On the solutions of the equation $p = x^2 + y^2 + 1$ in Lucas sequences

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Abstract—In 1970, Motohashi proved that there are an infinite number of primes having the form $x^2 + y^2 + 1$ for some nonzero integers x and y. In this research, we present a technique for studying the solutions of the equation $p = x^2 + y^2 + 1$, where the unknowns are derived from some Lucas sequences of the first kind $\{U_n(P,Q)\}$ or the second kind $\{V_n(P,Q)\}$ with P and Q are certain nonzero relatively primes integers. As applications to this technique, we apply our procedure in case of $(x, y, p) = (U_i(P,Q), U_j(P,Q), U_k(P,Q))$ or $(V_i(P,Q), V_j(P,Q), V_k(P,Q))$ with $i, j, k \geq 1, -2 \leq P \leq 3$ and $Q = \pm 1$.

Keywords—Lucas sequences, Diophantine equation, Prime numbers

1 Introduction and preliminaries

Several scientists are interested in the study of prime numbers due to its usage and applications in numerous scientific domains such as mathematics and computer science. In reality, there exists an infinite number of prime numbers with many well-known forms. For example, Edmund Landau [4] conjectured that there are an infinite number of primes of the type $p = x^2 + 1$. In addition, Shanks [9, 8] conjectured that there exist an infinite number of prime numbers of the forms $p = x^4 + 1$ and $p = \frac{1}{2}(x^2 + 1)$ for some integers x. Furthermore, Motohashi [6] proved that there are an infinite number of primes having the form $p = x^2 + y^2 + 1$. On the other hand, it is also known that the Lucas sequences of the first kind $\{U_n(P,Q)\}$ (simply $\{U_n\}$) or the second kind $\{V_n(P,Q)\}$ (simply $\{V_n\}$) which are defined by the following relations provide infinitely many prime numbers (see e.g. [5] and [10]):

$$U_0(P,Q) = 0, U_1(P,Q) = 1, U_n(P,Q) = PU_{n-1}(P,Q) - QU_{n-2}(P,Q) \text{ for } n \geq 2, \quad (1)$$

$$V_0(P,Q) = 2, V_1(P,Q) = P, V_n(P,Q) = PV_{n-1}(P,Q) - QV_{n-2}(P,Q) \text{ for } n \geq 2, \quad (2)$$

where P and Q are nonzero integers with $gcd(P,Q) = 1$.

Additionally, the first and second types of Lucas sequences are associated in the identity

$$V_n^2(P,Q) = DU_n^2(P,Q) + 4Q^n, \tag{3}$$

where $D = P^2 - 4Q$. Consequently, these sequences can be correspondingly expressed by the following formulas which are known as Binet's formulas:

$$U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for } n \geq 0 \tag{4}$$

and

$$V_n(P,Q) = \alpha^n + \beta^n \text{ for } n \geq 0, \tag{5}$$

where α is called the golden ratio and $\beta = \frac{-1}{\alpha}$. Note that α and β are the roots of these sequences' characteristic polynomial that is defined by

$$X^2 - PX + Q = 0,$$

where

$$\alpha = \frac{P + \sqrt{D}}{2} \text{ and } \beta = \frac{P - \sqrt{D}}{2}.$$

Thus, if α/β is not a root of unity then these sequences are said to be nondegenerate, and degenerate otherwise. As a result, they are degenerate only with $(P,Q) \in \{(\pm 1, 1), (\pm 2, 1)\}$, for more details see e.g. [7].

Considering these sequences, these are several well-known sequences obtained with certain values of P and Q. For instance, if $(P,Q) = (1,-1)$ we get the so called Fibonacci and Lucas sequences that are denoted by $\{F_n\}$ and $\{L_n\}$, respectively.

