Study of Local Dissipative Random Dynamical Systems

[1] **Asmahan A. yasir** and [2] **Ihsan J. Kadhim**

[1]Dep. of Mathematics, College of Education for Girls, University of Al-Kufa, Najaf/ Iraq.

[2]Dep. of Mathematics, College of Science, University of Al-Qadisiyah, Diwania/ Iraq.

[1]E-mail: asmahana.alshemary3@student.uokufa.edu.iq

Asmhan1983@gmail.com

[2]E-mail: Ihsan.kadhim@qu.edu.iq

[2]https://orcid.org/0000-0001-9035-2912?langen

Abstract: The main objective of this work is to study local dissipation in stochastic dynamical systems. The main properties of this concept were studied. In addition to presenting the necessary and sufficient conditions that make the random dynamic system a local dissipative. In Theorem 3.2, local dissipation is described in terms of Levinson center, and in Theorem 3.5, it is described in terms of $D^{+}_{\Gamma_X}(\omega)$ is a forward prolongation of the random set Γ_X and $J^{+}_{\Gamma_X}(\omega)$ is a forward limit prolongation of the random set Γ_X. Finally in Theorem 3.7 it is described by locally asymptotically condensing.

Key words: Random dynamical system (RDS), local dissipative, Compact dissipative, random attracter, random Levinson center (RLS)**.**

1.Introduction: It is well known that in the 1980s, efforts were made to develop sufficient mathematical models to account for turbulence phenomena, which greatly influenced the general theory of dissipative systems. Infinite-dimensional dissipative dynamics has made major strides in the past few years (see, for instance, Chueshov I. [6], Robinson J. [10], Temam R. [12] and Chiban D. [3] and the references therein). The dynamical systems encountered in physical or biological sciences can be loosely divided into two classes: conservative ones (including Hamiltonian systems) and those displaying some type of dissipation. These dynamical systems are often generated by partial differential equations and thus the underlying state space is an infinite dimensional. For interesting papers related to the theory of dissipativity on RDS, we cite the works by Hale K. [8], Christian Kuehn, Alexandra Neamt ¸u and Anne Pein [9], Arnold [1], Igor [7], amongst many others.

In Our work, we will continue the line of research introduced by Yasir and Kadhim [13,14,15]. A new relations of Local dissipativity on RDSs will be illustrated.

2. Notation and Preliminaries:

In this section, we will present some concepts and facts related to random dynamic systems that serve our work. Through our work the triple $(\Omega, \mathcal{F}, \mathbb{P})$ dente the probability space and $\mathbb{T} \coloneqq \mathbb{R}$, or $\mathbb{Z}$ (considered as an additive topological group) and (X, d) be a metric space.

Definition 2.1[7]: The **metric dynamical system (MDS)** is the triple $(\mathbb{T}, \Omega, \theta)$ where $\theta: \mathbb{T} \times \Omega \longrightarrow \Omega$ is an invariant measurable action. For convenience, it is denoted for the dynamic metric dynamic system by θ.

Definition 2.2[7]: The **random dynamical system (RDS)** is a pair (θ, φ) including an MDS θ and a mapping $\varphi : \mathbb{T} \times \Omega \times X \longrightarrow X$, that meets the following axioms for all $t, s \in \mathbb{T}$ and $\omega \in \Omega$,

(i) Continuity: $\varphi\ (\cdot, \omega, \cdot): \mathbb{T} \times \Omega \times X \longrightarrow X$ is continuous.
(ii) Cocycle property : the mapping $\varphi(t, \omega) := \varphi(t, \omega, \cdot)$ fulfill:

$$\varphi(0, \omega) = \mathrm{id}_X, \text{ and } \varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) .$$

Definition 2.3[7]

(a) The **random set** is a set-valued function $\mathrm{M}: \Omega \longrightarrow X$ such that the function $\Psi: \Omega \longrightarrow \mathbb{R}$ defined by $\Psi(\omega) \coloneqq \mathrm{d}(x, \mathrm{M}(\omega))$ is measurable for every $x \in X$.

