {1, 2, 3, . . .}, $a \in C$, assume $L[a, n] = \{ f : f \in L(\Omega), f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$ as well suppose that L = L[1,1]. Suppose that f and g belong to the class $L(\Omega)$ of analytic functions. We say that f is subordinate to g, expressed as follows:

$$f \prec g \text{ in } \Omega \text{ or } f(z) \prec g(z), (z \in \Omega)$$

if there exists a Schwarz function $\omega \in L(\Omega)$, which is analytic in Ω, with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \Omega)$, such that $f(z) = g(\omega(z)), (z \in \Omega)$.
Additionally, if g is univalent in Ω, we got (1):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Omega) \subset g(\Omega), (z \in \Omega).$$

Let L_p denote theclass of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1,2,3, \ldots\}) \tag{1}$$

and $g(z) \in L_p$ defined by:

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{2}$$

the Hadamard product(orconvolution) of $f(z), g(z)$ denoted by $f * g$ is defined by:

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k, \tag{3}$$

Differential subordination and superordination of third-order functions represent important concepts within complex analysis and geometric function theory. These ideas extend the notions of subordination and superordination, which concern the relationships between analytic functions.

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In the realm of complex analysis, functions are often studied with respect to their geometric properties and relationships. Subordination and superordination provide a means to compare and understand the behavior of functions in a geometric sense.
Differential subordination and superordination extend this comparison to the derivatives of functions. When we talk about third-order differential subordination or superordination, we are specifically examining the relationships between the third derivatives of functions
The concept of third-order differential subordination was recently introduced by Antonino and Miller (2), and Tang and Deniz have examined the outcomes (4), and You can see the results of third-order differential subordination in work for meromorphic function involving Liu-Srivastava linear operator (5). (see, for instance, (3,6, 7, 8, 9) on Third- order differential subordination and superordination drew the attention of several experts in this area. see, for instance (10,11,12).
In the work, we examine appropriate classes of admissible functions connected to a new differential operator $J^{m}_{\beta,\lambda}$ and define third order differential subordination and superordination features by means of a new $p-$valent functions differential operator in Ω and find fresh results of differential subordination and superordination with some corollaries are found .Finally , we get results of sandwich theorems by equations (43) and (44).
Here, we will go over a few more words and ideas from the idea of differential subordination .

Definition 1.1:(2) Assume K be the set of all functions k that are analytic and univalent on $\Omega \setminus E(k)$, where

$$E(k) = \{\xi : \xi \in \partial\, \Omega : \lim_{z\to\xi} k(z) = \infty\}, \tag{4}$$

and $\min|k'(z)| = p > 0$, for $\xi \in \partial\Omega \setminus E(k)$. Also, let the subclass of K for which k(0) = b, denoted by K(b) with

$$K(0) = K_0 \text{ and } K(1) = K_1. \tag{5}$$

Definition 1.2: (2) Let U be a collection in C and $k \in K$ and $n \in N\setminus\{1\}$. The class $\Psi_n[U, k]$ of admissible functions include of those functions ¥ : $C^4 \times \Omega \to C$, which meet the requirements for admissibility listed below:

$$¥\,(r, s, t, u;\ z) \notin U$$

Whenever

$$r = k(\xi),\ s = ɳ\, \xi k'(z),\ R\left(\frac{t}{s}+1\right) \geqq ɳ\ R\left(\frac{\xi\ k''(z)}{k'(z)} + 1\right)$$

and

$$R\left(\frac{u}{s}\right) \geqq ɳ^2\ R\left(\frac{\xi^2 k'''(z)}{k'(z)}\right),$$

in which $z \in \Omega$, $\xi \in \partial\,\Omega$ / E(k) and $ɳ \geqq 2$.

Definition 1.3:(2) Let ¥: $C^4 \times \Omega \to C$ and assume that the function h(z)is univalent in Ω. If the function p(z) is analytic in Ω and satisfies the following third-order differential subordination:

$$¥\,(\, p(z), z\, p'(z), z^2\, p''(z), z^3\, p'''(z); z) \prec h(z) \tag{6}$$

then p(z) is called asolution of the differential subordination . Additionally, a given univalent function k(z) is called a dominant of the solutions of (6) or more simply, a dominant if $p(z) \prec k(z)$ for all p(z) satisfying (6). A dominant $\hat{k}(z)$ that satisfies $\hat{k}\,(z) \prec k(z)$ for all dominants k(z) of (6) is said to be the best dominant.

Definition 1.4:(3) Let U be a set in C, $k \in L\,[a, n]$ with $k'(z) \neq 0$. The class of admissible functions $\Psi_n{}'[U, k]$ consists of those functions ¥: $C^4 \times \Omega \to C$ that satisfy the following admissiblity condition:

$$¥\,(r, s, t, u;\ z) \in U$$

whenever

$$r = k(z), s = \frac{z\,k'(z)}{m},\ R\left(\frac{t}{s}+1\right) \leqq \frac{1}{m}\ R\left(\frac{zk''(z)}{k'(z)}+1\right), R\left(\frac{u}{s}\right) \leqq \frac{1}{m^2}\ R\left(\frac{z^2\,k'''(z)}{k'(z)}\right),$$

where $z \in \Omega$, $\xi \in \partial\,\Omega$ and $m \geqq n \geqq 2$.
If ¥: $C^2 \times \overline{\Omega} \to C$ and $k \in L\,[a, n]$, then we get

$$¥\,\left(k(z), \frac{zk'(z)}{m}\,;\, \xi\right) \in U\ \ (z \in \Omega, \xi \in \partial\,\Omega\ \text{ and } m \geqq n \geqq 2\,).$$

If ¥: $C^3 \times \overline{\Omega} \to C$ and $k \in L\,[a, n]$, with $k'(z) \neq 0$ then we get

$$¥\,(r,s, t, u; z) \in U$$

whenever

$$r = k(z)\,,\, s = \frac{zk'(z)}{m},\ \text{ and } R\left(\frac{t}{s}+1\right) \leqq \frac{1}{m} R\left(\frac{zk''(z)}{k'(z)}+1\right)\ \ (z \in \Omega, \xi \in \partial\,\Omega\ \text{ and } m \geqq n \geqq 2\,).$$

Lemma 1.1:(4) Let $p \in L\,[a, n]$ $(n \geq 2)$. Also, let $k \in K(b)$ satisfy the conditions:

$$R\left(\frac{\xi\ k''(\xi)}{k'(\xi)}\right) \geqq 0,\quad \left|\frac{zp'(z)}{k'(\xi)}\right| \leqq ɳ\,,\ (z \in \Omega, \xi \in \partial\,\Omega \,/\, E(k) \text{ and } ɳ \geqq 2).$$

