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On T-continuous map

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ABSTRACT: Topology, as a mathematical discipline, relies heavily on the concept of open sets to define and study the properties of topological spaces. In this paper, we introduce two new classes of open sets, termed T-open and T-closed sets, which generalize the classical notion of openness in topological spaces. We provide precise definitions for these sets and investigate their basic properties. Furthermore, we introduce the notion of T-continuity, which characterizes mappings between spaces preserving the T-open and T-closed sets. Through a series of examples and applications, we demonstrate the utility of these new concepts in various branches of mathematics and theoretical computer science.

Keywords: T-continuous map, δ-Continuous map, T-open set and T-closed set.

1. INTRODUCTION

The concept of open sets lies at the heart of topology, providing a fundamental framework for defining the topology of a space and studying its properties. In this paper, we introduce a new type of open set, known as T-open sets, and explore their applications in topological spaces. In 1968, Velico [1] introduced the concept of δ-interior, δ-closure, θ-interior and θ-closure, for the purpose of studying the important class of H-closed space. The family of all sets in a topological space forms a topology. In 1980, Noiri [2] also introduced and studied the notion of δ-continuous map. In 1989, Jingcheng Tong [3], defined the concept of t-set.

Throughout this work, by $(\mathbb{W}, \tau)$ or simply by $\mathbb{W}$ we mean a topological space. If A is a subset of $\mathbb{W}$, then A°, $\bar{A}$ and $\mathbb{W} - A$ denote, respectively, the interior set of A, the closure set of A and the complement set of A.

Now, we recall some of the basic definitions and results in a topological space.

Definition1.1 [4]: Let $(\mathbb{W}, \tau)$ be a topological space and $\subseteq \mathbb{W}$. The set A is called regular open if $A = \bar{A}^{\circ}$.

Definition1.2 [3]: Let $(\mathbb{W}, \tau)$ be a topological space and $\subseteq \mathbb{W}$. The set A is called t-set if $A^{\circ} = \bar{A}^{\circ}$.

Proposition 1.3 [3]: If A is a closed set in a topological space $\mathbb{W}$, then A is a t-set.

Proposition 1.4 [3]: If A is a regular open set in a topological space $\mathbb{W}$, then A is a t-set.

Definition 1.5: Let A be a set in a topological space $\mathbb{W}$, a point $x \in \mathbb{W}$ is called T-cluster point of A if for any t-set U contain x implies $U \cap A \neq \emptyset$.

The set of all T-cluster points of A is called T-closure of A and it's denoted by $\bar{A}^{T}$.

Definition 1.7: A subset A of a topological space $(\mathbb{W}, \tau)$ is called T-closed set if $A = \bar{A}^{T}$.

Note. T-open set the complement of T-closed set and the family of all T-open (T-closed) sets of $(\mathbb{W}, \tau)$ is denoted by TO($\mathbb{W}$) (res. TC($\mathbb{W}$)).

Proposition 1.8: Every t-set is T-open set.

Definition 1.9: Let $(\mathbb{W}, \tau)$ be a topological space, a family TO($\mathbb{W}$) forms a topology on $\mathbb{W}$ is called T- topology and denoted by τ^{T}. $(\mathbb{W}, \tau^{T})$ is called T-topological space.

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Examples 1.10: 1) The usual space is a T-topological space.

2) Any Discrete topology is T-topology.

Definition 1.11: Let $(\mathbb{W},\tau)$ be a topological space and $A \subseteq \mathbb{W}$, $x \in A$ is called T-interior if there exists a t-set U such that $x \in U \subseteq A$. The set of all T-interior of A is called T-interior set of A which is his symbol $(A^{\circ T})$.

Propositions 1.12: If A is a subset of a topological space $\mathbb{W}$, then:

i) T-interior set ($A^{\circ T}$) is largest t-set contain in A
ii) A is a T-open iff $A = A^{\circ T}$.
iii) Every t-set is T-open set.
iv) Every closed set is T-open.
v) Every regular open is T-open set.
vi) Every open and t-set is regular open set.
vii) A° is regular open set, if A° is t-set.
viii) A is regular open set, if A is open and T-open set.
ix) A° is regular open set, if A° is T-open set.