(b) For some $y \in \mathrm{X}$ and some random variable $r: \Omega \longrightarrow \mathbb{R}^+$, consider the set $\mathcal{A} \coloneqq \{x: \mathrm{dis}_X(x, y) \leq \mathrm{r}(\omega), \text{for all } \omega \in \Omega \}$. A **random set** $S(\omega)$ is called **tempered** if $S(\omega) \subset \mathcal{A}$ for every $\omega \in \Omega$,

and $r: \Omega \longrightarrow \mathbb{R}^+$ is called **tempered random variable** (TRV), this means

$$\sup_{t\in\mathbb{T}}\{e^{-\lambda|t|}\ |r(\theta_t\omega)|\} < \infty \text{, for every } \lambda > 0 \text{ and } \omega \in \Omega.$$

Definition 2.4 [7] In the RDS (θ, φ) the **trajectory** starting from a random set M is the set-valued function defined by

$$\gamma_M^t(\omega) \coloneqq \cup_{\tau\geq t}\, \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)$$

Definition 2.5[7]: The **omega-limit set** of $\gamma_M^t(\omega)$ is a set-valued function

$$\omega \longmapsto \Gamma_M(\omega) \coloneqq \cap\, \overline{\gamma_M^t(\omega)} = \cap \overline{\cup\, \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}, t > 0, \tau \geq t.$$

Definition 2.6 [15]: We will call the set $J_X(\omega)$ defined by equality

$$J_X(\omega) \coloneqq \Gamma_K(\omega) = \cap\, \{\varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega) | t \in T, \omega \in \Omega, K \text{ is compact random set}\},$$

the **random Levinson Center (**RLC) of the compact dissipative RDS (θ, φ).

In the following definition we will introduce the concept of uniformly attracting in RDS.

Definition 2.7: For an RDS (θ, φ). The random set $M(\omega) \subseteq X$ is called **uniformly attracting** if there is a **TRV** $\delta(\omega) > 0$ so that

$$\lim_{t\to+\infty} \sup \mathrm{d}(\varphi(t, \theta_{-t}\omega)x\,, M(\omega)) = 0\ , x \in B(M, \delta).$$

Definition 2.8 [7]: For an RDS (θ, φ). A random set $S(\omega)$ is called **forward invariant** (**backward invariant**) whenever for every $\omega \in \Omega$, $t > 0$ and we have $\varphi(t, \omega)S(\omega) \subseteq S(\theta_t\omega)$(rep. $S(\theta_t\omega) \subseteq \varphi(t, \omega)S(\omega)$).

Definition 2.9 [15]: Consider RDS (θ, φ). A random set $M(\omega)$ is said to be **orbitally stable** if for any TRV ε and any non-negative number t, there exists TRV δ such that

$$d\big(x, M(\omega)\big) < \delta(\omega) \text{ implies } d\big(\varphi(t, \theta_{-t}\omega)x, M(\omega)\big) < \varepsilon(\omega).$$

Definition 2.10[7] The **absorbing set** for the RDS (θ, φ) relative to universe $\mathcal{M}$ is a closed random set $A(\omega)$ with the property that for every $B \in \mathcal{M}$ and ω *there is* $t_0(\omega)$ so that

$$\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega),\ \forall t \geq t_0(\omega)\,.$$

Definition 2.11 [7]: The RDS (θ, φ) is called **dissipative** in the universe $\mathcal{M}$, if there is an absorbing set A for the RDS (θ, φ) in $\mathcal{M}$ contained in a closed random ball $B_{r(\omega)}(x_0)$.

Definition 2.12 [15]: If the RDS (θ, φ) satisfy the limit

$$\lim_{t\to+\infty} d\big(\varphi(t,\theta_{-t}\omega)x(\theta_{-t}\omega),K(\omega)\big)=0$$

for all $x \in X^{\Omega}$ and some random set K , is called **point dissipative.**

Definition 2.13 [15]: If the RDS (θ,φ) satisfy the limit

$$\lim_{t\to+\infty} \sup\left\{d\big(\varphi(t,\theta_{-t}\omega)x,K(\omega)\big): x\in B_{\delta_{y(\omega)}}(y)\right\}=0$$

for every $y \in X$ and some TRV $\delta_y(\omega) > 0$ and some random set K, is called **locally dissipative** .

Definition 2.14[5]: Let (X,d) be a metric space, $K \subset X$ is **precompact** or **totally bounded** if every sequence in K admits a subsequence converges to a point of X.

3. Local Dissipativity on RDSs:

This section includes a study of the basic properties of the concept of locally dissipative random dynamical systems, as it was described in terms of compact dissipation, uniform attractors, and the Levinson Center.