If U be a set in C, ¥ $\in \Psi_n[U, k]$ and

$$¥\,(\, p(z), zp'(z), z^2\, p''(z), z^3\, p'''(z); z\,) \in U$$

then

$$p(z) \prec k(z) \qquad (z \in \Omega\,).$$

Lemma 1.2:(3) Let ¥ $\in \Psi_n'$ [U, k]. If ¥ (p(z), zp′(z), z^2 p″(z), z^3 p‴(z); z)is univalent in Ω,p ∈ K(b) and k ∈ L [a, n] satisfy:

$$R\left(\frac{\xi\, k''(\xi)}{k'(\xi)}\right) \geqq 0, \quad \left|\frac{zp'(z)}{k'(\xi)}\right| \leqq m, \ (z \in \Omega,\ \xi \in \partial\Omega / E(k) \text{ and } m \geqq n \geqq 2).$$

then

$$U \subset \{¥\,(p(z), zp'(z), z^2 p''(z), z^3 p'''(z);\ z \in \Omega\},$$

means that

$$k(z) \prec p(z) \qquad (z \in \Omega).$$

Lemma 1.3:(2) Let p ∈ L [a, n] where n ≧ 2, k ∈ K(b) satisfying the following conditions:

$$R\left(\frac{\xi^2 k'''(\xi)}{k'(\xi)}\right) \geqq 0 \text{ as well } \left|\frac{zp'(z)}{k'(\xi)}\right| \leqq \eta$$

where z ∈ Ω, ξ ∈ ∂ Ω \E(k) and η ≧ 2. If U is a set in C, ¥ ∈ Ψ_n[U, k], and

$$¥\,(p(z), zp'(z), z^2 p''(z), z^3 p'''(z);\ z) \in U,$$

then

$$p(z) \prec k(z) \qquad (z \in \Omega).$$

In this work, we provide a novel operator that makes use of convolution.

Definition 1.5: Let $f, g \in L_p$, β, λ ∈ N = {1,2,3, … }, α ∈ N_0,0 < δ <1 ,we define the following operator

$$J^0_{\beta,\lambda}\,(f * g)\,(z) = (f * g)\,(z).$$

as well

$$J^1_{\beta,\lambda}: L_p \to L_p$$

$$J^1_{\beta,\lambda}\,(f*g)(z) = \frac{1-\beta\lambda-\alpha}{\beta\lambda-\delta} J^0_{\beta,\lambda}\,(f*g)(z) + \frac{2\beta\lambda+\alpha-\delta-1}{\beta\lambda-\delta} z^p (z^{-p+1} J^1_{\beta,\lambda}\,(f*g)(z))'$$

$$= \frac{1-\beta\lambda-\alpha}{\beta\lambda-\delta}\left[z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k\right] + \frac{2\beta\lambda+\alpha-\delta-1}{\beta\lambda-\delta} z^p \left(z^{-p+1}\left[z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k\right]\right)'$$

$$= \frac{1-\beta\lambda-\alpha}{\beta\lambda-\delta}\left[z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k\right] + \frac{2\beta\lambda+\alpha-\delta-1}{\beta\lambda-\delta} z^p \left(z^{-p+1} z^{p-1} p + z^p z^{-p}(-p+1) + \sum_{k=p+1}^{\infty}\left[k z^{k-1} z^{-p+1} + (-p+1)\, z^k z^{-p}\right] a_k b_k\right)$$

$$= \frac{1-\beta\lambda-\alpha+2\beta\lambda+\alpha-\delta-1}{\beta\lambda-\delta} z^p + \sum_{k=p+1}^{\infty}\left(\frac{1-\beta\lambda-\alpha}{\beta\lambda-\delta} + \frac{(2\beta\lambda+\alpha-\delta-1)(k-p+1)}{\beta\lambda-\delta}\right) a_k b_k z^k$$

$$= z^p + \sum_{k=p+1}^{\infty}\left(1 + \frac{(2\beta\lambda+\alpha-\delta-1)(k-p)}{\beta\lambda-\delta}\right)^1 a_k b_k z^k)$$

(where m=1) , In general

$$J^m_{\beta,\lambda}\,(f*g)(z) = \frac{1-\beta\lambda-\alpha}{\beta\lambda-\delta} J^{m+1}_{\beta,\lambda}\,(f*g)(z) + \frac{2\beta\lambda+\alpha-\delta-1}{\beta\lambda-\delta} z^p (z^{-p+1} J^m_{\beta,\lambda}\,(f*g)(z))' = z^p + \sum_{k=p+1}^{\infty}\left(1 + \frac{(2\beta\lambda+\alpha-\delta-1)(k-p)}{\beta\lambda-\delta}\right)^m a_k b_k z^k), \tag{7}$$

Through basic computation, we arrive at

$$\frac{2\beta\lambda+\alpha-\delta-1}{\beta\lambda-\delta}\, z\,(J^m_{\beta,\lambda}\,(f*g)(z))' = J^{m+1}_{\beta,\lambda}\,(f*g)(z) - \left(1 - \frac{p(2\beta\lambda+\alpha-\delta-1)}{\beta\lambda-\delta}\right) J^m_{\beta,\lambda}\,(f*g)(z)\,. \tag{8}$$

2. SUBORDINATION RESULTS

In this section, we begin with a given set U and a function k, and we define a set of allowable functions to ensure that (6) is true. For this aim, we create the new class of admissible functions that follows, which are necessary to prove the important third-order differential subordination theorems for the operator $J^m_{\beta,\lambda}\,(f*g)\,(z)$ as given by (7).

Definition 2.1: Assume U be a set in C and k ∈ $k_{\dot 0}$ ∩ L_0. The class Þ [U, k] of admissible functions consists of those functions $\mathscr{J}$: C^4 × Ω ⟶ C, that meet the following admission requirements:

$$\mathscr{J}\,(a, b, c, d;\ z) \notin U,$$

whenever

$$a = k(\xi)\,, \quad b = \frac{\eta(2\beta\lambda+\alpha-\delta-1)\,\xi\, k'(\xi) - p\,(2\beta\lambda+\alpha-\delta-1)k(\xi)}{(\beta\lambda-\delta)} + k(\xi)$$

$$R\left(\frac{(c-2b+a)(\beta\lambda-\delta)^2 + 2p\,(2\beta\lambda+\alpha-\delta-1)(\beta\lambda-\delta)(b-a) + p^2\,(2\beta\lambda+\alpha-\delta-1)^2\, a}{(2\beta\lambda+\alpha-\delta-1)^2\,(\,(\beta\lambda-\delta)(b-a) + p\,(2\beta\lambda+\alpha-\delta-1)a)}\right) \geqq \eta\, R\left(\frac{\xi k''(\xi)}{k'(\xi)} + 1\right)$$

and

$$R\left(\frac{(d-3c+3b)(\beta\lambda-\delta)^3 - (2-2p)(2\beta\lambda+\alpha-\delta-1)(c-2b+a)\,(\beta\lambda-\delta)^3 + (-1+3p-p^2)(\beta\lambda-\delta)(b-a)(2\beta\lambda+\alpha-\delta-1)^2 + (p^2-p)(2\beta\lambda+\alpha-\delta-1)^3\, a}{(\beta\lambda-\delta)(2\beta\lambda+\alpha-\delta-1)(b-a) + p\,(2\beta\lambda+\alpha-\delta-1)^2 a}\right) \geqq \eta^2 R\left(\frac{\xi^2\, k'''(\xi)}{k'(\xi)}\right),$$

as well as z ∈ Ω, ξ ∈ ∂ Ω / E(k) and η ≧ 2.