If $(P,Q) = (2,-1)$, we similarly obtain the Pell and Pell-Lucas sequences that are respectively denoted by $\{P_n\}$ and $\{Q_n\}$. Note that the Binet's formulas for the sequences $\{F_n\}, \{L_n\}, \{P_n\}$ and $\{Q_n\}$ are given as follows:

$$F_n = \frac{\alpha_1^n - \beta_1^n}{\sqrt{5}}, L_n = \alpha_1^n + \beta_1^n, \tag{6}$$

$$P_n = \frac{\alpha_2^n - \beta_2^n}{2\sqrt{2}}, Q_n = \alpha_2^n + \beta_2^n, \tag{7}$$

where $(\alpha_1, \beta_1) = (\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$ and $(\alpha_2, \beta_2) = (1+\sqrt{2}, 1-\sqrt{2})$. Note that for simplicity and later use, we denote the Lucas sequences with the pairs $(P,Q) \in \{(3,1),(3,-1)\}$ by the following notation: $U_n(3,1) = M_n$, $V_n(3,1) = N_n$, $U_n(3,-1) = D_n$, and $V_n(3,-1) = E_n$.

Here, we present some results concerning of the Fibonacci numbers, Lucas numbers, Pell numbers and Pell-Lucas numbers (that we use later in the proofs of our main results) for which they respectively satisfy the following inequalities:

$$\alpha_1^{n-2} \leq F_n \leq \alpha_1^{n-1} \text{ for } n \geq 1, \tag{8}$$

$$\alpha_1^{n-1} \leq L_n \leq \alpha_1^{n+1} \text{ for } n \geq 1, \tag{9}$$

$$\alpha_2^{n-2} \leq P_n \leq \alpha_2^{n-1} \text{ for } n \geq 1, \quad (10)$$

$$\alpha_2^{n-1} \leq Q_n \leq \alpha_2^{n+1} \text{ for } n \geq 1. \quad (11)$$

Furthermore, Hashim, Szalay, and Tengely [3] proved that if $P \geq 2$ and $-P-1 \leq Q \leq P-1$ then Lucas sequences of the first and second kind satisfy the following inequalities:

$$\alpha^{n-2} \leq U_n \leq \alpha^{2n} \text{ for } n \geq 1, \quad (12)$$

$$2\alpha^{n-1} \leq V_n \leq \alpha^{2n} \text{ for } n \geq 1. \quad (13)$$

Considering that these sequences with the above mentioned forms of prime numbers, one may be interested in knowing whether or not there are infinitely many prime numbers of the forms $p = x^2+1, x^4+1, \frac{1}{2}(x^2+1)$ and x^2+y^2+1, where p, x and y represent terms in $\{U_n(P,Q)\}$ or $\{V_n(P,Q)\}$.

In [1, 2], we studied such special solutions of the equations: $p = x^2+1, p = x^4+1$ and $p = \frac{1}{2}(x^2+1)$. In fact, we found out that these equations have a finite number of solutions, and that leads to having a finite number of such primes.

The remaining equation is our main interest in this paper, namely, we investigate the solutions of the equation

$$p = x^2 + y^2 + 1, \quad (14)$$

where $(x,y,p) \in \{(U_i(P,Q), U_j(P,Q), U_k(P,Q)), (V_i(P,Q), V_j(P,Q), V_k(P,Q))\}$ with $i, j, k \geq 1$ for certain values of P and Q.

In the rest of this section, we present our procedure for investigating the solutions $(x,y,p) = (R_i, R_j, R_k)$ with $i,j,k \geq 1$ of equation (14), where R_i denotes a generalized Lucas number of the first or second kind, namely $R_i = U_i$ or V_i. We indeed apply the technique of this procedure in case of $-2 \leq P \leq 3$ and $Q = \pm 1$, where $\{U_n\}$ and $\{V_n\}$ are nondegenerate sequences. It is clear that the equation (14) has such special solutions only if $i < k$ and $j < k$. Hence, in order to find all the solutions $(x,y,p) = (R_i, R_j, R_k)$ with $i,j,k \geq 1$, we fix the condition that $i \leq j < k$. Then after determining the corresponding solutions, we permute the first two components of the these solutions. In the following, we summarize the general steps of the procedures under the condition $1 \leq i \leq j < k$ for finding the solutions $(x,y,p) = (R_i, R_j, R_k)$ of equation (14), with $i,j,k \geq 1$. In other words, we consider the equation

$$R_k = R_i^2 + R_j^2 + 1 \quad (15)$$

with $1 \leq i \leq j < k$ and $(R_i, R_j, R_k) = (U_i, U_j, U_k)$ or (V_i, V_j, V_k). Then we proceed follow the following stapes:

- We first divide equation (15) by R_j with using Binet's formulas presented by (4) or (5) and the inequalities presented by (8)-(13). Then after some simplifications, we obtain an upper bound for i, say $i \leq L$.

