Fuzzy Soc–Semi T–ABSO Modules and Related Conepts

Mustafa B. Salman
mustafaalghazalialghazali@gmail.com
Wafaa H. Hanoon
Dept of Mathematics-College of Education- University of Kufa, Najaf, Iraq.
wafaah.hannon@uokufa.edu.iq

Abstract— in this study, the connection between the fuzzy semi-T-ABSO module and the fuzzy socle is compared. To address this query, we provide an idea known as the fuzzy socle semi-TABSO modules. Several characteristics were found in the research that supported the new theory. Also, using simple algebraic methods, the connection between the fuzzy socle-semi T-ABSO module and the fuzzy socle-T-ABSO module was found. Also, we looked into the structure of the fuzzy socle semi-TABSO module for the fuzzy direct sum. The connections between the fuzzy socle semi-TABSO module and other fuzzy module types were also discussed as a divisible module and a comultiplication module. The definition of the new fuzzy system semi-TABS concepts will benefit from the research's conclusions.

Keywords: Fuzzy Soc-T–ABSO modules; Fuzzy Soc–Semi T–ABSO modules; Fuzzy Soc–Semi T–ABSO ideal.

1 INTRODUCTION

A fuzzy subset $A\ of\ X$ was described in 1965 by Zadeh [1]. Since then, other fields get developed the theory of fuzzy sets. Historical branches of pure mathematics and algebra presented the concept of fuzzy sets. Rosenfeld [2] introduced fuzzy subgroups and the fuzzy sub-groupoid in 1971. Negoita and Ralescu first introduce the ideas of fuzzy modules and fuzzy submodules in 1975 [3]. Fuzzy quotient modules and fuzzy finitely produced submodules [4]. Therefore, Saikia and Kalita [5] constructed fuzzy essential submodules and investigated their properties. Also looked into this In this article, all rings are unitary and commutative with identity. Hadi [6] 2004 proposed the idea of a semiprime fuzzy submodule. Saad [7], discussed the fuzzy socle semiprime submodule

In (2019), Wafaa [8] proposed the idea of a semi-TABSO fuzzy submodule (2019) In 2022, Saad [7] introduced the fuzzy Soc-T-ABSO submodule concept. Mustafa and Wafaa [9], presented the concept of socle semi-T-ABSO submodule

In this article, the notion of semi T-ABSO fuzzy module is generalized to a fuzzy Socle semi-T-Absorbing module. Two sections make up this article. We give a few basic definitions and attributes that we will need in the following section in section one. The fuzzy Socle semi-T-ABSO module's many fundamental characteristics, outputs, and findings are examined in section two.

Note: o.w., $F-set, F-submo$, $F-ideal$, $F-module, F-singleton$, and $F-Socle\ T-ABSO\ submodule$ are abbreviations for $otherwise$, fuzzy sets, submodules, ideals, modules, and singletons.

2 Basic concept

There are many fundamental ideas, and this part outlines them along with any traits they require for the next section.

Definition 2.1. [1]: let $x_1: D \to I$ be an F-set in D, where $x \in D$, $t \in I$ defined by:

$$x_t\ (y) = \begin{cases} t & if\ x = y \\ 0 & if\ x \neq y \end{cases}$$

For all y ∈ D. x_t is said to be an F-singleton or F-point in D.

If x =0 and v =1, then $0_1(y) = \begin{cases} 1 & if\ y = 0 \\ 0 & if\ y \neq 0 \end{cases}$, [9]

Definition 2.2 [10]: let B be an F-set is D, for all $t \in I$ the set $B_t = \{x \in D; B(X) \geq t\}$ is said to be a level subset of B. Keep in mind that, B_t is a subset of D in the ordinary sense.

Definition 2.3. [11]: If C is an F-module of a W-module G, then the submod Ct of M is called the level submod of G where $t \in I$

Remark 2.4. [12]: Let A and B be two F sets in S, then:

1- $A = B\ iff\ A(x) = B(x)$.
2- $A \subseteq B\ iff\ A(x) \leq B(x)$.
3- $A = B\ iff\ A_t = B_t$

if $A < B$ and there exist $x \in S$ such that $A(x) < B(x)$, then A is a proper F-subset of B and written as $A < B$.