Theorem 2.1: Suppose $\mathscr{J}$ ∈ Þ [U, k]. If the functions $f, g \in L_p$ and k ∈ $k_{\dot 0}$, it meets the following conditions:

$$R\left(\frac{\xi k''(\xi)}{k'(\xi)}\right) \geqq 0, \quad \left|\frac{J_{\beta,\lambda}^{m}(f*g)(z)}{k'(\xi)}\right| \leqq \eta, \tag{9}$$

and

$$\mathscr{J}\left(J_{\beta,\lambda}^{m}(f*g)(z), J_{\beta,\lambda}^{m+1}(f*g)(z), J_{\beta,\lambda}^{m+2}(f*g)(z), J_{\beta,\lambda}^{m+3}(f*g)(z); z\in\Omega\right) \subset U, \tag{10}$$

thus

$$J_{\beta,\lambda}^{m}(f*g)(z) \prec k(z)$$

Proof: Define the analytic function $p(z)$ in Ω by:

$$p(z) = J_{\beta,\lambda}^{m}(f*g)(z). \tag{11}$$

Shaped equations (8) and (11), we got

$$J_{\beta,\lambda}^{m+1}(f*g)(z) = \frac{(2\beta\lambda+\alpha-\delta-1)zp'(z) - p\,(2\beta\lambda+\alpha-\delta-1)p(z)}{(\beta\lambda-\delta)} + p(z), \tag{12}$$

Using a comparable justification, we obtain

$$J_{\beta,\lambda}^{m+2}(f*g)(z) = \frac{2(2\beta\lambda+\alpha-\delta-1)zp'(z) - 2p\,(2\beta\lambda+\alpha-\delta-1)p(z)}{(\beta\lambda-\delta)} + p(z) + \frac{(2\beta\lambda+\alpha-\delta-1)^2 z^2 p''(z) + (1-2p)(2\beta\lambda+\alpha-\delta-1)^2 zp'(z)}{(\beta\lambda-\delta)^2} + \frac{p^2\,(2\beta\lambda+\alpha-\delta-1)^2 p(z)}{(\beta\lambda-\delta)^2}, \tag{13}$$

and

$$J_{\beta,\lambda}^{m+3}(f*g)(z) = \frac{3(2\beta\lambda+\alpha-\delta-1)zp'(z) - 3p\,(2\beta\lambda+\alpha-\delta-1)p(z)}{(\beta\lambda-\delta)} + \frac{3(2\beta\lambda+\alpha-\delta-1)^2 z^2 p''(z) + (3-6p)(2\beta\lambda+\alpha-\delta-1)^2 zp'(z)}{(\beta\lambda-\delta)^2} + \frac{3p^2\,(2\beta\lambda+\alpha-\delta-1)^2 p(z)}{(\beta\lambda-\delta)^2} + \frac{(2\beta\lambda+\alpha-\delta-1)^3 z^3 p'''(z) + (2-2p)(2\beta\lambda+\alpha-\delta-1)^3 z^2 p''(z) + (3p^2-3p+1)(2\beta\lambda+\alpha-\delta-1)^3 zp'(z) - p^3\,(2\beta\lambda+\alpha-\delta-1)^3 p(z)}{(\beta\lambda-\delta)^3}. \tag{14}$$

We define the transformation from C^4 to C by

$$a(r,s,t,u) = r,\quad b(r,s,t,u) = \frac{(2\beta\lambda+\alpha-\delta-1)s - p\,(2\beta\lambda+\alpha-\delta-1)r}{\beta\lambda-\delta} + r,$$

$$c(r,s,t,u) = \frac{2(2\beta\lambda+\alpha-\delta-1)s - 2p\,(2\beta\lambda+\alpha-\delta-1)r}{(\beta\lambda-\delta)} + r + \frac{(2\beta\lambda+\alpha-\delta-1)^2 t + (1-2p)(2\beta\lambda+\alpha-\delta-1)^2 s + p^2\,(2\beta\lambda+\alpha-\delta-1)^2 r}{(\beta\lambda-\delta)^2} \tag{15}$$

$$d(r,s,t,u) = \frac{3(2\beta\lambda+\alpha-\delta-1)s - 3p\,(2\beta\lambda+\alpha-\delta-1)r}{(\beta\lambda-\delta)} + \frac{3(2\beta\lambda+\alpha-\delta-1)^2 t + (3-6p)(2\beta\lambda+\alpha-\delta-1)^2 s + 3p^2\,(2\beta\lambda+\alpha-\delta-1)^2 r}{(\beta\lambda-\delta)^2} + \frac{(2\beta\lambda+\alpha-\delta-1)^3 u + (2-2p)(2\beta\lambda+\alpha-\delta-1)^3 t + (3p^2-3p+1)(2\beta\lambda+\alpha-\delta-1)^3 s - p^3\,(2\beta\lambda+\alpha-\delta-1)^3 r}{(\beta\lambda-\delta)^3} \tag{16}$$

$$¥(r,s,t,u;z) = \mathscr{J}(a,b,c,d;z)$$

$$= \mathscr{J}\left(\begin{array}{c} r,\ \frac{(2\beta\lambda+\alpha-\delta-1)s - p\,(2\beta\lambda+\alpha-\delta-1)r}{\beta\lambda-\delta} + r \\ ,\frac{2(2\beta\lambda+\alpha-\delta-1)s - 2p\,(2\beta\lambda+\alpha-\delta-1)r}{(\beta\lambda-\delta)} + r + \frac{(2\beta\lambda+\alpha-\delta-1)^2 t + (1-2p)(2\beta\lambda+\alpha-\delta-1)^2 s + p^2\,(2\beta\lambda+\alpha-\delta-1)^2 r}{(\beta\lambda-\delta)^2}, \\ \frac{3(2\beta\lambda+\alpha-\delta-1)s - 3p\,(2\beta\lambda+\alpha-\delta-1)r}{(\beta\lambda-\delta)} + \frac{3(2\beta\lambda+\alpha-\delta-1)^2 t + (3-6p)(2\beta\lambda+\alpha-\delta-1)^2 s + 3p^2\,(2\beta\lambda+\alpha-\delta-1)^2 r}{(\beta\lambda-\delta)^2} \\ + \frac{(2\beta\lambda+\alpha-\delta-1)^3 u + (2-2p)(2\beta\lambda+\alpha-\delta-1)^3 t + (3p^2-3p+1)(2\beta\lambda+\alpha-\delta-1)^3 s - p^3\,(2\beta\lambda+\alpha-\delta-1)^3 r}{(\beta\lambda-\delta)^3} \end{array}\right). \tag{17}$$

Equations from (11) to (14) and Equation (17) are used to get the following:

$$¥(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) = \mathscr{J}\left(J_{\beta,\lambda}^{m}(f*g)(z), J_{\beta,\lambda}^{m+1}(f*g)(z), J_{\beta,\lambda}^{m+2}(f*g)(z), J_{\beta,\lambda}^{m+3}(f*g)(z); z\right). \tag{18}$$

