On Semi Feebly Separation Axioms

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Abstract
The goal of this work is to present some new separation axioms based on the concept of defining new types of open sets namely semi-feeably open set. We investigate their fundamental features.

Keywords: semi-feeably-$T_0$, semi-feeably-$T_1$

1 Introduction

Separation qualities are a standout amongst the most vital and fascinating concepts in topology. In 1963, N. Levin [10] proposed concept of a semi-open set. S.N Maheshwari and R. Prasad [9], used semi-open sets to characterize and investigate new partition aphorisms known as semi-detachment aphorisms. In 1975, Maheshwari and et.al. [8] created semi-$R_0$. P. Bahattacharya B. K. Lahiri [7] summarized up of shut sets to semi-summed up shut sets using semi-receptiveness in 1987. Cueva M. C characterized the idea of new type of topological space called semi-$T_{1/2}$ in 2000 [6] (i.e. the space where the semi-closed sets and semi-summed up sets classes meet). Although none these applications reversible, it is proved that each semi-$T_1$ space is semi-$T_{1/2}$ and each semi-$T_{1/2}$ is semi-$T_0$. Maheshwari and et. al. [5] initiated the study of feeably open in 1978. Aad Aziz Hussan Abdulla in [1] presented the idea of semi-feeably open (sf-open) set. “the goal of this study is to provide some characterizations of semi-feeably separation axioms”.

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2 Preliminaries

Definition 2.1.[1]
Let \((X, \tau)\) be a topological space. A subset \(A\) of \(X\) is said to be
(1) semi-feebly open set if \(\overline{A} \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open set.
i.e. \(\forall U\) is semi-open in \(X\) \((A \subseteq U \longrightarrow \overline{A} \subseteq U)\).
(2) the complement semi-feebly open set is said semi-feebly closed set.

Remark 2.2.[1]
If \(A\) is \(f\)-closed set, then \(A\) is \(sf\)-open set.

Proof.
let \(A\) be \(f\)-closed set in a topological space \(X\). \(A \subseteq U\). \(U\) s-open,
Since \(A\) is \((f\)-closed\) set then \(A = \overline{A^f}\) and \(A = \overline{A} \subseteq U\)
Hence \(A\) is \((sf\)-open\) set.

Remark 2.3[1]
If \(A\) is closed set, then \(A\) is \(sf\)-open set.

Proposition 2.4.[1]
If \(A_\lambda\) is a family of \(sf\)-open set, then \(\bigcup A_\lambda\) is \(sf\)-open set.

Proposition 2.5.[1]
Let \(X\) is a topological space and \(A, B \subseteq X\), then
1. \(A\) is an \(sf\)-closed set if and only if \(A = \overline{A^{sf}}\).
2. \(\overline{A^{sf}} \subseteq \overline{A}\).
3. \(\overline{A^{sf}} = \overline{(\overline{A})^{sf}}\).
4. If \(A \subseteq B\) then \(\overline{A^{sf}} \subseteq \overline{B^{sf}}\).

Lemma 2.6.
Let $X$ is a topological space and $A \subseteq X$, then

$$\bar{A}^s \subseteq \bar{A}^{sf}$$

**Proof.**

Let $x \in \bar{A}^s$ and $A$ is a $s$-closed set, then $A = \bar{A}^s \Rightarrow x \in A \subseteq \bar{A}^{sf}$.

Then $x \in \bar{A}^{sf}$.

Therefore $\bar{A}^s \subseteq \bar{A}^{sf}$.

3 **Lower separation axioms**

**Definition 3.1.**

A topological space $(X, \tau)$ is $sf-R_0$ if for each $sf$-open set $U, x \in U$ implies that $\overline{\{x\}}^{sf} \subset U$.

