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On Structure Fuzzy Fibration

Bushra Zidan Khalaf [1]*, Daher Al Baydli [2]

[1]Department of Mathematics, College of Education for Pure Science, University of Wasit, IRAQ

[2]Department of Mathematics, College of Education for Pure Science, University of Wasit, IRAQ

*Corresponding Author: Bushra Zidan Khalaf

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ABSTRACT: The Fuzzy Homotopy Lifting Property (FHLP) is a fundamental concept in algebraic topology, providing a framework for understanding continuous mappings between topological spaces. This property extends the classical notion of homotopy lifting to a more flexible and versatile setting, allowing for a nuanced analysis of maps that may not strictly preserve exact topological structures. This abstract explores the essence of the Fuzzy Homotopy Lifting Property, delving into its theoretical underpinnings, applications, and significance in contemporary mathematics. Beginning with a concise definition of FHLP, we elucidate its key features and establish connections to related concepts such as homotopy theory and topological invariance. We then survey prominent results and developments in the field, highlighting the interplay between FHLP and various areas of mathematics, including differential geometry, category theory, and algebraic topology. Moreover, we discuss practical implications of FHLP in diverse mathematical contexts, ranging from the study of fiber bundles to the analysis of topological data. By examining concrete examples and illustrating fundamental theorems, we illustrate the utility of FHLP in solving theoretical problems and addressing real-world challenges. Furthermore, we explore open questions and avenues for future research, envisioning potential extensions and refinements of FHLP theory to enrich our understanding of topological spaces and mappings.

Keywords: Fuzzy set, Fuzzy covering space, Fuzzy Homotopy Lifting Property, Fuzzy fiber structure, Fuzzy Fibration and Fuzzy lifting function

1. INTRODUCTION AND PRELIMINARIES

The field of fuzzy topology extends classical set theory and topology to accommodate degrees of membership and uncertainty. Fuzzy sets, introduced by Lotfi A. Zadeh in 1965, provide a mathematical framework to handle imprecision and vagueness in data representation. Fuzzy sets generalize classical sets by assigning membership values between 0 and 1, allowing elements to belong to a set to varying degrees [1].

Building upon the notion of fuzzy sets, the concept of fuzzy covering spaces emerges as a natural extension of classical covering space theory. In traditional topology, covering spaces play a pivotal role in understanding the fundamental group and homotopy theory. Fuzzy covering spaces extend this framework to incorporate fuzzy sets, enabling a more flexible treatment of topological structures and mappings [2][3].

Central to the study of fuzzy covering spaces is the notion of fuzzy lifting, which generalizes the classical lifting property in covering space theory. A fuzzy lifting function assigns fuzzy sets to points in the base space, capturing the uncertainty inherent in the mapping between spaces. This concept serves as a cornerstone in defining the Fuzzy Homotopy Lifting Property (FHLP) [4].

The Fuzzy Homotopy Lifting Property, with respect to a fuzzy measure μx, extends the classical homotopy lifting property to fuzzy settings. In essence, FHLP ensures that homotopies between fuzzy maps lift to fuzzy maps in the covering space, preserving the fuzzy structures induced by the mappings. This property is essential for studying the behavior of fuzzy mappings under continuous deformations and holds significant implications for understanding the topology of fuzzy spaces [4].

In the context of fuzzy topology, fibrations play a crucial role in capturing the local and global structure of spaces. A fuzzy fibration generalizes the notion of a fibration to fuzzy settings, providing a framework to analyze the behavior of mappings between fuzzy spaces. Various propositions and theorems characterize the properties of fuzzy fibrations, shedding light on the interplay between fuzzy maps, fuzzy covering spaces, and fuzzy lifting functions [2].

*Corresponding author: bushra1984b@gmail.com

In summary, the concepts of fuzzy sets, fuzzy covering spaces, fuzzy lifting, FHLP, and fuzzy fibrations form the basis of fuzzy topology, offering a versatile framework to study topological spaces in the presence of uncertainty and imprecision. These concepts bridge the gap between classical topology and fuzzy mathematics, opening avenues for exploring the rich interplay between structure and fuzziness in mathematical modeling and analysis.