- For each the values of $i = 1, 2, \ldots, L$, we substitute a particular i in equation (15) to get

$$R_k = R_j^2 + c \text{ for } c = R_i^2 + 1. \quad (16)$$

- We then substitute equation (16) in identity (3), we obtain equations of the form

$$y_1^2 = A_1 x_1^4 + B_1 x_1^2 + C_1, \quad (17)$$

where ($x_1 = U_j$ or V_j). Note that equation (17) can be written in the form

$$Y^2 = X^3 + B_1 X^2 + A_1 C_1 X, \quad (18)$$

where $Y = A_1 x_1 y_1$ and $X = A_1 x_1^2$.

- The values of X (with $X = A_1 x_1^2$) in the curve (18) can be found using the SageMath [11] function integral_points().
- From the obtained value of x_1, we get the values of j.
- Plugging the obtained value of j in equation (16), we get the values of k. Hence, all the values of (i, j, k) in which the equation (15) is satisfied are obtained.
- After obtaining all the possible solutions (i, j, k) of equation (15) under the condition $1 \leq i \leq j < k$, it remains to permute the components i and j in (i, j, k) in order to have all the possible solutions of equation (14) with $i, j, k \geq 1$.

2 Main results

Theorem 1. Suppose that $(x, y, p) = (F_i, F_j, F_k)$ with $i, j, k \geq 1$, then equation (14) has no more solutions other then $(x, y, p) = (1, 1, 3)$.
Proof: To begin the proof, we first determine an upper bound for i in the equation

$$F_k = F_i^2 + F_j^2 + 1, \quad (19)$$

for $i, j, k \geq 2$, as $F_1 = F_2 = 1$ we assumed that $i \geq 2$. Below is a summary of the steps required to achieve this bound under the assumption of $i \leq j < k$:

- We divide equation (19) by F_j , yielding

$$\frac{F_k}{F_j} = \frac{F_i^2}{F_j} + F_j + \frac{1}{F_j}.$$

As $i \leq j$, we deduce that

$$\frac{F_k}{F_j} \leq \frac{F_i^2 + 1}{F_i} + F_j\ . \quad (20)$$

- By substituting inequality (8) and identity (6) into inequality (20) (with $(\alpha_1, \beta_1) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2))$, we get that

$$\alpha_1^k - \beta_1^k \leq \frac{1}{\alpha_1^{i-2}}(\alpha_1^{i+2j-3} + \alpha_1^{2i+j-2} + \alpha_1^j - \beta_1^j \alpha_1^{i+j-3} - \beta_1^j \alpha_1^{2i-2} - \beta_1^j).$$

- Based on the assumption of $2 \leq i \leq j < k$ with $\beta_1 = \frac{-1}{\alpha_1}$, the last inequality leads to

$$\alpha_1^k \leq \frac{1}{\alpha_1^i}(\alpha_1^{3j+2}) \pm \frac{1}{\alpha_1^i}.$$

- Applying the absolute value to the latter inequality yields

$$|\alpha_1^k| \leq |\tfrac{1}{\alpha_1^i}(\alpha_1^{3j+2})| + |\tfrac{1}{\alpha_1^i}|.$$

Subsequently,

$$\alpha_1^k \leq \tfrac{2}{\alpha_1^i}(\alpha_1^{3j+2}), \quad (21)$$

because of $1 < \alpha_1^{3j+2}$.

- By dividing the inequality (21) by α_1^{3j+2} , we get that

$$\alpha_1^{k-3j-2} \leq \frac{2}{\alpha_1^i},$$

which may also be written as

$$|\alpha_1^i| \leq \left|\tfrac{2}{\alpha_1^I - 2}\right| \quad (22)$$

such that $I = k - 3j - 2$.