**Lemma 3.2.**

If a space $X$ is $sf-R_0$, then for every $sf$-open set $U$ and each $x \in U$,

$$\overline{\{x\}}^{s} \subset U.$$

**Proof.**

Let $X$ be $sf-R_0$. Then for every $sf$-open set $U$ and each $x \in U$, $\overline{\{x\}}^{sf} \subset U$. But by $\overline{\{x\}}^{sf} \subset \overline{\{x\}}^{s}$ and $\overline{\{x\}}^{s} = \overline{\{x\}}^{s} \cup \overline{\{x\}}^{s}$ by [[11] Proposition(1.1.19)], this implies that $\overline{\{x\}}^{s} \subset U$.

**Definition 3.3.**

A topological space $(X, \tau)$ is $sf-R_1$ if for each $x, y \in X$ with $\overline{\{x\}}^{sf} \neq \overline{\{y\}}^{sf}$, there exist disjoint $sf$-open set $U$ and $V$ such that $\overline{\{x\}}^{sf} \subset U$ and $\overline{\{y\}}^{sf} \subset V$.

**Theorem 3.4.**
If a topological space \((X, \tau)\) is sf-\(R_1\), then \((X, \tau)\) is sf-\(R_0\).

**Proof.**

Let \(U\) be a sf-open set such that \(x \in U\). If \(y \notin U\), then \(x \notin \overline{\{y\}}\) \(\text{sf}\), therefore \(\overline{\{x\}} \neq \overline{\{y\}}\). So, there exists a sf-open set \(V\) such that \(\overline{\{y\}} \subset V\) and \(x \in V\), which implies \(y \notin \overline{\{x\}}\). Hence \(\overline{\{x\}} \subset U\). Therefore \((X, \tau)\) is sf-\(R_0\).

**Theorem 3.5.**

A topological space \((X, \tau)\) is sf-\(R_0\) if and only if for every sf-closed set \(F\) and \(x \notin F\), there exist a sf-open set \(G\) such that \(F \subset G\) and \(x \notin G\).

**Proof.**

Let \((X, \tau)\) is sf-\(R_0\) and \(F\) is sf-closed set in \(X\) and \(x \notin F\). Then \(X \setminus F\) is sf-open set containing \(x\), since \((X, \tau)\) is sf-\(R_0\) implies that \(\overline{\{x\}} \subset X \setminus F\) and then \(F \subset X \setminus \overline{\{x\}}\).

Now let \(G = X \setminus \overline{\{x\}}\), then \(G\) is sf-open set not contains \(x\) and \(F \subset G\).

Conversely: Let \(x \in G\) where \(G\) is sf-open set in \(X\). Then \(X \setminus G\) is sf-closed set and \(x \notin X \setminus G\) implies that by hypothesis there exists sf-open set \(U\) such that \(x \notin U\) and \(X \setminus G \subset U\). Now \(X \setminus U \subset G\) and \(x \in X \setminus U\), but \(X \setminus U\) sf-closed set then \(\overline{\{x\}} \subset X \setminus U \subset G\) this implies that \((X, \tau)\) is sf-\(R_0\).

**Theorem 3.6.**

For a space \(X\), the following are equivalent:

1. \(X\) is sf-\(R_0\).

2. For any two points \(x\) and \(y\) in \(X\), \(x \notin \overline{\{y\}}\) if and only if \(y \notin \overline{\{x\}}\).

**Proof.**

(1) \(\Rightarrow\) (2). Let \(X\) is sf-\(R_0\) and \(x \in \overline{\{y\}}\). To show \(y \notin \overline{\{x\}}\), let \(V\) be any sf-open set containing \(y\). Since \(X\) is sf-\(R_0\) so \(\overline{\{y\}} \subset V\) implies that \(x \in V\), hence every sf-open set which containing \(y\) contains \(x\) this implies that \(y \notin \overline{\{x\}}\). By the same way we can prove that if \(y \notin \overline{\{x\}}\), then \(x \notin \overline{\{y\}}\).
(2) ⇒ (1). Let the hypothesis be satisfied and $U$ be any sf-open set and $x \in U$. To show \([x]^{sf} \subset U\), let $y \in [x]^{sf}$ implies that by hypothesis $x \in [y]^{sf}$, and then $U \cap \{y\} \neq \emptyset$ this implies that $y \in U$. Thus \([x]^{sf} \subset U\), therefore $X$ is sf-$R_0$.