Theorem 3.1: The RDS (θ,φ) is locally dissipative if and only if the following limit satisfied

$$\operatorname{Lim}_{t\to+\infty} d_X\big(\varphi(t,\theta_{-t}\omega)\mathrm{B}(\mathrm{p},\delta),J_X(\omega)\big)=0, \omega\in\Omega \qquad (3.1)$$

for all $\mathrm{p} \in X^{\Omega}$ and some TRV $\delta(\omega) > 0$.

Proof: The equation (3.1) lead to be (θ,φ) locally dissipative.

Conversely, let (θ,φ) be a locally dissipative RDS. Hence there is a nonvoid random compact subset $K(\omega) \subseteq X$ so that for all $p \in X^{\Omega}$, there is a TRV $\delta(p,\omega)$ such that:

$$\lim_{t\to+\infty} d_X(\varphi(t,\theta_{-t}\omega)\mathrm{B}(\mathrm{p},\delta),K(\omega)) = 0 \qquad (3.2)$$

Theorem **3.5** [15] lead to that the set $K_p(\omega) := \Gamma(\mathrm{B}(\mathrm{p},\delta_p(\omega)))$ is nonvoid, invariant compact, random set with

$$\lim_{t\to+\infty} d_X(\varphi(t,\theta_{-t}\omega)B\big(p,\delta_p(\omega))\big),K_p(\omega))=0 \qquad (3.3)$$

From equality (3.2) follows the inclusion $K_p(\omega) \subseteq K$.So, $\Gamma_{K_p}(\omega) \subseteq \Gamma_K(\omega) \subseteq J_X(\omega)$. Since the set $K_p(\omega)$ is an invariant, we have $K_p(\omega) = \Gamma_{K_p}(\omega)$; hence $K_p(\omega) \subseteq J_X(\omega)$. From the last inclusion and equality (3.3) we get equality (3.1). ■

Theorem 3.2: A compact dissipative RDS is local dissipative if and only if its random Levinson center is uniformly attracting .

Proof: Suppose that $J_X(\omega)$ is a random Levinson center of a local dissipative RDS (θ, φ) and $p \in J_X(\omega)$.

Theorem 3.1 lead to existence of a TRV $\delta(\omega)$ so that equality (3.1) is valid . Since $J_X(\omega)$ is compact, then the open covering $\{B\left(p, \delta_p(\omega)\right) : p \in J_X(\omega)\}$ admits a finite subcovering

$$\left\{B\left(p_i, \delta_{p_i}(\omega)\right)\right) \Big| i = 1, \dots, m\}.$$

Now, Lemma 3.3 [13] lead to existence of a TRV $\alpha(\omega)$ so that

$$B(J_X(\omega), \alpha(\omega)) \subset \cup\{B(p_i, \delta_{p_i}(\omega)))|\ i = 1, \dots, m\}.$$

It is clear that for a TRV $\alpha > 0$, the equality

$$\lim_{t\to+\infty} d_X\left(\varphi(t, \theta_{-t}\omega) B\left(J_X(\omega), \delta_p(\omega)\right), J_X(\omega)\right) = 0 \qquad (3.4)$$

valid, i.e., $J_X(\omega)$ is a uniformly attracting.

Conversely, suppose that the Levinson center of a compact dissipative RDS (θ, φ) is uniformly attracting , so, there is a TRV $\alpha(\omega)$ make (3.4) valid . If $x \in X$, then there is $l = l(x) > 0$ so that

$$\mathrm{d}\left(\varphi(t, \theta_{-t}\omega)x, J_X(\omega)\right) < \alpha(\omega) \qquad (3.5)$$

for all $t \geq l$. According to (3.4), for every random variable $\varepsilon(\omega) > 0$ we can choose $L(\varepsilon) > 0$ such that

$$\mathrm{d}\left(\varphi(t, \theta_{-t}\omega)y, J_X(\omega)\right) < \varepsilon(\omega) \qquad (3.6)$$

for all $t \geq L(\varepsilon)$ and $y \in B(J_X(\omega), \alpha(\omega))$.By (3.5) $B(J_X(\omega), \alpha(\omega))$ is open, then we may choose $\eta = \eta(x) > 0$, so that the inclusion $B(\varphi(l, \theta_{-l}\omega)x, \eta) \subset B(J_X(\omega), \alpha(\omega))$ hold.