Thus, obviously (10) become

$$¥(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \in U,$$

We see that

$$\frac{t}{s} + 1 = \frac{(c-2b+a)(\beta\lambda-\delta)^2 + 2p\,(2\beta\lambda+\alpha-\delta-1)(\beta\lambda-\delta)(b-a) + p^2\,(2\beta\lambda+\alpha-\delta-1)^2 a}{(2\beta\lambda+\alpha-\delta-1)^2\left((\beta\lambda-\delta)(b-a) + p\,(2\beta\lambda+\alpha-\delta-1)a\right)},$$

and

$$\frac{u}{s} = \frac{(d-3c+3b)(\beta\lambda-\delta)^3 - (2-2p)(2\beta\lambda+\alpha-\delta-1)(c-2b+a)(\beta\lambda-\delta)^3 + (-1+3p-p^2)(\beta\lambda-\delta)(b-a)(2\beta\lambda+\alpha-\delta-1)^2 + (p^2-p)(2\beta\lambda+\alpha-\delta-1)^3 a}{(\beta\lambda-\delta)(2\beta\lambda+\alpha-\delta-1)(b-a) + p\,(2\beta\lambda+\alpha-\delta-1)^2 a}.$$

So obviously, the admissibility condition for $\mathscr{J} \in Þ\,[U,k]$ in Definition 1.4, is the same as the admissibility condition $¥ \in \Psi_2[U,k]$, as stated in the Definition 1.2 with $n = 2$.
Thus, by making use of equation (9) and Lemma 1.3, we got

$$J_{\beta,\lambda}^{m}(f*g)(z) \prec k(z)$$

The proof is now complete.

The implication of Theorem 2.1 is our next finding, when it is unknown how k(z) will behave on $\partial\Omega$.

Corollary 2.1: Let $U \subset C$ and suppose k be univalent function in Ω with $k(0) = 1$. Assume $\mathscr{J} \in Þ\,[U,k]$ for a given $p \in (0,1)$, in which $k_p(z) = k(pz)$. If the function $f*g \in L_p$ and k_p meet the requirements listed below:

$$\mathrm{R}\left(\frac{\xi k_p''(\xi)}{k_p'(\xi)}\right) \geqq 0, \left|\frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{k_p'(\xi)}\right| \leqq \eta, \ (z \in \Omega;\ \eta \geqq 2;\ \xi \in \partial\Omega \setminus E(k)),$$

and

$$\mathscr{J}\left(J_{\beta,\lambda}^{m}(f*g)(z), J_{\beta,\lambda}^{m+1}(f*g)(z), J_{\beta,\lambda}^{m+2}(f*g)(z), J_{\beta,\lambda}^{m+3}(f*g)(z); z\right) \in U,$$

thus

$$J_{\beta,\lambda}^{m}(f*g)(z) \prec k(z), \quad (z \in \Omega).$$

Proof: From Theorem 2.1, we obtain

$$J_{\beta,\lambda}^{m}(f*g)(z) \prec k_p(z), \ (z \in \Omega).$$

Now, as follows subordination property leads to the conclusion stated by Corollary 2.1

$$k_p(z) \prec k(z).$$

This completes the Corollary 2.1 proof.
If $U \neq C$, is a simply connected domain, the $U = h(\Omega)$ in relation to certain conformal mapping $h(z)$ of Ω onto U. Here, however the class Þ [$h(\Omega)$, k] is expressed as Þ [h, k]. This results in Theorem 2.1 is immediate conclusion being as follows.

Theorem 2.2: Let $\mathscr{J} \in$ Þ [h, k].If the function $f*g \in L_p$ and $k \in k_1$, meet the requirements listed below:

$$\mathrm{Re}\left(\frac{\xi k_p''(\xi)}{k_p'(\xi)}\right) \geqq 0, \left|\frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{k_p'(\xi)}\right| \leqq \eta, \quad (z \in \Omega;\ \eta \geqq 2;\ \xi \in \partial\Omega \setminus E(k)) \tag{19}$$

and

$$\mathscr{J}\left(J_{\beta,\lambda}^{m}(f*g)(z), J_{\beta,\lambda}^{m+1}(f*g)(z), J_{\beta,\lambda}^{m+2}(f*g)(z), J_{\beta,\lambda}^{m+3}(f*g)(z); z\right) \prec h(z), \tag{20}$$

thus

$$J_{\beta,\lambda}^{m}(f*g)(z) \prec k(z), \quad (z \in \Omega).$$

Corollary 2.2: suppose $U \subset C$ and suppose k be univalent function in Ω also $k(0) = 1$. Assume $\mathscr{J} \in$ Þ [Ω, k_p] for a given $p \in (0,1)$, in which $k_p(z) = k(pz)$. If the function $f*g \in L_p$ and k_p meet the requirements listed below:

$$\mathrm{Re}\left(\frac{\xi k_p''(\xi)}{k_p'(\xi)}\right) \geqq 0, \left|\frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{k_p'(\xi)}\right| \leqq \eta, \quad (z \in \Omega;\ \eta \geqq 2;\ \xi \in \partial\Omega \setminus E(k_p))$$

and

$$\mathscr{J}\left(J_{\beta,\lambda}^{m}(f*g)(z), J_{\beta,\lambda}^{m+1}(f*g)(z), J_{\beta,\lambda}^{m+2}(f*g)(z), J_{\beta,\lambda}^{m+3}(f*g)(z); z\right) \prec h(z),$$

thus

$$J_{\beta,\lambda}^{m}(f*g)(z) \prec (f*g)(z),$$

The best dominant of differential subordination (20) is produced by the following result.

Theorem 2.3: Suppose h be univalent function in Ω. Also let $\mathscr{J}: C^4 \times \Omega \to C$ and ¥ given by (17). Assume the differential equation that follows

$$¥\left(k(z), zk'(z), z^2 k''(z), z^3 k'''(z); z\right) = h(z), \tag{21}$$

thus

$$J_{\beta,\lambda}^{m}(f*g)(z) \prec k(z),$$

Where the best dominating is $k(z)$.
Proof: Making use of Theorem 2.1, therefore, k must be a dominant of (20). k is also a solution of (20) as it satisfies (21). As a result, every dominant will dominate k. k is therefore the most dominant.
Considering the definition 2.1, in the specific instance where $k(z) = Mz, (M > 0)$, the class Þ [U, k] of admissible functions, and it is symbolized by Þ [Ω, M] and stated as bellow.