**Theorem 3.7.**

A space $X$ is sf-$R_0$ if and only if for any $x$ and $y$ in $X$ if \([x]^{sf} \neq [y]^{sf}\), then \([x]^{sf} \cap [y]^{sf} = \emptyset\).

**Proof.**

Let $X$ be sf-$R_0$ and $x, y \in X$ such that \([x]^{sf} \neq [y]^{sf}\). Then there exists $z \in [x]^{sf}$ such that $z \notin [y]^{sf}$ implies that there exists an sf-open set $U$ containing $z$ but not $y$, hence $x \in [x]^{sf}$. Therefore we have $x \notin [y]^{sf}$ implies that $x \in X \setminus [y]^{sf}$ which is an sf-open set, but $X$ is sf-$R_0$ so \([x]^{sf} \subset X \setminus [y]^{sf}\) this implies that \([x]^{sf} \cap [y]^{sf} = \emptyset\).

Conversely. Let the hypothesis be satisfied and let $U$ be any sf-open set in $X$ and $x \in U$. If $U = X$, then clearly \([x]^{sf} \subset U\), but if $U \neq X$, then there exists $y \in X$ such that $y \notin U$. Now $x \neq y$ and $x \notin [y]^{sf}$ implies that \([x]^{sf} \neq [y]^{sf}\), then by hypothesis \([x]^{sf} \cap [y]^{sf} = \emptyset\) implies that $y \notin [x]^{sf}$. Thus if $y \notin U$, then $y \notin [x]^{sf}$ this implies that \([x]^{sf} \subset U\). Hence $X$ is sf-$R_0$.

**Definition 3.8.**

Let \((X, t)\) is a topological space. If for each $a, b \in X$ where $a \neq b$ there exists a semi-feebly-open set $W$ of $X$ containing $a$ but not $b$, we say that $X$ is semi-feebly-$T_0$ space.

**Theorem 3.9.**

Let \((X, t)\) is a topological space. We say that $X$ is sf-$T_0$-space if and only if

for every $x, y \in X, x \neq y$. Implies \([x]^{sf} \neq [y]^{sf}\)

**Proof.**

Let $x, y \in X$ with $x \neq y$ and $X$ is sf-$T_0$-space. We shall show that \([x]^{sf} \neq [y]^{sf}\). Since $X$ is sf-$T_0$-space, there exists a sf-open set $U$ such that $x \in U$ but $y \notin U$. Also $x \notin X \setminus U$ and $y \notin X \setminus U$ where $X \setminus U$ is sf-closed set in $X$. Now by definition \([y]^{sf}\) is the intersection of all sf-closed set which contain $y$. Hence, $y \in [y]^{sf}$ but $x \notin [y]^{sf}$ as
$x \not\in X \setminus U$. Therefore, $\overline{x}^{sf} \neq \overline{y}^{sf}$.

Conversely, for any $x, y \in X, x \neq y$. And $\overline{x}^{sf} \neq \overline{x}^{sf}$. Then there exists at least one point such that $z \in X$ such that $z \in \overline{x}^{sf}$ but $z \notin \overline{y}^{sf}$.

We claim that $x \notin \overline{y}^{sf}$. If $x \in \overline{y}^{sf}$ then $\{x\} \subseteq \overline{y}^{sf}$ implies $\overline{x}^{sf} \subseteq \overline{y}^{sf}$. So, $z \in \overline{y}^{sf}$, which is a contradiction. Hence, $x \notin \overline{y}^{sf}$. Now, $x \notin \overline{y}^{sf}$ implies $x \in X \setminus \overline{y}^{sf}$ is sf-open in $X$ but $y \notin X \setminus \overline{y}^{sf}$ Observe $X$ that is sf-$T_0$-space.