Definition (1.1): [5][6][7] A fuzzy set in $\boldsymbol{X}$ is a function with domain $\boldsymbol{X}$ and the codomain is values in $\boldsymbol{I} = [\mathbf{0}, \mathbf{1}]$, that is an element of $\boldsymbol{I}^{\boldsymbol{X}}$. The membership is denoted by: $\widetilde{\boldsymbol{A}} = \{(\boldsymbol{x}, \boldsymbol{\mu}_{\widetilde{\boldsymbol{A}}}(\boldsymbol{x})) | \boldsymbol{x} \in \boldsymbol{X}\}$.

Definition (1.2): [8][9] Let $\boldsymbol{\mu_X}$ and $\boldsymbol{\mu_Y}$ be a fuzzy topological spaces and $\boldsymbol{P}: \boldsymbol{\mu_X} \to \boldsymbol{\mu_Y}$ be a fuzzy continuous mapping .A fuzzy set $\boldsymbol{A} \subset \boldsymbol{\mu_Y}$ is said to be evenly fuzzy covered by $\boldsymbol{P}$ if $\boldsymbol{A}$ is fuzzy connected and open fuzzy set ,and each fuzzy component of $\boldsymbol{P}^{-\mathbf{1}}(\boldsymbol{A})$ is an open set that is mapped fuzzy homeomorphically onto $\boldsymbol{A}$ by $\boldsymbol{P}$.

Definition (1.3): A fuzzy fiber structure is a triple ($\boldsymbol{\mu_X}, \boldsymbol{P}, \boldsymbol{\mu_Y}$) consisting of two fuzzy topological spaces $\boldsymbol{\mu_X}$, $\boldsymbol{\mu_Y}$ and a fuzzy continuous surjection $\boldsymbol{P}: \boldsymbol{\mu_X} \to \boldsymbol{\mu_Y}$. The space $\boldsymbol{\mu_X}$ is called fuzzy total [or fuzzy fibered] space,$\boldsymbol{P}$ is termed the projection, and $\boldsymbol{\mu_Y}$ is a fuzzy base space for each $\in \boldsymbol{\mu_Y}$. Then $\boldsymbol{F} = \boldsymbol{P}^{-\mathbf{1}}(\boldsymbol{y})$and $\boldsymbol{F}$ is called fuzzy fiber over $\boldsymbol{y}$ .We refer to $(\boldsymbol{\mu_X}, \boldsymbol{P}, \boldsymbol{\mu_Y})$ as a fuzzy fiber structure over $\boldsymbol{P}$. [10], [11], [12]

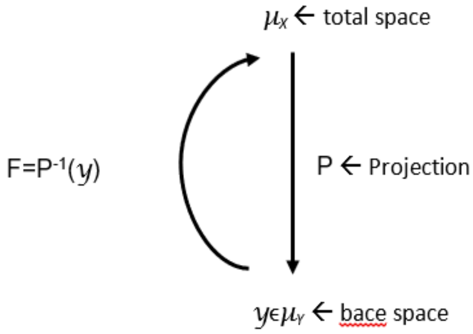

FIGURE 1. Fuzzy Fiber Structure

Definition (1.4): Let $\boldsymbol{P}: \boldsymbol{\mu_X} \to \boldsymbol{\mu_Y}$ be a fuzzy map. We say that $\boldsymbol{P}$ has Fuzzy Homotopy Lifting Property (F.H.L.P) with respect to $\boldsymbol{\mu_Y}$ if and only if given a fuzzy map $\boldsymbol{u}: \boldsymbol{\mu_Z} \to \boldsymbol{\mu_X}$ and a fuzzy homotopy $\boldsymbol{h_t}: \boldsymbol{\mu_Z} \to \boldsymbol{\mu_Y}$ such that $\boldsymbol{P} \circ \boldsymbol{u} = \boldsymbol{h_0}$. Then there exists a fuzzy homotopy $\boldsymbol{h_t^*}: \boldsymbol{\mu_Z} \to \boldsymbol{\mu_X}$ such that:

$$h_0^* = u \tag{1}$$

$$P \circ h_0^* = h_t \qquad \forall y \in \mu_Y \qquad and\ t \in I \tag{2}$$

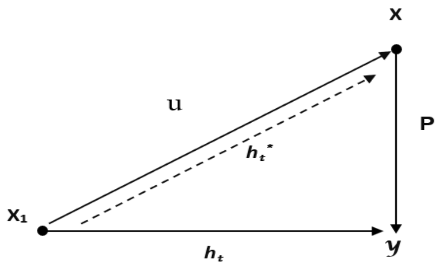

FIGURE 2. Fuzzy Homotopy Lifting Property

Definition (1.5): Let $\boldsymbol{\mu_X}, \boldsymbol{\mu_Y}$ be two fuzzy topological spaces. A fuzzy fiber structure ($\boldsymbol{\mu_X}, \boldsymbol{P}, \boldsymbol{\mu_Y}$) is called fuzzy fiber space or "fuzzy fibration" for class $\boldsymbol{\Re}$ of fuzzy spaces if $\boldsymbol{P}$ has the "fuzzy homotopy lifting property" (F.H.L.P) for each $\boldsymbol{\mu_Y} \in \boldsymbol{\Re}$.

Example (1.6): Let $\boldsymbol{P}: \boldsymbol{\mu_Y} \times \boldsymbol{F} \to \boldsymbol{\mu_Y}$ be a fuzzy projection. Then $\boldsymbol{P}$ is a fuzzy fibration. Moreover for $\boldsymbol{y} \in \boldsymbol{\mu_Y}$, the fuzzy fiber over $\boldsymbol{\mu_Y}$ is a fuzzy homeomorphic to $\boldsymbol{F}$ .A fuzzy fibration can be used to fuzzy lift a fuzzy path in $\boldsymbol{\mu_Y}$ to a fuzzy in $\boldsymbol{\mu_X}$ as the following theorem shows.

Theorem (1.7): Let $\boldsymbol{\mu_X}, \boldsymbol{\mu_Y}$ be two fuzzy topological spaces. If $\boldsymbol{P}: \boldsymbol{\mu_X} \to \boldsymbol{\mu_Y}$ is a fuzzy fibration, then any fuzzy path $\boldsymbol{u}$ in $\boldsymbol{\mu_Y}$ with $\boldsymbol{u}(\mathbf{0}) \in \boldsymbol{P}(\boldsymbol{\mu_X})$ can be fuzzy lifted to a fuzzy path in $\boldsymbol{\mu_X}$.

Proof: Suppose that p is one fuzzy point space. We regard u as a fuzzy homotopy.$u: p \times C \to \mu_Y$ where a fuzzy point $x \in \mu_X$ such that $P(x) = u(0)$ corresponds to a fuzzy map:$p \to \mu_X$ such that $p\big(u(\alpha) = u(\alpha, 0)\big)$,where $\alpha \in p$.
Since P is a fuzzy fibration, it has the fuzzy homotopy lifting property and so there exists a fuzzy path v in μ_X such that $v(0) = x$ and $pv = u$ .Hence ,v is a fuzzy lifting of u .

2. THE CONCEPT OF UNIQUE FUZZY PATH LIFTING

Definition (2.1): A fuzzy map $\boldsymbol{P}: \boldsymbol{\mu_X} \to \boldsymbol{\mu_Y}$ is said to have unique fuzzy path lifting if for a fuzzy paths $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{\mu_X}$ such that $\boldsymbol{Pu} = \boldsymbol{Pv}$ and $\boldsymbol{u(0)} = \boldsymbol{v(0)}$,we have $\boldsymbol{u} = \boldsymbol{v}$.

Lemma (2.2): A fuzzy covering map has unique fuzzy path lifting. This result was proven in the second chapter from lemma (2.6.3).

Lemma (2.3): Suppose that $\boldsymbol{P}: \boldsymbol{\mu_X} \to \boldsymbol{\mu_Y}$ has unique fuzzy path lifting. Then $\boldsymbol{P}$ has the unique fuzzy lifting property for fuzzy path connected spaces.