By hypothesis, the mapping $\varphi(t, \theta_{-t}\omega): X \to X$ is continuous, We can find a TRV $\delta(\omega) = \delta_x(\omega) > 0$ such that

$$\varphi(l, \theta_{-l}\omega)y \in B(\varphi(l, \theta_{-l}\omega)x, \eta)\ and\ \varphi(l, \theta_{-l}\omega)y \in B(J_X(\omega), \alpha(\omega)) \text{ for all } y \in B(x, \delta_x(\omega)).$$

By virtue of (3.6), we have

$$y(t+l) \in B(J_X(\omega), \varepsilon(\omega)) \text{ for all } t \geq L(\varepsilon) \;\; and \;\; y \in B(x, \delta_x(\omega)).$$

Set $L(\varepsilon, x) := l(x) + L(\varepsilon)$. Then $\varphi(t, \theta_{-t}\omega)y \in B(J_X(\omega), \varepsilon(\omega))$ for all $t \geq L(\varepsilon, x)$ and $y \in B(x, \delta_x)$. Consequently (θ, φ) is local dissipative. ■

Lemma 3.3: The nonvoid forward invariant compact random set is orbitally stable . If it is uniformly attracting.

Proof*:* Suppose M be a random set with the given property. Assume, if possible, that $M(\omega)$ is not orbitally stable. So there is TRV $\varepsilon_0(\omega)$ and a positive real sequence $\delta_n \longrightarrow 0$ and $x_n \in B(M(\omega), \delta_n)$ and $\{t_n\}$ with $t_n \longrightarrow +\infty$ such that

$$d(\varphi(t_n, \theta_{-t_n}\omega)x_n, M(\omega)) \geq \varepsilon_0(\omega) \tag{3.7}$$

Since M is uniformly attracting, for a TRV $\varepsilon_0(\omega)$ there exists $L(\varepsilon_0) > 0$ such that

$$d(\varphi(t, \theta_{-t}\omega)x, M(\omega)) < \frac{\varepsilon_0(\omega)}{2} \tag{3.8}$$

for all $x \in B(M, \alpha)$ and $t \geq L(\varepsilon_0)$, where a TRV $\alpha(\omega) > 0$ such that

$$\lim_{t \to +\infty} d_X(\varphi(t, \theta_{-t}\omega)B(M, \alpha(\omega)), M(\omega)) = 0$$

Since $x_n \in B(M, \delta_n)$ and $\delta_n \longrightarrow 0$, then $\{x_n\}$ convergent. Set $x_0 = \lim_{t \to +\infty} x_n$, thus $x_0 \in M$ and $x_n \geq L(\varepsilon_0)$ as $n \longrightarrow +\infty$. Then inequality (3.8) leads to

$$d(\varphi(t_n, \theta_{-t_n}\omega)x_n, M(\omega)) < \frac{\varepsilon_0(\omega)}{2} \tag{3.9}$$

Both (3.7) and (3.9) are contradictory. ■

Assume $\Gamma_X(\omega) := \overline{\cup \{\Gamma_x(\omega) | x \in X\}}$, $\omega \in \Omega$. Let (θ, φ) be a compact dissipative RDS and $J_X(\omega)$ its random Levinson center (see, for more details [15]). It is clear that $\Gamma_X(\omega) \subseteq J_X(\omega)$. The set $\Gamma_X(\omega)$ is an essential to distinguishing of a dissipative RDS, (see [13] Theorem **3.5**).

Definition 3.4 [14]: Let $\{\varepsilon_s : \Omega \to \mathbb{R},\ s \in \mathbb{R}^+\}$ be a family of tempered random variables, and define the forward prolongation and the forward limit prolongation of the random set M respectively as follows :

$$D_M^+(\omega) := \cap_{s>0} \overline{\cup \{\varphi(t, \theta_{-t}\omega)B(M, \varepsilon_s) | t \geq 0, \omega \in \Omega\}} ,$$

$$J_M^+(\omega) := \cap_{s>0} \cap_{t \geq 0} \overline{\cup \{\varphi(\tau, \theta_{-\tau}\omega)B(M, \varepsilon_s) | \omega \in \Omega, \tau \geq t\}}$$

In particular, if $M = \{x\}$, then we set

$$D_x^+(\omega) := D^+(\{x\}), \text{ and } J_x^+(\omega) := J^+(\{x\}).$$

The set $D_M^+(\omega)$ is called the **first forward prolongation of a random set** M and $J_M^+(\omega)$ is called **first forward prolongational limit set of a random set** M.