**Proposition 3.10.**

Whenever $X$ is sf-$T_0$-space, then each subspace of $X$ is sf-$T_0$-space.

**Proof.**

Consider $X$ as a sf-$T_0$-space and $Y \subseteq X$. Take $\alpha$ and $\beta$ as unequal points of $Y$. As $Y \subseteq X$, $\alpha$ and $\beta$ are also unequal points of $X$. As per given, $X$ is sf-$T_0$-space, we have a sf-open set $K$ so that $\alpha \in K, \beta \notin K$. Then we have $Y \cap K$ is sf-open in $Y$ having $\alpha$ but not $\beta$. Thus $Y$ is sf-$T_0$-space.

**Definition 3.11.**

A subset $A$ of a topological space $(X, \tau)$ is called to be semi-feebly generalized closed set (written in short as sfg-closed) if, $\overline{A}^{sf} \subseteq O$ hold whenever $A \subseteq O$ and $O$ is sf-open.

**Proposition 3.12.**

Every sf-closed set is sfg-closed set.

**Definition 3.13.**

A topological space $(X, \tau)$ is called sf-$T_{1/2}$ if every sfg-closed set in $(X, \tau)$ is sf-closed set in $(X, \tau)$.

**Definition 3.14.**

That for any subset $E$ of $(X, \tau), \overline{E}^{sf} = \cap\{A : A \subseteq (\in sfD(X, \tau))\}$, where
sfD(X, τ) = \{ A : A \subset X and A is sfg-closed in (X, τ) \} and
sfO(X, τ)^* = \{ B : \overline{E^{sf}} = E^c \}.

**Theorem 3.15.**

A topological space (X, τ) is a sf-T_{1/2} space if and only if

\[ sfO(X, τ) = sfO(X, τ)^* \]

**Proof.**

Necessity: Since the sf-open sets and the sfg-closed sets coincide by the assumption, \( E^{sf} = \overline{E^{sf}} \) holds for every subset E of (X, τ).

Therefore, we have \( sfO(X, τ) = sfO(X, \tau)^* \).

Sufficiency: Let \( A \) be a sfg-closed set of (X, \tau). Then, we have \( A = \overline{A}^{sf} \) and \( A^c \) is sf-open set in (X, \tau). Thus \( A \) is sf-closed set. Therefore (X, \tau) is sf-T_{1/2}.

**Theorem 3.16.**

A topological space is sf-T_{1/2} space if and only if each \( x \in X, \{ x \} \) is sf-open or \( \{ x \} \) is sf-closed.

**Proof.**

Necessity: Suppose that for some \( x \in X, \{ x \} \) is not sf-closed. Since X is the only sf-open set containing \( \{ x^c \} \), the set \( \{ x^c \} \) is sfg-closed and so it is sf-closed in the sf-T_{1/2} space (X, τ). Therefore \( \{ x \} \) is sf-open.

Sufficiency: Since \( sfO(X, \tau) \subset sfO(X, \tau)^* \) holds, we show \( sfO(X, \tau)^* \subset sfO(X, \tau) \). Let \( E \in sfO(X, \tau)^* \)

Suppose that \( E \notin sfO(X, \tau) \). Then, \( \overline{E^{sf}} = E^c \) and \( \overline{E^{sf}} \neq E^c \) hold. There exists a point \( x \) of X such that \( x \in E^c \) and \( x \notin E^c = \overline{E^{sf}} \). Since \( x \notin \overline{E^{sf}} \), there exists a sfg-closed set \( A \) such that \( x \notin A \) and \( A \supset E^c \). By the hypothesis,
the singleton \{x\} is sf-open or sf-closed.