- Let

$$B = \min_{I \in Z} |\alpha_1^I - 2|.$$

  - If $I = 0$, we obtain $B = 1$.
  - If $I \geq 1$, then $\alpha_1^I \geq \alpha_1^1 > 1.618$. This indicates that $B > 0.382$.
  - Similarly, if $I \leq -1$ then $B > 1.382$. Hence, we have that $B \geq 0.382$. Thus, inequality (22) gives that

$$\alpha_1^i < \frac{2}{B} < \frac{2}{0.382} < 5.236,$$

and this shows that

$$i \leq \frac{ln(5.236)}{ln(\alpha_1)} < \frac{ln(5.236)}{ln(1.618)} < 3.441.$$

Therefore, $i \leq 3$.
In equation (19), we substitute the values of i such that $i \in \{2, 3\}$ to obtain the corresponding values of j and k. This can be done in the following steps:

- For $i = 2$, we obtain the equation

$$F_k = F_j^2 + 2. \tag{23}$$

- For $i = 3$, we have

$$F_k = F_j^2 + 5. \tag{24}$$

Lastly, we have to acquire the values of j and k corresponding to each values of $i = 1$ and 2.

- Firstly, we combine equation (23) with identity (3) to obtain the elliptic curves

$$Y^2 = X^3 + 20X^2 + 120X \text{ and } Y^2 = X^3 + 20X^2 + 80X$$

with $X = 5x_1^2$ and $x_1 = F_j$. By using the SageMath function integral points(), we get $(X, Y) \in \{(0, 0), (5, 35), (24, 168)\}$. We choose $X = 5$ and 24 since we are only interested in positive numbers of X.
  - If $5 = X = 5x_1^2 = 5F_j^2$. Hence, $j = 2$. Then by substituting $j = 2$ into (23), we find $F_k = F_2^2 + 2 = 3$ which yields $k = 4$. Hence, the corresponding solution of equation (14) for $(i, j, k) = (2, 2, 4)$, with $i \leq j < k$, is $(x, y, p) = (F_2, F_2, F_4) = (1, 1, 3)$.
  - For $24 = 5F_j^2$; that is impossible.
- Finally, we consider equation (24) in combination with identity (3) to generate the elliptic curves

$$Y^2 = X^3 + 50X^2 + 645X \text{ and } Y^2 = X^3 + 50X^2 + 605X,$$

where $X = 5x_1^2$ such that $x_1 = F_j$. Similarity, using the integral_points() we obtain $X \in \{0, 80\}$. For $X = 5F_j^2 = 80$, no solution exists.

Thus, the solutions to equation (14) with $i, j, k \geq 2$ are $(x, y, p) = (1, 1, 3)$. So, the proof of Theorem 1 is achieved.

Theorem 2. Assume that $(x, y, p) = (L_i, L_j, L_k)$ with $i, j, k \geq 1$, then the solutions to equation (14) are given by

$$(x, y, p) \in \{(1, 1, 2), (1, 2, 5), (2, 1, 5)\}.$$

Proof: To begin the proof, we must first determine the upper bound for i in the equation

$$L_k = L_j^2 + L_i^2 + 1 \tag{25}$$

such that $i, j, k \geq 1$ and $i \leq j < k$. Initially, we divide equation (25) by L_j to obtain

$$\frac{L_k}{L_j} = \frac{L_i^2}{L_j} + L_j + \frac{1}{L_j}.$$

Subsequently,

$$\frac{L_k}{L_j} \leq \frac{L_i^2+1}{L_i} + L_j \tag{26}$$

as $i \leq j$. Thus, by replacing the inequality (9) and (identity (6) in the case of the Lucas sequence) in inequality (26), we obtain that

$$\alpha_1^k + \beta_1^k \leq \frac{1}{\alpha_1^{i+2}}\left(\alpha_1^{i+2j+2} + \alpha_1^{2i+j+2} + \alpha_1^j + \beta_1^j \alpha_1^{i+j+2} + \beta_1^j \alpha_1^{2i+2} + \beta_1^j\right).$$

Now, by applying the assumption that $1 \leq i \leq j < k$ with the fact $\beta_1 = \frac{-1}{\alpha_1}$ in the later inequality, we conclude that

$$\left|\alpha_1^k\right| \leq \left|\frac{7}{\alpha_1^i}\left(\alpha_1^{3j-1}\right)\right| + \left|\frac{1}{\alpha_1^i}\right|.$$

This implies that

$$\alpha_1^k \leq \frac{8}{\alpha_1^i}\left(\alpha^{3j-1}\right).$$

So, by dividing the last inequality by α_1^{3j-1}, we discover that

$$\alpha_1^I \leq \frac{8}{\alpha_1^i}$$

such that $I = k - 3j + 1$. By adopting the same strategy in the proof of Theorem 1, we indeed obtain that $i \leq 3$. Now, we substitute the values of $i = 1, 2$ and 3 into equation (25).