Proof: Obvious.

Proposition 1.13: If $(\mathbb{W},\tau)$ is a T_1-space and $A \in \mathbb{W}$, then A is T-open.
Proof: Let $(\mathbb{W},\tau)$ be a T_1-space and $A \in \tau$. Then for any a $\in A$, $\{a\}$ is closed. Thus $\{a\}$ is T-open by Proposition 1.12 (iv), so $A = \bigcup_{a\in A}\{a\}$ is T-open. Hence $A \in \tau^T$.

Corollary 1.14: If $(\mathbb{W},\tau)$ is a T_1-space, then $\tau \subseteq \tau^T$.

Definition 1.15 [2]: Let $(\mathbb{W},\tau)$ be a top. sp. and $A \subseteq X$. A point $x \in X$ is called δ-cluster point of A if for any regular open set U contain x implies $U \cap A \neq \emptyset$.
The set of all δ-cluster points of A is called δ-closure of A and its denoted by $\bar{A}^{\delta}$

Definition 1.16 [2]: Let $(\mathbb{W},\tau)$ be a top. sp. and $A \subseteq \mathbb{W}$. The set A is said to be δ-closed set if $A = \bar{A}^{\delta}$.
Note. δ-open set is the complement of δ-closed set and δO($\mathbb{W}$) (res. δC($\mathbb{W}$)) refer the family of all T-open (res. T-closed) sets.

Definition 1.17 [2]: A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is called strongly δ-continuus if for any $x \in \mathbb{W}$ and any open neighborhood V containing $\mathcal{L}(x)$, there exists an δ-open neighborhood U of x suchthat $\mathcal{L}(U) \subseteq V$.

Definition 1.17 [4]: Let $(\mathbb{W},\tau)$ be a top. sp. and $A \subseteq \mathbb{W}$. The set A is called compact if every cover of it by open subsets of $\mathbb{W}$, has a finite subcover.

Definition 1.18 [5]: A space $\mathbb{W}$ is said to be Pa-closed if every paracompact subset of $\mathbb{W}$ is closed.

Definition 1.19 [6]: A surjective continuous map $\mathcal{L}: (\mathbb{W},\tau) \to (\mathbb{M},\acute{\tau})$is said to be paracompact if the pre-image for any paracompact set in $\mathbb{M}$ is paracompact set in $\mathbb{W}$.

Proposition 1.20 [7]: Every compact space is paracompact.

Proposition 1.21 [7]: Every closed subset of T_2-space is compact.

2. T-continuous map

Definition 2.1: A map $\mathcal{L}: (\mathbb{W},\tau) \to (\mathbb{M},\acute{\tau})$ is said to be T-continuous if the inverse image of any open set in $\mathbb{M}$ is T-open subset of $\mathbb{W}$.

Proposition 2.2: Let $\mathcal{L}: (\mathbb{W},\tau) \to (\mathbb{M},\acute{\tau})$ be a map, then the following statements are equivalent:

i- $\mathcal{L}$ is T-continuous map.
ii- $\forall x \in \mathbb{W}$ and $V \in \acute{\tau}$ s.t $\mathcal{L}(x) \in V$, $\exists\, U \in TO(\mathbb{W})$ containing x s.t $\mathcal{L}(U) \subseteq V$.
iii- The inverse image of any closed set in $\mathbb{M}$ is T-closed subset of $\mathbb{W}$.