Case (1). \{x\} is sf-open. Since \{x^c\} is a sf-closed set with \(E \subseteq \{x^c\}\),
we have \(E^{sf} \subseteq \{x^c\}\), i.e., \(x \notin E^{sf}\). This contradicts the fact that \(x \in E^{sf}\).

Therefore \(E \in sfO(\mathbb{X}, \tau)\).

Case (2). \{x\} is sf-closed. Since \(\{x^c\}\) is a sf-open set containing the sf-closed
set \(A \supset E^c\), we have \(\{x^c\} \supset A^{sf} \supset E^{sf}\). Therefore \(x \notin E^{sf}\). This is a
contradiction. Therefore \(E \in sfO(\mathbb{X}, \tau)\).

Hence in both cases, we have \(E \in sfO(\mathbb{X}, \tau)\), i.e., \(sfO(\mathbb{X}, \tau)^* \subseteq sfO(\mathbb{X}, \tau)\).

**Corollary 3.17.**

\(\mathbb{X}\) is sf-\(T_{1/2}\) if and only if every subset of \(\mathbb{X}\) is the intersection of all sf-open
sets and all sf-closed sets containing it.

**Proof.**

Necessity: let \(\mathbb{X}\) is sf-\(T_{1/2}\) with \(B \subseteq \mathbb{X}\) arbitrary. Then \(B = \{\{x\}\}, x \notin B\), an
intersection of sf-open and sf-closed[Theorem(3.16)]. The result follows.

Sufficiency: for \(x \in \mathbb{X}, \{x\}^c\) is the intersection of all sf-open sets and all
sf-closed sets containing it. Thus \(\{x\}^c\) is either sf-open or sf-closed and
\(\mathbb{X}\) is sf-\(T_{1/2}\).

**Definition 3.18.**

A space \((\mathbb{X}, \tau)\) is called a sf-\(T_{1/4}\) space if for every finite subset \(F \subseteq \mathbb{X}\) and
every point \(y \notin F\) there exists a subset \(A \subseteq \mathbb{X}\) such that \(F \subseteq A, y \notin A\)
and \(A\) is sf-open or sf-closed.

**Proposition 3.19.**

Let \((\mathbb{X}, \tau)\) be a sf-\(T_{1/4}\) space. Then every subspace of \(\mathbb{X}\) is a sf-\(T_{1/4}\) space.

Recall that a subset \(F\) of a space \((\mathbb{X}, \tau)\) sf-locally finite if every point
has an sf-open neighborhood \( U_x \) such that \( F \cap U_x \) is at most finite. \( x \in X \)

**Theorem 3.20.**

For a space \((X, \tau)\) the following are equivalent:

1. \((X, \tau)\) is a sf-\( T_{1/2} \) space,
2. For every sf-locally finite subset \( F \subset X \) and every point \( y \notin F \) there exists a subset \( A \subset X \) such that \( F \subset A \), \( y \notin A \) and \( A \) is sf-open or sf-closed.

**Definition 3.21.**

Let \((X, \tau)\) is a topological space. Then \( X \) is sf-\( T_1 \)-space if for each \( a, b \in X \) such that \( a \neq b \) there exists a sf-open set \( W \) of \( X \) containing \( a \) but not \( b \) and a sf-open set \( U \) of \( X \) containing \( b \) but not \( a \).

**Remark 3.22.**

Every sf-\( T_1 \)-space is sf-\( T_0 \) space.

**Proof.**

From the definition of sf-\( T_1 \)-space it is follows that it is sf-\( T_0 \), since there exists a sf-open set \( G \) such that \( x \in G \) but \( y \notin G \) the converse is not true.

**Corollary 3.23.**

Every sf-\( T_0 \)-space is not sf-\( T_1 \) space.

The following example supports this.