- For $i = 1$, we obtain the equation

$$L_k = L_j^2 + 2.$$

- For $i = 2$, then

$$L_k = L_j^2 + 10.$$

- If $i = 3$, we get

$$L_k = L_j^2 + 17.$$

Finally, we combine the above equations with identity (3) to get their corresponding curves as shown in (18) whose integral points can be obtained with SageMath function integral_points(). As a result, with all of these equations we obtain the set of solutions $(x, y, p) = (L_i, L_j, L_k) \in \{(1,1,2),(1,2,5)\}$ with $i \le j < k$. Hence, the complete set of solutions to equation (14) with $i, j, k \ge 1$ is given by

$$(x, y, p) = (L_i, L_j, L_k) \in \{(1,1,2),(1,2,5),(2,1,5)\}$$

Hence, the proof of Theorem 2 is concluded.

Theorem 3. Suppose that $i, j, k \ge 1$. If $(x, y, p) = (P_i, P_j, P_k)$ represents a solution to equation (14), then it has no such solution.
Proof:. We start by determining an upper bound for i in the equation

$$P_k = P_j^2 + P_i^2 + 1, \tag{27}$$

where $i, j, k \ge 1$ under the condition that $i \le j < k$. By following the argument applied the proofs of Theorems 1 and 2, we first find the solutions of (27) under the condition that $1 \le i \le j < k$. Dividing the latter equation by P_j with using the fact of inequality (10) and Binet's formula of Pell sequence presented in (7) imply that

$$\alpha_2^k - \beta_2^k \le \frac{1}{\alpha_2^{i-2}}\left(\alpha_2^{i+2j-2} + \alpha_2^{2i+j-2} + \alpha_2^j - \beta_2^j \alpha_2^{i+j-2} - \beta_2^j \alpha_2^{2i-2} - \beta_2^j\right).$$

It follows that

$$|\alpha_2^k| \le \left|\frac{6}{\alpha_2^i}\left(\alpha_2^{3j+2}\right)\right| + \left|\frac{1}{\alpha_2^i}\right|.$$

Hence,

$$\alpha_2^{k-3j-2} \le \frac{7}{\alpha_2^i}.$$

Thus, we get that

$$\alpha_2^i \le \frac{7}{B}$$

such that $B = \min_{I \in Z} |\alpha_2^I - 3|$ and $I = k - 3j - 2$. Indeed, we find that $B \ge 0.586$. Hence, the last inequality indicates that

$$i \le \frac{ln(11.945)}{ln(2.414)} < 2.814.$$

This gives $i \le 2$. We plug the values of i into equation (27) to get the values of j and k for $i = 1$ and 2.

- If $i = 1$, we get that

$$P_k = P_j^2 + 2.$$

Now, we substitute the latter equation in identity (3) to have the elliptic curves
$$Y^2 = X^3 + 32X^2 + 288X \text{ and } Y^2 = X^3 + 32X^2 + 224X$$

with $X = 8x_1^2$ and $x_1 = P_j$. By using the SageMath function integral points(), we get $(X, Y) = (0, 0)$. Therefore, it has nonzero solution.

- If $i = 2$, we have that

$$P_k = P_j^2 + 5.$$

Next, we combine the last equation with the identity (3) to obtain the elliptic curves

$$Y^2 = X^3 + 80X^2 + 1632X \text{ and } Y^2 = X^3 + 80X^2 + 1568X,$$

where $X = 8x_1^2$ such that $x_1 = P_j$. We get $X = -36$ and 0, we once again find no solution.

So, equation (14) with $i, j, k \geq 1$ has no solutions, and this proves Theorem 3.

Theorem 4. The equation (14) has no solution of the form $(x, y, p) = (Q_i, Q_j, Q_k)$ if $i, j, k \geq 1$.