Definition 2.2: Suppose U be set in C and $M > 0$. The class Þ [Ω, M] of admissible functions include for those functions $\mathscr{J}: C^4 \times \Omega \to C$, so that

$$\mathscr{J}\left(\begin{array}{c} Me^{i\theta}, \ \left(\frac{(2\beta\lambda+\alpha-\delta-1)\eta-p\,(2\beta\lambda+\alpha-\delta-1)}{\beta\lambda-\delta}+1\right)Me^{i\theta}, \\ \left(\frac{2(2\beta\lambda+\alpha-\delta-1)Q-2p\,(2\beta\lambda+\alpha-\delta-1)}{(\beta\lambda-\delta)}+1\right)Me^{i\theta}+\frac{(2\beta\lambda+\alpha-\delta-1)^2 L+((1-2p)(2\beta\lambda+\alpha-\delta-1)^2 Q+p^2\,(2\beta\lambda+\alpha-\delta-1)^2)Me^{i\theta}}{(\beta\lambda-\delta)^2}, \\ \frac{(3(2\beta\lambda+\alpha-\delta-1)Q-3p\,(2\beta\lambda+\alpha-\delta-1))\,Me^{i\theta}}{(\beta\lambda-\delta)}+\frac{3(2\beta\lambda+\alpha-\delta-1)^2 L+((3-6p)(2\beta\lambda+\alpha-\delta-1)^2 Q+3p^2\,(2\beta\lambda+\alpha-\delta-1)^2)\,Me^{i\theta}}{(\beta\lambda-\delta)^2} \\ +\frac{(2\beta\lambda+\alpha-\delta-1)^3 N+(2-2p)(2\beta\lambda+\alpha-\delta-1)^3 L+((3p^2-3p+1)\,(2\beta\lambda+\alpha-\delta-1)^3 Q-p^3\,(2\beta\lambda+\alpha-\delta-1)^3)\,Me^{i\theta}}{(\beta\lambda-\delta)^3} \end{array}\right), \tag{22}$$

$(z \in \Omega)$.

$$\mathrm{R}\left(Le^{-i\theta}\right) \geqq (Q-1)\,Q\,M,$$

and

$$\mathrm{R}\left(N e^{-i\theta}\right) \geqq 0, \forall\, \theta \in R;\ \eta \geqq 0.$$

Corollary 2.3: Assume $\mathscr{J} \in$ Þ [U, M]. If the function $f, g \in L_p$ meets all requirements listed below

$$\left|J_{\beta,\lambda}^{m+1}(f*g)(z)\right| \leqq \eta M, \ (z \in \Omega;\ \eta \geqq 2 \quad M > 0),$$

As well

$$(J^{m}_{\beta,\lambda}(f*g)(z), J^{m+1}_{\beta,\lambda}(f*g)(z), J^{m+2}_{\beta,\lambda}(f*g)(z), J^{m+3}_{\beta,\lambda}(f*g)(z); z) \in \Omega,$$

thus

$$\left|J^{m}_{\beta,\lambda}(f*g)(z)\right| < M.$$

Particularly, whenever $U = k(\Omega) = \{\omega : |\omega| < M\}$, the class Þ [U, M] is simply represented by Þ[M]. It is present possible to rewrite Corollary 2.3 as follows from.

Corollary 2.4: Let $\mathscr{J} \in$ Þ [M]. If $f * g \in L_p$ meets all requirements listed below:

$$\left|J^{m+1}_{\beta,\lambda}(f*g)(z)\right| \geqq \eta M, (z \in \Omega; \eta \geqq 2; M > 0),$$

As well

$$|J^{m}_{\beta,\lambda}(f*g)(z), J^{m+1}_{\beta,\lambda}(f*g)(z), J^{m+2}_{\beta,\lambda}(f*g)(z), J^{m+3}_{\beta,\lambda}(f*g)(z); z| < M,$$

then

$$\left|J^{m}_{\beta,\lambda}(f*g)(z)\right| < M.$$

Definition 2.3: Suppose U be a set in C and $k \in K_1 \cap L_1$, the class Þ [U, k]of admissible functions consists of those functions $\mathscr{J} : C^4 \times \Omega \to C$, which satisfy the following admissibility conditions

$$\mathscr{J}(a, b, c, d; z) \notin U,$$

whenever

$$a = k(\xi),\ b = \frac{\eta(2\beta\lambda+\alpha-\delta-1)\,\xi k'(\xi)}{(\beta\lambda-\delta)} + k(\xi),$$

$$R\left(\frac{(\beta\lambda-\delta)(c-2b+a)}{(b-a)(2\beta\lambda+\alpha-\delta-1)}\right) \geqq \eta R\left(\frac{\xi k''(\xi)}{k'(\xi)} + 1\right),$$

and

$$R\left(\frac{(\beta\lambda-\delta)^2(d+3b-3c-a)}{(b-a)(2\beta\lambda+\alpha-\delta-1)^2} - \frac{(\beta\lambda-\delta)(p(a-b)+3c-7b+4a)}{(b-a)(2\beta\lambda+\alpha-\delta-1)} - 2\right) \geqq \eta^2 R\left(\frac{\xi^2 k'''(\xi)}{k'(\xi)}\right),$$

in which $z \in \Omega$, $\xi \in \partial\Omega / E(k)$ as well $\eta \geqq 2$.

Theorem 2.4: Suppose $\mathscr{J} \in$ Þ [U , k]. If the function $f * g \in L_p$ as well $k \in K_1$, meet the requirements listed below:

$$R\left(\frac{\xi k''(\xi)}{k'(\xi)}\right) \geqq 0,\ \left|\frac{J^{m+1}_{\beta,\lambda}(f*g)(z)}{z^p k'(\xi)}\right| \leqq \eta, \tag{23}$$

as well

$$\left\{\mathscr{J}\left(\frac{J^{m}_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+1}_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+2}_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+3}_{\beta,\lambda}(f*g)(z)}{z^p}; z\right)\right\} \subset U(z \in \Omega). \tag{24}$$

Thus

$$\frac{J^{m}_{\beta,\lambda}(f*g)(z)}{z^p} \prec k(z).$$

Proof: Define p(z)in Ω, through

$$p(z) = \frac{J^{m}_{\beta,\lambda}(f*g)(z)}{z^p} \tag{25}$$

according to equations (8) and (11), we obtain

$$\frac{J^{m+1}_{\beta,\lambda}(f*g)(z)}{z^p} = \frac{(2\beta\lambda+\alpha-\delta-1)\,z p'(z)}{(\beta\lambda-\delta)} + p(z). \tag{26}$$

By a same account, we obtain

$$\frac{J^{m+2}_{\beta,\lambda}(f*g)(z)}{z^p} = \frac{(2\beta\lambda+\alpha-\delta-1)^2 z^2 p''(z) + (2\beta\lambda+\alpha-\delta-1)^2 z p'(z)}{(\beta\lambda-\delta)^2} + \frac{2(2\beta\lambda+\alpha-\delta-1)\,z p'(z)}{(\beta\lambda-\delta)} + p(z), \tag{27}$$

and

$$\frac{J^{m+3}_{\beta,\lambda}(f*g)(z)}{z^p} = \frac{3(2\beta\lambda+\alpha-\delta-1)^2 z^2 p''(z) + (2-p)(2\beta\lambda+\alpha-\delta-1)^2 z p'(z)}{(\beta\lambda-\delta)^2} + \frac{3(2\beta\lambda+\alpha-\delta-1)\,z p'(z)}{(\beta\lambda-\delta)} + p(z) + \frac{(2\beta\lambda+\alpha-\delta-1)^3 z^3 p'''(z) + 3(2\beta\lambda+\alpha-\delta-1)^3 z^2 p''(z) + (2\beta\lambda+\alpha-\delta-1)^3 z p'(z)}{(\beta\lambda-\delta)^3}. \tag{28}$$