By part (2), we can deduce that $x_t \subseteq A$ iff $A(x) \geq t$

Definition 2.5. [12]: If G is a W-module. An F-set X of M is called an F-module of a W-module G if:

1- $X(x-y) \geq \min(X(x), X(y), forall x, y \in G\}$.
2- $X(rx) \geq X(x) forall x \in G \ and \ r \in W$.
3- $X(0) = 1$.

Definition 2.6. [13]: Let X and A be two F-modules of W-module G. A is said to be an F-submod of X if $A \subseteq X$.

Proposition 2.7. [10]: Let A be an F-set of a W-module G. Then the level subset $A_t, t \in I$ is a submod of G if A is an F-submod of X where X is an F-module of a W-module G.

Definition 2.8. [12]: Let $A \ and \ B$ be two $F-submods \ of \ an \ F-module \ X$. The residual quotient of $A \ and \ B$ denoted by $(A : B)$ is the F-subset of W defined by:
$(A : B)(r) = \sup\{t \in [0:1]: r_t B \subseteq A\}, for \ all \ r \in R$.
$That \ (A : B) = \{r_t: r_t B \subseteq A; r_t \ is \ a \ F-singleton o fR\}. If \ B = <x_k>, then \ (A :<x_k>) = \{r_t: r_t x_k \subseteq A: r_t \ is \ a \ F-singletono \ \{R\}$.

Lemma 2.9. [14]: Let A be an $F-submod \ of \ F-module \ X$, $(A_t : X_t) \supseteq (A : X)_t, For \ all \ t \in I$.
It follows that if, $X = A \oplus B, for \ A, B \leq X \ then \ X_t = (A \oplus B)_t = A_t \oplus B_t$.

Definition 2.10. [13]: An $F-subset \ K$ of a ring W is called an $F-ideal \ of \ W$, if $\forall x, y \in R$:
1) $K(x-y) \geq min\{K(x), K(y)\}$.
2) $K(xy) \geq \max\{K(x), K(y)\}$.

Proposition 2.11. [13]: Let A and B be two F-submods of an F-module X of a W-module G. Then $(A : B)$ is an F-ideal of W.

Definition 2.12. [15]: Let $X \ be \ an \ F-module \ of \ a \ W-module \ M, X$ is called F-simple iff X has no proper F-submods (in fact X is F-simple iff X has only itself and 0_1).
Moreover, the $F-Socle \ of \ X$ is sum of simple $F-submod \ of \ X$ and denoted by $F-Soc(X)$. That is X is called Semi-simple if $X = F-Soc(X)$.

Lemma 2.13. **[9]:** $(F-Soc(X))_t = Soc(X_t)$ for any F-module X for each $t \in I$ with $(F-Soc(X))_t \neq X_t$

Definition 2.14. [16]: Let X be an F-module of a W-module G. X is called a multiplication F-module iff for each F-submod $A \ of \ X$, there exists an F-ideal K of W such that $A = KX$.

Definition 2.15. [12]: An $F-module\ X$ of a W-module G is called a cyclic F-module if there exists $x_v \subseteq X\ such\ that\ y_k \subseteq X\ written\ as$ $y_k = r_l x_v$ for some F-singleton $r_l\ of\ R,$ where $k, l, v \in L$ in this case, write $X =< x_v >$ to denote the cyclic F-module generated by x_v.