Proof: First, we find an upper bound for i in the equation

$$Q_k = Q_j^2 + Q_i^2 + 1, \tag{28}$$

for $i, j, k \geq 1$. Following the same strategy as in the proofs of the previous theorems with the assumption of $1 \leq i \leq j < k$, we ultimately conclude that

$$|\alpha_2^i| \leq \left|\frac{15}{\alpha_2^I - 3}\right|. \tag{29}$$

Suppose that

$$B = \min_{I \in Z} |\alpha_2^I - 3|.$$

- If $I = 0$, we obtain $B = 2$.
- If $I \geq 1$, then $\alpha_2^I \geq \alpha_2^1 > 2.414$. This indicates that $B > 0.586$.
- Similarly, if $I \leq -1$ then $B > 2.586$.

Therefore, we deduce that $B = 0.586$. As a result, inequality (29) indicates that $\alpha_2^i < 25.579$, and it means that

$$i \leq \frac{ln(25.579)}{ln(\alpha_2)} < \frac{ln(25.579)}{ln(2.414)} < 3.678.$$

Hence, $i \leq 3$. Next, we calculate the values of j and k corresponding to the values of $i \in \{1, 2, 3\}$ in equation (28) with $i, j, k \geq 1$. Firstly, we establish the values of X obtained from the integral points (X, Y) of the curves shown by (18). The specifics of calculations for the solutions $(x, y, p) = (Q_i, Q_j, Q_k)$ of equation (14) under the assumption of $1 \leq i \leq j < k$ are given in the table below, emphasizing that the triples

$[1, B_1, A_1C_1]$ representing the coefficients of elliptic curves of the form (18) such that the values of $X = A_1x_1^2$ with $x_1 = Q_j$ (Observe that the third column includes just the positive values of X):

i	$[1, B_1, A_1C_1]$	X	(x_1, j)	k	$\{(x, y, p)\}$
1	[1, 10, 1864]	-	-	-	{ }
	[1, 10, 1344]	-	-	-	{ }
2	[1, 592, 87872]	-	-	-	{ }
	[1, 592, 87360]	80	-	-	{ }
		1092	-	-	{ }
3	[1, 3152, 2458880]	-	-	-	{ }
	[1, 3152, 2458368]	16	-	-	{ }
		21296	-	-	{ }
		153648	-	-	{ }

As a result, equation(14) with $i, j, k \geq 1$ has no solution, and Theorem 4 is proved.

Theorem 5. If $x = M_i, y = M_j$ and $p = M_k$ with $i, j, k \geq 1$, then the only solution to equation (14) is $(x, y, p) = (1, 1, 3)$.
Proof: To begin the proof, we first determine an upper bound for i in the equation

$$M_k = M_j^2 + M_i^2 + 1, \tag{30}$$

where $i, j, k \geq 1$ such that $i \leq j < k$. Next, we divide the above equation by M_j with using $i \leq j$ to get

$$\frac{M_k}{M_j} = \frac{M_i^2}{M_j} + M_j + \frac{1}{M_j}. \tag{31}$$

Consequently, we conclude that

$$\alpha^k + \beta^k \leq \frac{1}{\alpha^{i-2}}(2\alpha^{i+2j-2} + 2\alpha^{2i+j} + \alpha^j - 2\beta^j\alpha^{i+j-2} - 2\beta^j\alpha^{2i} - \beta^j)$$

by substituting inequality (12) and identity (4) into inequality (31) (with $(\alpha, \beta) = ((3 + \sqrt{5})/2, (3 - \sqrt{5})/2)$). By replacing the assumption $1 \leq i \leq j < k$ with the fact $\beta = \frac{-1}{\alpha}$ in the last inequality, we arrive to the inequality

$$\alpha^k \leq \frac{3}{\alpha^i}(\alpha^{3j+2}).$$

Hence, by dividing the last inequality by α^{3j+2} we get that

$$\alpha^{k-3j-2} \leq \frac{3}{\alpha^i}.$$

Therefore,

$$\alpha^i \leq \frac{3}{B}, \tag{32}$$

where $B = \min_{I \in Z} |\alpha^I - 2.5|$ with $I = k - 3j - 2$.

- For $I = 0$, then $B = 1.5$.
- For $I \geq 1$, we obtain $\alpha^I \geq \alpha^1 > 2.618$. Hence, $B > 0.118$.
- Similarly, if $I \leq -1$ then $B > 2.118$.