Currently, we define the transformation from C^4 to C by

$$a(r,s,t,u) = r,\ b(r,s,t,u) = \frac{(2\beta\lambda+\alpha-\delta-1)s}{\beta\lambda-\delta} + r,$$

$$c(r,s,t,u) = \frac{(2\beta\lambda+\alpha-\delta-1)^2 t + (2\beta\lambda+\alpha-\delta-1)^2 s}{(\beta\lambda-\delta)^2} + \frac{2(2\beta\lambda+\alpha-\delta-1)\,s}{(\beta\lambda-\delta)} + r, \tag{29}$$

and

$$d(r,s,t,u) = \frac{3(2\beta\lambda+\alpha-\delta-1)^2 t + (2-p)(2\beta\lambda+\alpha-\delta-1)^2 s}{(\beta\lambda-\delta)^2} + \frac{3(2\beta\lambda+\alpha-\delta-1)\,s}{(\beta\lambda-\delta)} + r + \frac{(2\beta\lambda+\alpha-\delta-1)^3 u + 3(2\beta\lambda+\alpha-\delta-1)^3 t + (2\beta\lambda+\alpha-\delta-1)\,s}{(\beta\lambda-\delta)^3}. \tag{30}$$

Suppose

$$¥(r, s, t, u; z) = \mathscr{J}(a, b, c, d; z)$$

$$= \mathscr{J}\left(\begin{array}{c} r,\ \frac{(2\beta\lambda+\alpha-\delta-1)s}{\beta\lambda-\delta}+r \\ ,\frac{(2\beta\lambda+\alpha-\delta-1)^2 t+(2\beta\lambda+\alpha-\delta-1)^2 s}{(\beta\lambda-\delta)^2}+\frac{2(2\beta\lambda+\alpha-\delta-1)s}{(\beta\lambda-\delta)}+r\ , \\ \frac{3(2\beta\lambda+\alpha-\delta-1)^2 t+(2-p)(2\beta\lambda+\alpha-\delta-1)^2 s}{(\beta\lambda-\delta)^2}+\frac{3(2\beta\lambda+\alpha-\delta-1)s}{(\beta\lambda-\delta)}+r+\ \frac{(2\beta\lambda+\alpha-\delta-1)^3 u+3(2\beta\lambda+\alpha-\delta-1)^3 t+(2\beta\lambda+\alpha-\delta-1 s}{(\beta\lambda-\delta)^3} \end{array}\right). \tag{31}$$

Lemma 1.1 will be used in the proof. Equations (25)–(27) and equation (31), together, allow us to get at

$$¥\,(p(z), z p'(z), z^2 p''(z), z^3 p'''(z) ; z) =$$
$$\mathscr{J}\left(\frac{J_{\beta,\lambda}^{m}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+2}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+3}(f*g)(z)}{z^p}\ ; z\right). \tag{32}$$

Thus, obviously, (24) becomes into

$$¥\,(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) \in U,$$

so

$$\frac{t}{s}+1 = \frac{(\beta\lambda-\delta)(c-2b+a)}{(b-a)(2\beta\lambda+\alpha-\delta-1)},$$

and

$$\frac{u}{s} = \frac{(\beta\lambda-\delta)^2(d+3b-3c-a)}{(b-a)(2\beta\lambda+\alpha-\delta-1)^2} - \frac{(\beta\lambda-\delta)(p(a-b)+3c-7b+4a)}{(b-a)(2\beta\lambda+\alpha-\delta-1)} - 2.$$

Therefore, the admissibility condition for $\mathscr{J} \in Þ\,[U, k]$ in Definition 2.3 is comparable to the requirements for admission for $¥ \in \Psi_2 2[U, k]$, as stated in the definition1.2 when $n \geqq 2$.
Consequently, using equation (23) with Lemma 1.3 gives us

$$\frac{J_{\beta,\lambda}^{m}(f*g)(z)}{z^p} \prec k(z).$$

Theorem 2.5: Let $\mathscr{J} \in Þ\,[U, k]$.If the function $f*g \in L_p$as well $k \in K_1$,meet the requirements listed below:

$$R\left(\frac{\xi k''(\xi)}{k'(\xi)}\right) \geqq 0, \left|\frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{z^p k'(\xi)}\right| \leqq \eta\ , (z \in \Omega) \tag{33}$$

and

$$\mathscr{J}\left(\frac{J_{\beta,\lambda}^{m}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+2}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+3}(f*g)(z)}{z^p}; z\right) < h(z), \tag{34}$$

.
thus

$$\frac{J_{\beta,\lambda}^{m}(f*g)(z)}{z^p} \prec k(z).$$

Definition 2.8: Suppose U be set in C and $M > 0$. The class Þ [U,M]of admissible functions consists of those function $\mathscr{J}: C^4 \times \Omega \to C$, such that

$$\mathscr{J}\left(\begin{array}{c} Me^{i\theta},\left(\frac{(2\beta\lambda+\alpha-\delta-1)\eta}{\beta\lambda-\delta}+1\right)\ Me^{i\theta} \\ ,\frac{(2\beta\lambda+\alpha-\delta-1)^2 L+(2\beta\lambda+\alpha-\delta-1)^2 Q}{(\beta\lambda-\delta)^2}+\left(\frac{2(2\beta\lambda+\alpha-\delta-1)Q}{(\beta\lambda-\delta)}+1\right)Me^{i\theta}\ , \\ \frac{3(2\beta\lambda+\alpha-\delta-1)^2 L+(2-p)(2\beta\lambda+\alpha-\delta-1)^2 Q}{(\beta\lambda-\delta)^2}+\left(\frac{3(2\beta\lambda+\alpha-\delta-1)Q}{(\beta\lambda-\delta)}+1\right)Me^{i\theta}+\ \frac{(2\beta\lambda+\alpha-\delta-1)^3 N+3(2\beta\lambda+\alpha-\delta-1)^3 L+(2\beta\lambda+\alpha-\delta-1)Q}{(\beta\lambda-\delta)^3} \end{array}\right) \tag{35}$$

where $(z \in \Omega)$,

$$R(Le^{-i\theta}) \geqq (Q-1)QM,$$

and

$$R(N e^{-i\theta}) \geqq 0, \forall\, \theta \in R\,; \eta \geqq 0.$$

Corollary 2.5: Let $\mathscr{J} \in Þ\,[U, M]$. If the function $f*g \in L_p$, meet the requirements listed below:

$$\left|\frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{z^p}\right| \leqq \eta M, (z \in \Omega;\ \eta \geqq 2;\quad M > 0),$$

As well

$$\mathscr{J}\left(\frac{J_{\beta,\lambda}^{m}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+2}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+3}(f*g)(z)}{z^p}; z\right) \in U,$$

then

$$\left|\frac{J_{\beta,\lambda}^{m}(f*g)(z)}{z^p}\right| < M$$

Particularly, when $U = k(\Omega) = \{\omega: |\omega| < M\}$, the class Þ [U, M] is simply shown by Þ [M]. Currently possible to rewrite Corollary 2. 5 as follows from.
Corollary 2.6: Let $\mathscr{J} \in [U, M]$. If the function $f*g \in L_p$, meet the requirements listed below:

$$\left|\frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{z^p}\right| \leqq \eta M, (z \in \Omega;\ \eta \geqq 2;\quad M > 0),$$

As well

$$\left|\mathscr{J}\left(\frac{J_{\beta,\lambda}^{m}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+1}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+2}(f*g)(z)}{z^p}, \frac{J_{\beta,\lambda}^{m+3}(f*g)(z)}{z^p};\ z\right)\right| < M,$$

then

$$\left|J^{m}_{\beta,\lambda}\,(f * g\,)(z)\right| < M,$$

3.SUPERORDINATION RESULTS

This section examines the properties of differential superordination.