Proof: i → ii

Let $x \in \mathbb{W}$ and let V be open in $\mathbb{M}$ containing $\mathcal{L}(x)$. Since $\mathcal{L}$ is T-continuous map, then $\mathcal{L}^{-1}(V)$ is T-open set in $\mathbb{W}$, so $\mathcal{L}(\mathcal{L}^{-1}(V)) \subseteq V$.

ii→iii

Let F be a closed subset of $\mathbb{M}$, then $(\mathbb{M} - F) \in \acute{\tau}$. From (ii), For any $x \in \mathbb{W}$ and $\mathcal{L}(x) \in (\mathbb{M} - F)$, there exists a T-open set U containing x such that $\mathcal{L}(U) \subseteq (\mathbb{M} - F)$, thus $U \subseteq \mathcal{L}^{-1}(\mathbb{M} - F)$. But U is T-open, then there is a t-set D such that $x \in D \subseteq U \subseteq \mathcal{L}^{-1}(\mathbb{M} - F)$, therefore $\mathcal{L}^{-1}(\mathbb{M} - F)$ is T-open set. Hence $\mathbb{M} - (\mathcal{L}^{-1}(\mathbb{M} - F))= \mathcal{L}^{-1}(F)$ is T-closed.

iii→i

Clear.

Proposition 2.3: Let $(A, \tau_{|A})$ be a subspace of $(\mathbb{W}, \tau)$ and A be a closed set in $\mathbb{W}$. $B \subseteq A$ is T-open in A if and only if B is T-open in X.

Proof: Let $B \in TO(A)$, then there exists $U \in TO(\mathbb{W})$ such that $B = A \cap U$. But A is a closed subset of $\mathbb{W}$, thus A is a T-open set. Since $TO(\mathbb{W})$ is a topology, hence $A \cap U$ is T-open in $\mathbb{W}$. The convers is obvious.

Proposition 2.4: A map $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ is a T-continuous if and only if $\overline{\mathcal{L}^{-1}(B)}^{T} \subseteq \mathcal{L}^{-1}(\bar{B}), \forall B \subseteq \mathbb{M}$.

Proof: Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a T-continuous map and $B \subseteq \mathbb{M}$, then $f^{-1}(\bar{B})$ is T-closed subset of X. Since $B \subseteq \bar{B}$, thus $\mathcal{L}^{-1}(B) \subseteq \mathcal{L}^{-1}(\bar{B})$. Therefore, $\overline{\mathcal{L}^{-1}(B)}^{T} \subseteq \overline{\mathcal{L}^{-1}(\bar{B})}^{T}=\mathcal{L}^{-1}(\bar{B})$. Conversely, Let $\overline{\mathcal{L}^{-1}(B)}^{T} \subseteq \mathcal{L}^{-1}(\bar{B})$, for any $B \subseteq \mathbb{M}$ and F is a closed set in $\mathbb{M}$ $(F = \bar{F})$. Thus, from hypothesis $\overline{\mathcal{L}^{-1}(F)}^{T} \subseteq \mathcal{L}^{-1}(F)$. We have $\mathcal{L}^{-1}(F) \subseteq \overline{\mathcal{L}^{-1}(F)}^{T}$, thus $\mathcal{L}^{-1}(F) = \overline{\mathcal{L}^{-1}(F)}^{T}$. Hence $\mathcal{L}^{-1}(F)$ is T-closed and so, $\mathcal{L}$ is a continuous map.

Proposition 2.5: Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a map and $\mathbb{W}$ be T_1-space. Then $\mathcal{L}$ is T-continuous map if $\mathcal{L}$ is continuous.

Proof: Let $\mathcal{L}$ be a continuous map and U be open set in $\mathbb{M}$. Thus $\mathcal{L}^{-1}(U)$ is open set in $\mathbb{W}$, but we have $\mathbb{W}$ is T_1-space, then $\mathcal{L}^{-1}(U)$ is T-open from Corollary 1.13. Hence $\mathcal{L}$ is T-continuous map.

Proposition 2.6: Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a paracompact map. Then $\mathcal{L}$ is T-continuous, where $\mathbb{W}$ is a pa-closed space and $\mathbb{M}$ is a discrete space.