Thus, we conclude that $B \geq 0.118$. Hence, inequality (32) indicates that

$$i < \frac{ln(25.423)}{ln(2.618)} < 3.362.$$

Therefore, $i \leq 3$. In the following, we substitute the values of $i \in \{1, 2, 3\}$ into equation (30):

Δ If $i = 1$, we get the equation

$$M_k = M_j^2 + 2.$$

Now, we replace the latter equation into identity (3) to obtain the elliptic curve

$$Y^2 = X^3 + 20X^2 + 120X,$$

where $X = 5x_1^2$ such that $x_1 = M_j$. By using the SageMath function integral points(), we get $X = 5$ and 24.

- For $5 = X = 5M_j^2$, then $M_j^2 = 1$. Therefore, $j = 1$. Then by substituting $j = 1$ in (30), we get $\mathrm{k} = 2$. So, the corresponding solution is $(\mathrm{x}, \mathrm{y}, \mathrm{p}) = (1, 1, 3)$.
- For $24 = 5M_j^2$; that is impossible.

Δ If $i = 2$, we have that

$$M_k = M_j^2 + 10.$$

Combining the last equation with the identity (3) yields the equation

$$Y^2 = X^3 + 100X^2 + 2520X$$

with $X = 5x_1^2$ and $x_1 = M_j$. We obtain $X = 0$. Thus, no solution exists.

Δ Finally, for $i = 3$ we obtain

$$M_k = M_j^2 + 65.$$

We use the preceding equation together with identity (3) to produce the elliptic curves

$$Y^2 = X^3 + 20X^2 + 120X$$

such that $X = 5M_j^2$. Similarly, it has no solution.

Hence, the solution to equation (14) with $i, j, k \geq 1$ is $(x, y, p) = (1, 1, 3)$. So, Theorem 5 is completely proved.

Theorem 6. If $(x, y, p) = (N_i, N_j, N_k)$ with $i, j, k \geq 1$, then equation (14) has no solution.
Proof: Again, we begin by determining an upper bound for i in the equation

$$N_k = N_j^2 + N_i^2 + 1, \quad (33)$$

where $i, j, k \geq 1$. Using the same technique used in the proofs of the preceding theorems with the condition $1 \leq i \leq j < k$, we finally conclude that

$$\alpha^i \leq \frac{8}{B}, \quad (34)$$

where $B = \min_{I \in Z} |\alpha^I - 2.5|$ and $\alpha = (3 + \sqrt{5})/2$. Following some computations, we determine that $B = 0.118$. As a consequence, inequality (34) demonstrates that $\alpha^i \leq 67.796$, which means that

$$i \leq \frac{ln(67.796)}{ln(\alpha)} < \frac{ln(67.796)}{ln(2.618)} < 4.381.$$

Therefore, $i \leq 4$. Next, we identify the values of j and k that correspond to the values of $1 \leq i \leq 4$ with $j, k \geq 1$. The first step is determining the values of X derived from the integral points (X, Y) of the corresponding curves presented by (18). The details of the computations for the solutions $(x, y, p) = (N_i, N_j, N_k)$ of equation (14) under the assumption of $i \leq j < k$ are shown in the table below, with $[1, B_1, A_1C_1]$ representing the coefficients of elliptic curves such that $X = A_1 x_1^2$ for $x_1 = N_j$:

i	$[1, B_1, A_1C_1]$	X	(x_1, j)	k	$\{(x, y, p)\}$
1	[1, 100, 2600]	0	-	-	{}
		5	-	-	{}
		520	-	-	{}
2	[1, 500, 62600]	0	-	-	{}
3	[1, 3250, 2640725]	0	-	-	{}
		980	-	-	{}
4	[1, 22100, 122102600]	0	-	-	{}

Thus, equation (14) has no solutions with $i, j, k \geq 1$, and this proves Theorem 6.