Theorem 3.5: A point dissipative RDS (θ, φ) is local dissipative if and only if the set $D^+_{\Gamma_X}(\omega)$ (resp., $J^+_{\Gamma_X}(\omega)$) is compact and uniformly attracting .

Proof: Suppose that (θ, φ) is point dissipative and $D^+_{\Gamma_X}(\omega)$ (resp., $J^+_{\Gamma_X}(\omega)$) is compact and uniformly attracting. By Lemma 3.3, $D^+_{\Gamma_X}(\omega)$ (resp., $J^+_{\Gamma_X}(\omega)$) is orbitally stable, so Theorem 3.23 [14] tell us that the RDS (θ, φ) is compact dissipative and $D^+_{\Gamma_X}(\omega)$ (resp., $J^+_{\Gamma_X}(\omega)$) agrees with its random Levinson center $J_X(\omega)$. Hence by Theorem 3.2 the proof is finished.

Conversely, suppose that (θ, φ) is local dissipative. By Proposition 3.17 [15], (θ, φ) is compact dissipative. Consequently $D^+_{\Gamma_X}(\omega)$ (resp., $J^+_{\Gamma_X}(\omega)$) is compact and orbitally stable ([14] Theorem 3.22). So that $D = D^+_{\Gamma_X}(\omega)$ (respectively, $J = J^+_{\Gamma_X}(\omega)$) ([14]Theorem 3.15 and Corollary 3.16).
From Theorem 3.2 yield $D^+_{\Gamma_X}(\omega)$ (resp., $J^+_{\Gamma_X}(\omega)$)) is an uniformly attracting set. ■

Definition 3.6: The RDS (θ, φ) is called a **locally asymptotically condensing** if the limit

$$\lim_{t\to+\infty} d_X(\varphi(t, \theta_{-t}\omega)B_{p,\delta}(\theta_{-t}\omega), K_p(\omega)) = 0 \qquad (3.10)$$

holds for all $p \in X$ and some TRV $\delta_p(\omega)$ and a nonvoid compact random set $K_p(\omega) \subseteq X$, where $B_{p,\delta} \equiv B\left(p, \delta_p(\omega)\right)$.

Theorem 3.7: The point dissipative RDS is local dissipative if and only if it is locally asymptotically condensing.
Proof: Suppose that (θ, φ) is locally dissipative, then it is easy to see that (θ, φ) is asymptotically condensing.
Conversely, suppose that (θ, φ) is point dissipative and locally asymptotically condensing,
Let us show that $\Gamma_K^+(\omega)$ is precompact for any compact random subset K. Let $p \in K$ and $\delta_p(\omega)$ be a TRV and $K_p(\omega)$ compact random subset satisfy (3.10) . Since K is compact, then open covering $\{B(p, \delta_p(\omega)) | \, p \in K\}$ admits a finite subcovering $\left\{B\left(pi, \delta_{p_i}(\omega)\right) \middle| \, i = 1,2, \dots, n\right\}$.

Set $W(\omega) := K_{p_1}(\omega) \cup K_{p_2}(\omega) \cup \dots \cup K_{p_n}(\omega)$. Hence W is compact and

$$\lim_{t\to+\infty} d_X(\varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega), W(\omega)) = 0 \qquad (3.11)$$

From equality (3.11) follows the set $\Gamma_K^+(\omega)$ is relative compact. According to Theorem 3.22 [14], (θ, φ) is compact dissipative. Now , let $p \in J_X(\omega)$, $\delta_p(\omega)$ be a TRV and $K_p(\omega)$ be compact set satisfy (3.10). By Lemma 3.5 [15], the set $\Gamma(B(p,\delta_p(\omega)))$ is nonvoid invariant compact, and the equality

$$\lim_{t\to+\infty} d_X(\varphi(t,\theta_{-t}\omega)B(p,\delta_p(\omega)),\Gamma(B(p,\delta_p(\omega)))) = 0 \qquad (3.12)$$

holds. So $\Gamma(B(p,\delta_p)) \subseteq J_X(\omega)$ because $J_X(\omega)$ is a maximal invariant compact set of (θ,φ) and from equality (3.12) follows (3.1). According to Theorem 3.1, (θ,φ) is local dissipative. ■

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