Proof :Let μ_Z be a fuzzy path connected .Let $u, v: \mu_Z \to \mu_X$ be a fuzzy maps such that $Pu = Pv$ .Let $z_0 \in \mu_Z$ such that $u(z_0) = v(z_0)$ .We have to show that $u = v$ .Let z be an arbitrary fuzzy element of μ_Z and let h be a fuzzy path in μ_Z beginning and ending at z_0 and z respectively .
Consider the fuzzy paths uh and vh in μ_X .These are fuzzy lifting of the some fuzzy path in μ_Y and have the some beginning. Since P has unique fuzzy path lifting, we observe that $uh = vh$ .Then $u(z) = (uh)(1) = (vh)(1) = v(z)$.
Since $z \in \mu_Z$ is a fuzzy arbitrary,$u = v$. The following theorem shows certain connection between a fuzzy fibration and unique fuzzy path lifting.

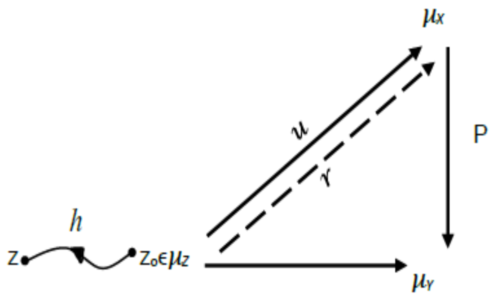

FIGURE 3. Unique Fuzzy Path Lifting

Theorem (2.4): A fuzzy fibration has unique fuzzy path lifting if and only if every fuzzy fiber has no non-null fuzzy path.

Proof :Let μ_X and μ_Y be any fuzzy topological spaces .Let $P: \mu_X \to \mu_Y$ be a fuzzy fibration with unique fuzzy path lifting .Let $y \in \mu_Y$ and assume that u be a fuzzy path in the fuzzy fiber $P^{-1}(y)$ .Let v be a null fuzzy path in $P^{-1}(y)$such that $u(0) = v(0)$ .Then $Pu = Pv$ and this implies $u = v$ and so u is a null fuzzy path . Conversely: Suppose that $P: \mu_X \to \mu_Y$ is a fuzzy fibration such that every fuzzy fiber has no non-null fuzzy path. Let u and v be a fuzzy path in μ_X such that $Pu = Pv$ and $u(0) = v(0)$ .For $t \in C$,let h_t be the fuzzy path in μ_X defined by:

$$h_t = \begin{cases} u\big((1-2x)t\big) & , \quad 0 \le x \le ½ \\ v\big((2x-1)t\big) & , \quad ½ \le x \le 1 \end{cases} \qquad (3)$$

In this way obtain a fuzzy path h_t in μ_X from $u(t)$to $v(t)$such that Ph_t becomes a closed fuzzy path in μ_Y which is fuzzy homotopic relative to C to the null fuzzy path at $P\big(u(t)\big)$ .From the fuzzy homotopy lifting property of P ,we see that there is a fuzzy map $F': C \times C \to \mu_X$ such that $F'(t', 0) = h_t(t')$ and such that F'fuzzy maps $(0 \times C) \cup (C \times 1) \cup (1 \times C)$ to the fuzzy fiber $P^{-1}\Big(P\big(u(t)\big)\Big)$ .By hypotheses , $P^{-1}\Big(P\big(u(t)\big)\Big)$ has no non-null fuzzy paths . Hence, F' fuzzy maps $0 \times C, C \times 1$ and $1 \times C$ to a single fuzzy point and this implies that $F'(0,0) = F'(1,0)$. Thus $h_t(0) = h_t(1)$ and $u(t) = v(t)$. This proves the theorem.
Theorem (2.5): Let $(\boldsymbol{\mu_X}, \boldsymbol{P}, \boldsymbol{\mu_Y})$ and $(\boldsymbol{\mu_L}, \boldsymbol{q}, \boldsymbol{\mu_C})$ be fuzzy fibration then $(\boldsymbol{\mu_X} \times \boldsymbol{\mu_L}, \boldsymbol{P} \times \boldsymbol{q}, \boldsymbol{\mu_Y} \times \boldsymbol{\mu_C})$ is also fuzzy fibration.