**Example 3.24.**

Let \( X = \{1, 2, 3\} \), \( \tau = \{X, \emptyset, \{1\}\} \) be a topology defined on \( X \). Here sf-open sets are \( \{X, \emptyset, \{2\}, \{3\}, \{2, 3\}\} \). It is clear \( X \) is sf-\( T_0 \) space but is not sf-\( T_1 \) since \( 1 \neq 2 \) and there exist sf-open set contain 2 but there is not exist sf-open set such that containing 1 but not 2.

**Proposition 3.25.**
$X$ is sf-$T_1$ if and only if for all $x \in X$ implies $\{x\}$ is sf-closed sets.

**Proof.**

Let $\{z\}$ sf-closed set for every $z \in X$. Let $x, y \in X$ such that $x \neq y$.

Then $x \in \{y\}^c$ and $\{y\}$ is sf-closed set. Therefore $\{y\}^c$ sf-open set containing $x$ but not $y$ and, and $\{x\}^c$ sf-open set containing $y$ but not $x$.

Then $X$ is sf-$T_1$

Conversely, let $X$ be a sf-$T_1$ space and $y \in X$. To prove $\{y\}$ is sf-closed set. Let $x \in \{y\}^c$ then $x \neq y$. Since $X$ sf-$T_1$, then there exists sf-open set in $X, U$ such that $x \in U$ and $x \notin U$. Then $x \in U \subset \{y\}^c = \bigcup \{U_x : x \in \{y\}^c\}$ which is sf-open set. Hence $\{x\}$ is sf-closed set.

**Theorem 3.26.**

A space $X$ is sf-$T_1$ if and only if it is sf-$T_0$ and sf-$R_0$.

**Proof.**

Let $X$ be sf-$T_1$ space. Then from [Remark (3.22)] $X$ is sf-$T_0$ and by [Proposition (3.25)] every singleton set in $X$ is sf-closed. Now $X$ is sf-$R_0$ space since for any $x \in U$, where $U$ is sf-open set, $\overline{x}^{sf} = \{x\} \subset U$.

Thus the space $X$ is sf-$R_0$.

Conversely, let $x, y \in X$ be any two distinct points. Since $X$ is sf-$T_0$ so there exists an sf-open set $U$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

Now let $x \in U$ and $y \notin U$ and since $X$ is sf-$R_0$ space so $\overline{x}^{sf} \subset U$ and we have $y \notin U$ implies that $y \notin \overline{x}^{sf}$, then $y \in X \setminus \overline{x}^{sf}$ which is sf-open set so take $V = X \setminus \overline{x}^{sf}$. Thus $U$ and $V$ are sf-open sets in $X$ such that $x \in U, y \in V$ and $x \notin V$ and $y \notin U$, implies that $X$ is sf-$T_1$ space.
Definition 3.27[2]
A subset $A$ of a topological space $(X, \tau)$ is called sg-closed set if,
\[ \overline{A}^s \subset O \text{ hold whenever } A \subset O \text{ and } O \text{ is s-open of } (X, \tau), \]
the complement of a sg-closed set is called a sg-open set.

Definition 3.28[3]
A subset $A$ of a topological space $(X, \tau)$ is called $\Psi$-closed set if,
\[ \overline{A}^\Psi \subset O \text{ hold whenever } A \subset O \text{ and } O \text{ is sg-open of } (X, \tau). \]

Theorem 3.29[4]
Let $A$ be a subset of topological space $(X, \tau)$, then
1) $A$ is $\Psi$-closed if and only if $\overline{A}^\Psi \setminus A$ does not contain any non-empty sg-closed set.
2) If $A$ is $\Psi$-closed and $A \subset B \subset \overline{A}^\Psi$, then $B$ is $\Psi$-closed

Definition 3.30.
A space $(X, \tau)$ is said to be a sf-$T_{1/3}$ space if every $\Psi$-closed set in $(X, \tau)$ is sf-closed.