Proof: Let U be open set in $\mathbb{M}$ and so, U is closed. Since $\mathbb{M}$ is T_2-space, then U is compact. By Proposition 1.20, U is paracompact set. Therefore, $\mathcal{L}^{-1}(U)$ is paracompact due to, $\mathcal{L}$ is paracompact map. Now, $\mathcal{L}^{-1}(U)$ is closed by Proposition 1.18, thus $\mathcal{L}^{-1}(U)$ is T-open set owing to Proposition 1.12 (iv). Hence $\mathcal{L}$ is T-continuous map.

Proposition 2.7: Every strongly δ-continuous map is T-continuous.

Proof: Let $\mathcal{L}: (\mathbb{W}, \tau) \to (\mathbb{M}, \acute{\tau})$ and $x \in \mathbb{W}$, if V is an open set in $\mathbb{M}$ such that $\mathcal{L}(x) \in V$, then there exists an δ-open set D containing x such that $\mathcal{L}(D) \subseteq V$. Thus, there exists a regular open U contains x such that $x \in U \subseteq D$ and so, $\mathcal{L}(x) \in \mathcal{L}(U) \subseteq \mathcal{L}(D) \subseteq V$. But U is T-open by Proposition 1.12 (v), therefore $\mathcal{L}$ is T-continuous by Proposition 2.2.

The Converse of the above Proposition need not true. The following example shows that.

Example 2.8: Let $\mathbb{W} = \{a, b, c\}$, $\mathbb{M} = \{1,2,3\}, \tau = \{\mathbb{W}, \emptyset, \{a\}, \{a, b\}\}$ and $\acute{\tau} = \{\mathbb{M}, \emptyset, \{1,2\}\}$. Define a map $\mathcal{L}: (\mathbb{W}, \tau) \to (\mathbb{M}, \acute{\tau})$ by $\mathcal{L}(c) = 1$, $\mathcal{L}(\{a, b\}) = 3$, since $TC(\mathbb{W}) = \{\mathbb{W}, \emptyset, \{a\}, \{a, b\}\}$, then $\mathcal{L}$ is T-continuous because $\mathcal{L}^{-1}(3)$={a,b}, $\mathcal{L}^{-1}(Y) = X$, $\mathcal{L}^{-1}(\emptyset) = \emptyset$ and $\mathcal{L}$ is not strongly δ-continuous because $\mathcal{L}^{-1}(\{1,2\}) = \{c\} \notin \delta O(X)$.

Proposition 2.9: Let $\mathcal{L}: \mathbb{W} \to \mathbb{M}$ be a T-continuous map and $\mathbb{M}$ be a Hausdorff space. Then the subset $\Delta = \{(x, y): \mathcal{L}(x) = \mathcal{L}(y)\}$ is a T-closed.

Proof: Let $(x, y) \notin \Delta$, then $\mathcal{L}(x) \neq \mathcal{L}(y)$. Since Y is Hausdorff, then there is two open sets V_1, V_2 in $\mathbb{M}$ containing $\mathcal{L}(x)$ and $\mathcal{L}(y)$ respectively, such that $V_1 \cap V_2 = \emptyset$. We have $\mathcal{L}$ is T-continuous map, thus there exists $U_1, U_2 \in TO(\mathbb{W})$ containing x and y respectively, such that $\mathcal{L}(U_1) \subseteq V_1$ and $\mathcal{L}(U_2) \subseteq V_2$. It follows that $(x, y) \in U_1 \times U_2 \subseteq (X \times X) - \Delta$, and so, $(x, y) \in \Delta^{c^{\circ T}}$, therefore $(x, y) \notin (X \times X) - \Delta^{c^{\circ T}} = \bar{\Delta}^{T}$. Consequently, $\bar{\Delta}^{T} \subseteq \Delta$ and hence Δ is T-closed.

Definition 2.10: Let $(\mathbb{W}, \tau)$ be a top. sp. and $A \subseteq \mathbb{W}$. The set A is called T-compact if every cover of it by T-open subsets of $\mathbb{W}$, has a finite subcover.