Proof: Let μ_X , μ_Y , μ_Z, μ_L and μ_C are fuzzy topological spaces. Let $u: \mu_Z \to \mu_X$ and $u': \mu_Z \to \mu_L$ be any fuzzy maps. Define $u^*: \mu_Z \to \mu_X \times \mu_L$ be a fuzzy map by $u^*(z) = \big(u(z), u'(z)\big)$, and $h_t: \mu_Z \to \mu_Y$ and $h'_t: \mu_Z \to \mu_C$ be any fuzzy maps. Define $h^*_t: \mu_Z \to \mu_Y \times \mu_C$ by $h^*_t(z) = \big(h(z)\, , h'(z)\big)$ and $(P \times q) \circ u^* = h^*_0$. Since P , q are fuzzy fibrations, then there exists $u_t: \mu_Z \to \mu_X$ such that $P \circ u_t = h_t$, $u_0 = u$ and $u'_t: \mu_Z \to \mu_L$ such that $q \circ u'_t = h_t$, $u'_0 = u'$. Now for h^*_t there exists $u^*_t: \mu_Z \to \mu_X \times \mu_L$ define as $u^*_t(z) = (u_t\,(z), u'_t(z))$ such that:

$$(P \times q) \circ u^*_t = h^*_t \qquad (4)$$
$$u^*_0 = u^* \qquad (5)$$

Then $P \times q: \mu_X \times \mu_L \to \mu_Y \times \mu_C$ has fuzzy homotopy lifting property with respect to μ_Z . Therefore $P \times q$ is a fuzzy fibration.

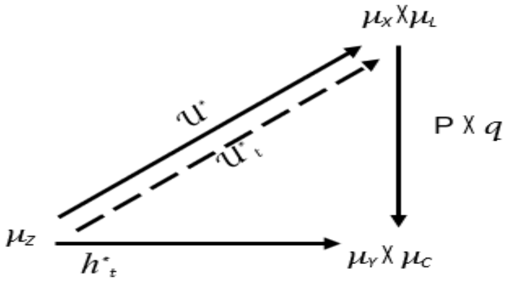

FIGURE 4. The Product Fuzzy Fibretion

Definition (2.6): Let $(\boldsymbol{\mu_X}, \boldsymbol{P}, \boldsymbol{\mu_Y})$ be a fuzzy fiber structure , and let $\boldsymbol{\mu_Z}: (\boldsymbol{\alpha}: \boldsymbol{I} \to \boldsymbol{\mu_Y})$, $\boldsymbol{\gamma_p} \subset \boldsymbol{\mu_X} \times \boldsymbol{\mu_Z}$ be a fuzzy subspace , $\boldsymbol{\gamma_p} = \{(\boldsymbol{x}, \boldsymbol{\alpha}) \in \boldsymbol{\mu_X} \times \boldsymbol{\mu_Z} \,|\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{\alpha}(\boldsymbol{0})\}$ of the Cartesian product . A fuzzy lifting function for $(\boldsymbol{\mu_X}, \boldsymbol{P}, \boldsymbol{\mu_Y})$ is a fuzzy continuous map $\boldsymbol{\lambda}: \boldsymbol{\gamma_P} \to \boldsymbol{\mu_N}$ such that $\boldsymbol{\lambda}(\boldsymbol{x}, \boldsymbol{\alpha})(\boldsymbol{0}) = \boldsymbol{x}$ ***and*** $\boldsymbol{P} \circ \boldsymbol{\lambda}(\boldsymbol{x}, \boldsymbol{\alpha})(\boldsymbol{t}) = \boldsymbol{\alpha}(\boldsymbol{t})$ for each $(\boldsymbol{x}, \boldsymbol{\alpha}) \in \boldsymbol{\gamma_P}$ ***and*** $\boldsymbol{t} \in \boldsymbol{I}$, we say that $\boldsymbol{\lambda}$ is a fuzzy regular if $\boldsymbol{\lambda}(\boldsymbol{x}, \boldsymbol{\alpha})$ is a constant fuzzy path whenever $\boldsymbol{\alpha}$ is a constant fuzzy path .