Theorem 3.31.
For a topological space $(X, \tau)$, the following conditions are equivalent:
(i) $(X, \tau)$ is a sf-$T_{1/3}$ space.
(ii) Every singleton of $X$ is either sg-closed or sf-open set.
(iii) Every singleton of $X$ is either sg-closed or open set.

Proof.
(i) $\Rightarrow$ (ii) let $x \in X$ and suppose that $\{x\}$ is not sg-closed of $(X, \tau)$. Then
$X\setminus\{x\}$ is not sg-open set. so, $X$ is the only sg-open set containing $X\setminus\{x\}$.
Hence $X\setminus\{x\}$ is $\Psi$-closed set. Since $(X, \tau)$ is sf-$T_{1/3}$ space, then $X\setminus\{x\}$ is
a $\text{sf}$-closed set or equivalently $\{x\}$ is $\text{sf}$-open set.

(ii) $\Rightarrow$ (i) let $A$ be a $\Psi$-closed set. clearly $A \subset \overline{A}$. let $x \in X$. By Assumption, $\{x\}$ is either $\text{sg}$-closed or $\text{sf}$-open.

Case(1) suppose $\{x\}$ is $\text{sg}$-closed.\[\text{Theorem(3.29)}\] $\overline{A} - A$ does not contain any non-empty $\text{sg}$-closed set. Since $x \in \overline{A}$, then $x \in A$.

Case(2) suppose $\{x\}$ is a $\text{sf}$-open set. Since $x \in \overline{A} \text{sf}$, then $\{x\} \cap A \neq \emptyset$.

So $x \in A$. Thus in any case $\overline{A} \text{sf} \subset A$.

Therefore $A = \overline{A} \text{sf}$ or equivalently $A$ is $\text{sf}$-closed set of $(X, \tau)$.

Hence $(X, \tau)$ is an $\text{sf}-T_{1/3}$ space.

(iii) $\iff$ (ii) Follows from the fact that a singleton is $\text{sf}$-open if and only if it is open.

4 Some new separation axioms

Definition 4.1.

Let $(X, \tau)$ be a topological space. Let $A \subset X$ we say that $A$ is semi-feebly $\text{-Difference (sf-D)}$ set if there exists $U, V$ are $\text{sf}$-open set such that $U \neq X$ and $A = U \setminus V$.

Remark 4.2.

Every $\text{sf}$-open set $U \neq X$ is $\text{sf-D}$-set if $A = U$ and $V = \emptyset$

Corollary 4.3.

Every $\text{sf-D}$-set is not $\text{sf}$-open set.

The following example shows.

Example 4.4.

Let $X = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$. So,
sf-open set are \( \emptyset, X, \{1, 3, 4\}, \{1, 2, 4\}, \{3, 4\}, \{1, 4\}, \{4\}, \{1, 3\}, \{1\}, \{3\}, \{1, 2\} \), then \( U = \{1, 2, 4\} \neq X \) and \( V = \{1, 3, 4\} \) are sf-open sets in \( X \) and \( A = U \setminus V = \{1, 2, 4\} \setminus \{1, 3, 4\} = \{2\} \), then we have \( A = \{2\} \) is a sf-\( D \)-set but it is not sf-open set.

**Definition 4.5.**

A topological space \((X, \tau)\) is said to be:

1. sf-\( D_0 \) if for any pair of distinct points \(x\) and \(y\) of \(X\) there exists a sf-\( D \)-set of \(X\) containing \(x\) but not \(y\) or a sf-\( D \)-set of \(X\) containing \(y\) but not \(x\).

2. sf-\( D_1 \) if for any pair of distinct points \(x\) and \(y\) of \(X\) there exists a sf-\( D \)-set of \(X\) containing \(x\) but not \(y\) and a sf-\( D \)-set of \(X\) containing \(y\) but not \(x\).