Definition 3.1: Let U be a set in C, $k \in K$ with $k'(z) \neq 0$ and $m \in N \setminus \{1\}$. The class of admissible functions Ƥ′[U , k] comprises those functions $\mathscr{J}: C^4 \times \Omega \to C$ which meet the requirements for admission

$$\mathscr{J}(a, b, c, d; \xi) \in U,$$

whenever

$$a = k(z), \quad b = \frac{(2\beta\lambda+\alpha-\delta-1)\, z\, k'(z) - mp\,(2\beta\lambda+\alpha-\delta-1)k(z)}{m(\beta\lambda-\delta)} + m\, k(z)$$

$$R\left(\frac{(c-2b+a)(\beta\lambda-\delta)^2+2p\,(2\beta\lambda+\alpha-\delta-1)(\beta\lambda-\delta)(b-a)+p^2\,(2\beta\lambda+\alpha-\delta-1)^2\,a}{(2\beta\lambda+\alpha-\delta-1)^2\,(\,(\beta\lambda-\delta)(b-a)+p\,(2\beta\lambda+\alpha-\delta-1)a)}\right) \geqq \frac{1}{m} R\left(\frac{zk''(z)}{k'(z)}+1\right)$$

and

$$R\left(\frac{(d-3c+3b)(\beta\lambda-\delta)^3-(2-2p)(2\beta\lambda+\alpha-\delta-1)(c-2b+a)\,(\beta\lambda-\delta)^3+(-1+3p-p^2)(\beta\lambda-\delta)(b-a)(2\beta\lambda+\alpha-\delta-1)^2+(p^2-p)(2\beta\lambda+\alpha-\delta-1)^3\,a}{(\beta\lambda-\delta)(2\beta\lambda+\alpha-\delta-1)(b-a)+p\,(2\beta\lambda+\alpha-\delta-1)^2 a}\right) \geqq \frac{1}{m^2} R\left(\frac{z^2k'''(z)}{k'(z)}\right),$$

where ($z \in \Omega$, and $\zeta \in \partial\,\Omega$, $m \in N\setminus\{1\}$).

Theorem 3.1: Let $\mathscr{J} \in$ Ƥ′[U , k]. If the functions $f * g \in L_p$ and $J^{m}_{\beta,\lambda}\,(f * g\,)(z) \in K_1$ meet the requirements listed below:

$$R\left(\frac{zk''(z)}{k'(z)}\right) \geqq 0, \quad \left|\frac{J^{m}_{\beta,\lambda}(f*g\,)(z)}{k'(z)}\right| \leqq m, \tag{36}$$

as well

$$\mathscr{J}(J^{m}_{\beta,\lambda}(f * g\,)\,(z), J^{m+1}_{\beta,\lambda}(f * g\,)\,(z)\,, J^{m+2}_{\beta,\lambda}(\,f * g\,)\,(z), J^{m+3}_{\beta,\lambda}(f * g\,)\,(z);\; z\,), \tag{37}$$

is univalent

$$U \subset \{\,\mathscr{J}(J^{m}_{\beta,\lambda}(f * g\,)\,(z), J^{m+1}_{\beta,\lambda}(\,f * g\,)\,(z), J^{m+2}_{\beta,\lambda}(f * g\,)\,(z), J^{m+3}_{\beta,\lambda}(\,f * g\,)(z);\; z\,);\; z \in \Omega\,)\}, \tag{38}$$

implies that

$$k(z) \prec J^{m}_{\beta,\lambda}\,(f * g\,)(\,z).$$

Proof: Suppose the function p(z) provided by (11) and ¥ provided by (17). As $\mathscr{J} \in$ Ƥ′[U , k], the Equations (18) and (38) imply that

$$U \subset \{¥\,(p\,(z), z\,p'\,(z), z^2p''\,(z), z^3p'''(z);\; z \in \Omega\,)\,\}\,.$$

This naturally flows from (17) the admissible condition for $\mathscr{J} \in$ Ƥ′[U , k] in Definition 3.1 is equivalent to the admissible condition for $\mathscr{J} \in$ Ƥ′[U , k] in accordance with Definition 1.4. for $n = 2$. Therefore, using the parameters found in (36) furthermore from Lemma 1.2, we get

$$k(z) \prec p(z)$$

or, in the same way

$$k(z) \prec J^{m}_{\beta,\lambda}(\,f * g\,)\,(z).$$

If $U \neq C$ is a domain with simple connections and = h(Ω) about a mapping that conforms h(z) of Ω onto U, then the class Ƥ′[h(Ω),k] by Ƥ′[h , k]. Following the same procedure as in the previous section, Theorem 3.1 leads to the following result.

Theorem 3.2: Let $\mathscr{J} \in$ Ƥ′[h , k]. and let h is analytic in h(Ω). If the functions $f * g \in L_p$ as well $J^{m}_{\beta,\lambda}(f * g\,)(\,z) \in K_1$ meet the requirements listed below (36) as well:

$$\mathscr{J}(J^{m}_{\beta,\lambda}(f * g\,)(\,z), J^{m+1}_{\beta,\lambda}(\,f * g\,)(\,z), J^{m+2}_{\beta,\lambda}(\,f * g\,)\,(z), J^{m+3}_{\beta,\lambda}(f * g\,)\,(z);\; z\,)$$

is univalent in Ω, then

$$h(z) \prec \mathscr{J}(J^{m}_{\beta,\lambda}(\,f * g\,)\,(z), J^{m+1}_{\beta,\lambda}(f * g\,)\,(z), J^{m+2}_{\beta,\lambda}(f * g\,)\,(\,z), J^{m+3}_{\beta,\lambda}(\,f * g\,)\,(\,z);\; z\,), \tag{39}$$

implies that

$$k(z) \prec J^{m}_{\beta,\lambda}\,(f * g\,)(z).$$

Theorem 3.1 is used to infer the proof.

The best subordinate of (39) exists for a good choice of $\mathscr{J}$, as demonstrated by the following theorem.