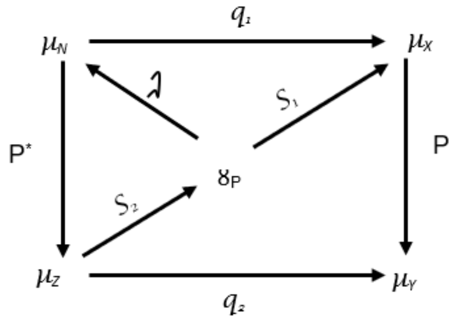

FIGURE 5. Fuzzy Lifting Function

Thus, a fuzzy lifting function associated to each $x \in \mu_X$ and a fuzzy path α in μ_Y starting at (x) , fuzzy path $\lambda(x, \alpha)$in μ_X starting at x , that is lift of α . Since the c-fuzzy topology is used in μ_N , the fuzzy continuity of λ is fuzzy equivalence to that of fuzzy associated $\lambda: \gamma_P \times I \to \mu_X$, by sampling : $\lambda(x, \alpha) \in \mu_N$and $\lambda(x, \alpha): I \to \mu_X$ $\lambda(x, \alpha)(0) = x$, $P \circ \lambda(x, \alpha)(t) = \alpha(t)$. $P^*(\alpha): I \to \mu_Y$, $P^*(\alpha)(t) = P\big(\alpha(t)\big)$ $\lambda: \gamma_P \times I \to \mu_X$ such that $(x, \alpha)(t) = \big(\lambda(x, \alpha)\big)(t)$.

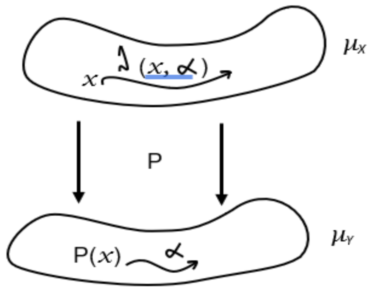

FIGURE 6. Fuzzy Path

Remark (2.7): A fuzzy map $\boldsymbol{P}: \boldsymbol{\mu_X} \to \boldsymbol{\mu_Y}$ is a fuzzy fibration if and only if there exist a fuzzy lifting function for $\boldsymbol{P}$.

Theorem (2.8): The fuzzy fiber structure $(\boldsymbol{\mu_X}, \boldsymbol{P}, \boldsymbol{\mu_Y})$ is a fuzzy fibration if and only if a fuzzy lifting function exists.

Proof: If P is a fuzzy fibration. Let $\mu_Z = \gamma_P$ and $u: \gamma_P \to \mu_X$ and $h_t: \gamma_P \to \mu_Y$, defined by $u(x, \alpha) = x$ *and* $h_t(x, \alpha) = \alpha(t)$ then $h_0(x, \alpha) = \alpha(0) = P(x) = P \circ u(x, \alpha)$.

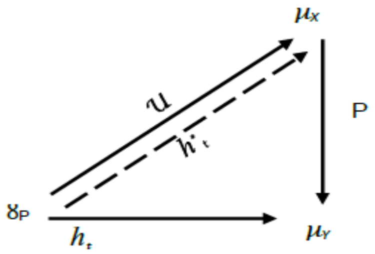

FIGURE 7. Fuzzy Fibration

There exist a fuzzy map $h_t^*: \gamma_P \to \mu_X$ be a fuzzy homotopy lifting such that $h_0^*(x,\alpha) = u(x,\alpha) = x$ and $P \circ h_t^* = h_t$. h_t^* defines a fuzzy lifting function λ for P by $\lambda(x,\alpha)(t) = h_t^*(x,\alpha)$. λ is a fuzzy lifting function which is whenever h_t^* is stationary with h_t .

Conversely: If P has a fuzzy lifting function. Let $u: \mu_Z \to \mu_X$ be given and $h_t: \mu_Z \to \mu_Y$ be a fuzzy homotopy such that $P \circ u = h_0$, $for\ each\ z \in \mu_Z$, let $\alpha_z: I \to \mu_Y$ be defined by $\alpha_z(t) = h_t(z)$. Defined a fuzzy map $h_t^*: \mu_Z \to \mu_X$ as follows: $h_t^*(z) = \lambda(u(z), \alpha_z)(t)$ then:

$$h_0^*(z) = u(z) \quad (6)$$

$$P \circ h_t^* = h_t \quad (7)$$

Therefore P has a fuzzy fibration.

CONFLICTS OF INTEREST

In this paper we have some result as shown below: We illustrated the concept of fuzzy set and fuzzy covering space in homotopy theory. Also, we gave a new definition of fuzzy homotopy lifting property and fuzzy fibration. In additions we proved a fuzzy fibration has unique fuzzy path lifting. And we have a new definition of fuzzy lifting function and prove the fuzzy fiber structure is a fuzzy fibration.

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