3. sf-\( D_2 \) if for any pair of distinct points \(x\) and \(y\) of \(X\) there exist disjoint sf-\( D \)-set \(G\) and \(E\) of \(X\) containing \(x\) and \(y\), respectively.

**Remark 4.6.**

For a topological space \((X, \tau)\), the following properties hold:

1. If \((X, \tau)\) is sf-\( T_k \), then it is sf-\( D_k \), for \(k = 0, 1, 2\).

2. If \((X, \tau)\) is sf-\( D_k \), then it is sf-\( D_{k-1} \), for \(k = 1, 2\).

**Proof.**

It follows from [Remark (4.2)] and [Definition (4.5)].

**Proposition 4.7.**

A space \(X\) is sf-\( D_0 \) if and only if it is sf-\( T_0 \).

**Proof.**

Suppose that \(X\) is sf-\( D_0 \). Then for each distinct pair \(x, y \in X\), at least one of \(x, y\), say \(x\), belongs to sf-\( D \)-set \(G\) but \(y \notin G\). Let \(G = U_1 \setminus U_2\) where \(U_1 \neq X\) and \(U_1, U_2\) are sf-open set. Then \(x \in U_1\), and for \(y \notin G\) we have two cases: (a) \(y \notin U_1\), (b) \(y \in U_1\) and \(y \in U_2\).

In case (a), \(x \in U_1\) but \(y \notin U_1\).

In case (b), \(y \in U_2\) but \(x \notin U_2\).
Thus in both the cases, we obtain that $X$ is sf-$T_0$.

Conversely, if $X$ is sf-$T_0$, by [Remark (4.6) (1)], $X$ is g sf-$D_0$.

**Proposition 4.8.**

A space $X$ is sf-$D_1$ if and only if it is sf-$D_2$.

**Proof.**

**Necessity.** Let $x, y \in X, x \neq y$. Then there exist sf-$D$-sets $G_1, G_2$ in $X$ such that $x \in G_1, y \notin G_1$ and $y \in G_2, x \notin G_2$. Let $G_1 = U_1 \setminus U_2$ and $G_2 = U_3 \setminus U_4$, where $U_1, U_2, U_3$ and $U_4$ are sf-open sets in $X$. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_4$. We discuss the two cases separately.

(i) $x \notin U_3$. By $y \notin G_1$ we have two sub-cases:

(a) $y \notin U_1$. Since $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$, and since $y \in U_3 \setminus U_4$, we have $x \in U_3 \setminus (U_1 \cup U_4)$. Therefore $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \emptyset$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \setminus U_2$, and $y \in U_2$. Therefore $(U_1 \setminus U_2) \cap U_2 = \emptyset$.

(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \setminus U_4$ and $x \in U_4$. Hence $(U_3 \setminus U_4) \cap U_4 = \emptyset$. Therefore $X$ is sf-$D_2$.

**sufficiency.** Follows from [Remark (4.6)(2)].

**Corollary 4.9.**

If $(X, \tau)$ is sf-$D_1$, then it is sf-$T_0$.

**Proof.**

Follows from [Remark (4.6) (2)] and [Proposition (4.7)].

**Remark 4.10.**

Here is an example which shows that the converse of [Corollary (4.9)] is not true in general.

**Example 4.11.**

Let $X = \{1, 2\}, \tau = \{\emptyset, \{1\}, X\}$ be a topology on $X$. 

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Then \((X, \tau)\) is sf-\(T_0\), but not sf-\(D_1\), since there is no sf-\(D\)-set containing 2 but not 1.

5 Conclusion

In topological space, separation axioms are very important. Through this paper, it was concluded that there is a relationship between semi-feebly-\(T_1\), semi-feebly-\(T_0\) and the axioms of separation of type semi-feebly-\(R_0\), semi-feebly-\(R_1\) there is also a relationship between semi-feebly-\(D_0\), semi-feebly-\(D_1\), semi-feebly-\(D_2\).

6 References