Theorem 7. If $(x, y, p) = (D_i, D_j, D_k)$ with $i, j, k \geq 1$, then the set of solutions to equation (14) is given by $(x, y, p) \in \{(1, 1, 3)\}$.
Proof: We begin by determining an upper bound for i in the equation

$$D_k = D_j^2 + D_i^2 + 1, \quad (35)$$

where $i, j, k \geq 1$ and $i \leq j < k$. Below is a summary of the necessary actions to achieve this bound:

- First, we divide equation (35) by D_j and then use $i \leq j$ to obtain

$$\frac{D_k}{D_j} \leq \frac{D_i^2+1}{D_i} + D_j \ . \quad (36)$$

- Therefore, we get that

$$\alpha^k + \beta^k \leq \frac{1}{\alpha^{i-2}}(2\alpha^{i+2j-2} + 2\alpha^{2i+j} + \alpha^j - 2\beta^j\alpha^{i+j-2} - 2\beta^j\alpha^{2i} - \beta^j),$$

by combining inequality (12) and (identity (4) with $(\alpha,\beta) = ((3+\sqrt{13})/2, (3-\sqrt{13})/2))$ into inequality (36). By replacing the assumption $i \leq j < k$ with the fact $\beta = \frac{-1}{\alpha}$ in the latter inequality, we find that

$$\alpha^k \leq \frac{3\alpha^{3j+2}}{\alpha^i}. \quad (37)$$

- Dividing inequality (37) by α^{3j+2}, we find that

$$\alpha^{k-3j-2} \leq \frac{3}{\alpha^i}.$$

With some simplification, we get that $\alpha^i < 15.151$ which implies that $i < 2.275$.

Hence, $i \leq 2$. We plug the values of i in equation (35) to obtain the values of j and k corresponding to $i = 1$ and 2.

Δ If $i = 1$, we get that

$$D_k = D_j^2 + 2.$$

Now, we replace the latter equation into identity (3) in order to get the elliptic curves

$$Y^2 = X^3 + 52X^2 + 728X \text{ and } Y^2 = X^3 + 52X^2 + 624X,$$

where $X = 13x_1^2$ such that $x_1 = D_j$. By using the SageMath software, we get $X = 13$ and 56.

- For $X = 13$, we obtain $j = 1$ which implies that $D_k = D_1 + 2 = 3$. Hence, $k = 2$. Thus, the corresponding solution is $(x,y,p) = (1,1,3)$
- For $56 = 13D_j^2$; this has no solution.

Δ If $i = 2$, we have that

$$D_k = D_j^2 + 10.$$

Lastly, we combine the later equation with the identity (3) to obtain the equations

$$Y^2 = X^3 + 260X^2 + 16952X \text{ and } Y^2 = X^3 + 260X^2 + 16898X,$$

for $= 5D_j^2$. We get $X = 0$. Thus, no solution exists.

Thus, the solutions to the equation (14) with $i, j, k \geq 1$ are $(x, y, p) = (1, 1, 3)$. So, Theorem 7 has been proven.

Theorem 8. Suppose that $x = E_i, y = E_j$ and $p = E_k$ with $i, j, k \geq 1$, then equation (14) is not solvable over the integers x, y and p.

Proof: We begin by getting an upper bound for i (with $1 \leq i \leq j < k$) in the equation

$$E_k = E_j^2 + E_i^2 + 1. \qquad (38)$$

Similarly, by using the argument applied the proofs of the previous theorems, we get $i \leq 2$.

To get the values of j and k that correspond to the values of $i \in \{1, 2\}$, we substitute the values of i into equation (38).

- For $i = 1$, we obtain the equation

$$E_k = E_j^2 + 10.$$

  Then, we combine the latter equation with the identity (3) to yield the elliptic curves

$$Y^2 = X^3 + 260X^2 + 17576X \text{ and } Y^2 = X^3 + 260X^2 + 16224X$$

  with $X = 13x_1^2$ and $x_1 = E_j$. By using the SageMath function integral points(), we have $X \in \{13, 1248\}$. But, $X = 13E_j^2 = 13$ or 1248 leads to no solution.

- If $i = 2$, we get that

$$E_k = E_j^2 + 122,$$

  which leads to the equations

$$Y^2 = X^3 + 3172X^2 + 2516072X \text{ and } Y^2 = X^3 + 3172X^2 + 2514720X$$

  such that $X = 13E_j^2$. Here, we have $X \in \{-1612, -1587, -1560, 0\}$. Again, we obtain no solution.

Hence, equation (14) has no solutions of the form $(x, y, p) = (E_i, E_j, E_k)$ with $i, j, k \geq 1$. Hence, the proof of Theorem 8 is achieved.

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