Proposition 2.11: Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a T-continuous map and $A \subseteq \mathbb{W}$ is be T-compact subset of $\mathbb{W}$. Then $\mathcal{L}(A)$ is a compact subset of $\mathbb{M}$.

Proof: Let $\{V_\alpha : \alpha \in \Lambda\}$ be open cover of $\mathcal{L}(A)$ in $\mathbb{M}$. For each $x \in A$, there exists $\alpha(x) \in \Lambda$ such that $\mathcal{L}(x) \in V_{\alpha(x)}$. Now, we have $\mathcal{L}$ is a T-continuous, then there exists $U_x \in TO(\mathbb{W})$ containing x such that $\mathcal{L}(U_x) \subseteq V_{\alpha(x)}$. Thus the family $\{U_x : x \in A\}$ is a T-open cover of A in $\mathbb{W}$. Hence there exists a finite subset A^* of A such that $A \subseteq \bigcup_{x \in A^*} U_x$, therefore we have $\mathcal{L}(A) \subseteq \bigcup_{x \in A^*} V_{\alpha(x)}$. This shows that $\mathcal{L}(A)$ is compact.

Proposition 2.12: Let $\mathcal{L}: \mathbb{W} \rightarrow \mathbb{M}$ be a map and $\mathbb{M}$ be regular space. Then $\mathcal{L}$ is strongly δ-continuous map if and only if $\mathcal{L}$ is T-continuous.

Proof: Let $x \in \mathbb{W}$ and V be a open in Y such that $\mathcal{L}(x) \in V$, then there exists an open set D in Y and $\mathcal{L}(x) \in D \subseteq \overline{D} \subseteq V$, Due to Y is regular space. Since $\mathcal{L}$ is T-continuous map, then there is $U \in TO(\mathbb{W})$ such that $\mathcal{L}(U) \subseteq D$ by Proposition 2.3 (ii). Assume that $y \notin \overline{D}$, then there exists an open set H contains y in $\mathbb{M}$ such that $H \cap D = \emptyset$, also $\mathcal{L}^{-1}$(H)$\cap \mathcal{L}^{-1}(D) = \emptyset$. Since $U \subseteq \mathcal{L}^{-1}(D)$ then, $\mathcal{L}^{-1}$(H)$\cap U = \emptyset$ and $\mathcal{L}^{-1}$(H)$\cap U^{\circ} = \emptyset$. Now, we obtain $H \cap \mathcal{L}(U^{\circ}) = \emptyset$, and $y \notin \mathcal{L}^{-1}(U^{\circ})$, this show that $\mathcal{L}(U^{\circ}) \subseteq \overline{D} \subseteq V$. But $U^{\circ}$is regular open set and so, it is δ-open set. Hence, $\mathcal{L}$ is strongly δ-continuous map. Conversely, By Proposition 2.6.

REFERENCES

[1] N. v., Velicko, *"H-closed Topological Space,"* Amer. Math. Soc. Transl.,(78), 103-118, 1968.

[2] T., Noiri, *"On δ-continuous Function,"* J. Korean Math Soc.,(16), 161-166, 1980.

[3] T., Jingchang, *"On Decomposition of Continuity In Topological Space,"* Acta Mathematica Hungarica, (48) (1-2), 51-55, 1989.

[4] J., Dugundji,"Topology," Allyn and Bacon, Boston, 1966.

[5] M. J., Saad and H. K., Hiyam, *"Paracompactly closed map,"* Wasit Journal for Pure sciences, vol. 1, Issue 3, pp. 81-89, 2022.

[6] M. J., Saad and H. K., Hiyam, *"Certain types of paracompact actions,"* AIP Conf. Proc. 2977, 040126, 2023.

[7] C. H., Dowker,*"On countably paracompact spaces,"* Canadian Journal of Mathematics, vol. 3, pp. 219 - 224, 1951.