Theorem 3.3: Suppose the function h be analytic in Ω, as well suppose $\mathscr{J}: C^4 \times \Omega \to C$ and ¥ defined by (17). Let us assume the differential equation

$$¥\,(\,k(z), zk'(z), z^2\,k''(z), z^3\,k'''(\,z);\; z\,) = h(z),$$

has an answer $k(z) \in K_1$. If $f * g \in L_p$ as well $J^{m}_{\beta,\lambda}\,(f * g\,)(z) \in K_1$ meet the requirements listed below (36) and:

$$\mathscr{J}(J^{m}_{\beta,\lambda}(\,f * g)\,(\,z), J^{m+1}_{\beta,\lambda}\,(\,f * g\,)\,(\,z), J^{m+2}_{\beta,\lambda}\,(\,f * g\,)\,(z), J^{m+3}_{\beta,\lambda}\,(\,f * g)\,(z);\; z\,)$$

is univalent in Ω, then

$$h(z) \prec \mathscr{J}(J^{m}_{\beta,\lambda}(f * g)(z), J^{m+1}_{\beta,\lambda}(f * g)(z), J^{m+2}_{\beta,\lambda}(f * g)(z), J^{m+3}_{\beta,\lambda}(f * g)(z); z)$$

implies that

$$k(z) \prec J^{m}_{\beta,\lambda}(f * g)(z).$$

so k(z) is the best subordinant.

Proof: Since Theorem 3.3 proof is identical to Theorem 2.3, so it's not mentioned here.

Theorem 3.4: Let $\mathscr{J} \in \text{Þ}'[U, k]$, If the functions $f, g \in L_p$ as well $\frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p} \in k_1$ satisfy

$$R\left(\frac{\xi k''(\xi)}{k'(\xi)}\right) \geqq 0, \left|\frac{J^{m+1}_{\beta,\lambda}(f * g)(z)}{z^p k'(z)}\right| \leqq m, \tag{40}$$

and

$$\mathscr{J}\left(\frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+1}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+2}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+3}_{\beta,\lambda}(f * g)(z)}{z^p}; z\right)$$

is univalent in Ω, then

$$U \subset \left\{\mathscr{J}\left(\left(\frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+1}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+2}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+3}_{\beta,\lambda}(f * g)(z)}{z^p}; z\right) z \in \Omega\right\}, \tag{41}$$

that means

$$k(z) \prec \frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p}.$$

Proof: Let p(z) and ¥ be the functions represented by (22) and (31). We may infer from (32) and (41) that $\mathscr{J} \in \text{Þ}'[U, k]$

$$U \subset \{¥(p(z), zp'(z), z^2p''(z), z^3p'''(z); z \in \Omega\}$$

What we discover, from (29) and (30), that the admissible condition for $\mathscr{J} \in \text{Þ}'[U, k]$ in Definition 3.1 is comparable to that for ψ as shown in Definition 1.4 when $n = 2$. Therefore, $¥ \in \Psi_2\ [U, k]$so by using 3.3 and Lemma 1.2, As we have

$$k(z) \prec p(z) \quad (z \in \Omega),$$

or

$$k(z) \prec \frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p}$$

If $U \neq C$ is a domain with simple connections and $U = h(\Omega)$ about a mapping h(z) in conformity of Ω onto U, then the class $\text{Þ}'[h(\Omega), k]$ is expressed by $\text{Þ}'[h, k]$.

Following the same procedure as in the previous section, Theorem 3.4 leads to the following result.

Theorem 3.5: Let $\mathscr{J} \in \text{Þ}'[h, k]$. suppose the function h be analytic in Ω. If the functions $f, g \in L_p$ as well $\frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p} \in k_1$ satisfy (40) with

$$\mathscr{J}\left(\frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+1}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+2}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+3}_{\beta,\lambda}(f * g)(z)}{z^p}; z\right)$$

is univalent in Ω, thus

$$h(z) \prec \mathscr{J}\left(\frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+1}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+2}_{\beta,\lambda}(f * g)(z)}{z^p}, \frac{J^{m+3}_{\beta,\lambda}(f * g)(z)}{z^p}; z\right)$$

implies that

$$k(z) \prec \frac{J^{m}_{\beta,\lambda}(f * g)(z)}{z^p}.$$

4. SANDWICH RUSULTS

We get the following sandwich-type conclusion by combining Theorems 2.2 and 3.2.

Theorem 4.1: Assume the functions h_1, k_1be analytic functions in Ω, as well assume the function h_2 be univalent in Ω, $k_2 \in K_1$ as well $k_1(0) = k_2(0) = 1$ also $\mathscr{J} \in \text{Þ}[h_2, k_2] \cap \text{Þ}'[h_1, k_1]$. If the function $f, g \in L_p$ and $J^{m}_{\beta,\lambda}(f * g)(z) \in K_1 \cap L$ and

$$\mathscr{J}(J^{m}_{\beta,\lambda}(f * g)(z), J^{m+1}_{\beta,\lambda}(f * g)(z), J^{m+2}_{\beta,\lambda}(f * g)(z), J^{m+3}_{\beta,\lambda}(f * g)(z); z)$$

is univalent in Ω, and the conditions (9), (36) are realized, thus

$$h_1(z) \prec \mathscr{J}(J^{m}_{\beta,\lambda}(f * g)(z), J^{m+1}_{\beta,\lambda}(f * g)(z), J^{m+2}_{\beta,\lambda}(f * g)(z), J^{m+3}_{\beta,\lambda}(f * g)(z); z) \prec h_2(z) \tag{42}$$

Implies that

$$k_1(z) \prec J^{m}_{\beta,\lambda}(f * g)(z) \prec k_2(z). \tag{43}$$

Proof: From Theorems 2.2 and 3.2, respectively, the conclusion is drawn.

We get the following sandwich-type conclusion by combining Theorems 2.5 and 3.5.

Theorem 4.2: Let the functions h_1 , k_1 be analytic functions in Ω, and let the function h_2 be univalent in Ω, $k_2 \in K_1$ with $k_1(0) = k_2(0) = 1$ and $\mathscr{J} \in$ Þ[h_2 , k_2]∩ Þ′[h_1 , k_1]. If the function $f, g \in L_p$ and $\frac{J^m_{\beta,\lambda}(f*g)(z)}{z^p} \in K_1 \cap L$ and

$$\mathscr{J}\left(\frac{J^m_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+1}_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+2}_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+3}_{\beta,\lambda}(f*g)(z)}{z^p}; z\right)$$

is univalent in Ω, and the conditions (23), (40) are realized, thus

$$h_1(z) \prec \mathscr{J}\left(\frac{J^m_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+1}_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+2}_{\beta,\lambda}(f*g)(z)}{z^p}, \frac{J^{m+3}_{\beta,\lambda}(f*g)(z)}{z^p}; z\right) \prec h_2(z)$$

implies that

$$k_1(z) \prec \frac{J^m_{\beta,\lambda}(f*g)(z)}{z^p} \prec k_1(z). \tag{44}$$

4. CONCLUSIONS

Using a specific differential operator of analytic p-valent functions in $\mathbb{U}$, we examine classes of admissible functions and determine the features of third-order differential subordination. We obtain some new results on differential subordination along with a few corollaries.
The sandwich theorems are formed by the symmetry between these features and outcomes and the properties of the differential superordination. Our results differ from the other authors' earlier results. We provided avenues for writers to apply the results obtained in the work to generalize our new subclasses and derive novel results in univalent and multivalent function theory.

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