Definition 2.16 [17]: A fuzzy submodule A of fuzzy module X is called a divisible fuzzy if for each F-singleton $x_v \subseteq A$ there exists a fuzzy singleton $y_h \subseteq A$ and for each $r \in R$, $r \neq 0$, $x_v = ry_h\ where\ (ry)_h = ry_h$, X is called a divisible fuzzy module if X is a fuzzy divisible submodule of itself

Proposition 2.17. [18]: Let X be an F $-$ module of a W-module G and P, $H\ be\ a\ F-\ submod\ \ of\ X$. Then P is F-Soc-semiT-ABSO submod iff $\mathrm{r}_s^2\mathrm{H} \subseteq \mathrm{P}$ for F-singleton r_s of W, implies $\mathrm{r}_\mathrm{s}\mathrm{H} \subseteq \mathrm{P} + \mathrm{F} - \mathrm{Soc}(\mathrm{X})$ or $\mathrm{r}_s^2 \subseteq (\mathrm{P} + \mathrm{F} - \mathrm{Soc}(\mathrm{X}) \colon \mathrm{X})$

Definition 2.18. [19]:

let X be a F-module of a W-module G. then X is called a semiprime F-module if for each non-empty F-submod A of X, F-ann A is a semiprime F-ideal of W

3 F-Soc-Semi T-ABSO Modules

Definition 3.1: let $X\ be\ an\ F-module\ of\ a\ W-module\ G$. Then X is called an $F-Soc-T-ABSO$ (F-Soc-prime) module if 0_1 is an $F-Soc-T-ABSO$ (F-Soc-prime) submod of X

Example 3.2: let $X = Z_4 \to I$ such that $X(y) = \begin{cases} 1 & if\ y \in Z_4 \\ 0 & o.w \end{cases}$

It's clear that X is an F-module of a Z-module Z_4.

$$0_{\frac{1}{4}}(y) = \begin{cases} \frac{1}{4} & if\ y = 0 \\ 0 & if\ \ y \neq 0 \end{cases}$$

$$F - Soc(X)(y) = \begin{cases} \frac{1}{2} & if\ y \in (\bar{2}) \\ 0 & if\ \ y \notin (\bar{2}) \end{cases}$$

$$0_{\frac{1}{4}} + F - Soc(X)(y) = \begin{cases} \frac{1}{2} & if\ y \in (\bar{2}) \\ 0 & if\ \ y \notin (\bar{2}) \end{cases}$$

$(0_1 + F - Soc(X):X)(y) = \begin{cases} 1 & if\ y \in 2Z \\ 0 & if\ \ y \notin 2Z \end{cases}$

X is an F-Soc-T-ABSO (F-Soc-prime) since $2_{\frac{1}{2}} \cdot 2_{\frac{1}{2}}\left(\bar{1}_{\frac{1}{4}}\right) = 0_{\frac{1}{4}}(4_{\frac{1}{4}} \cdot \left(\bar{1}_{\frac{1}{2}}\right) = 0_{\frac{1}{4}}$, then $2_{\frac{1}{2}}\left(\bar{1}_{\frac{1}{4}}\right) = 2_{\frac{1}{4}} \subseteq 0_{\frac{1}{4}} + F - Soc(X)$, where $0_{\frac{1}{4}} + F - Soc(X)(2) = \frac{1}{2} > \frac{1}{4}$ $(4_{\frac{1}{4}} = 0_{\frac{1}{4}} \subseteq 0_{\frac{1}{4}} + F - Soc(X)$ where $0_{\frac{1}{4}} + F - Soc(X)(4) = \frac{1}{2} > \frac{1}{4}$) and $2_{\frac{1}{2}} \cdot 2_{\frac{1}{2}} = 4_{\frac{1}{2}} \subseteq (0_1 + F - Soc(X):X)$, where $(0_1 + F - Soc(X):X)(4) = 1 > \frac{1}{2}$.

Definition 3.3: let $X\ be\ an\ F - module\ of\ a\ W - module\ G$. Then X is called an F-Soc-quasi module if 0_1 is an F-Soc-T-ABSO submod of X.

Remark and Example 3.4

1) Every F-semiprime module is an F-Soc-semi T-ABSO module, but the converse is not true in general, for example:

   Let $X: Z_{49} \rightarrow I$ like that $X(y) = \begin{cases} 1 & if\ y \in Z_{49} \\ 0 & o.w \end{cases}$

   It's clear that X is an F-module of a Z-module Z_{49}

   $0_{\frac{1}{3}}(y) = \begin{cases} \frac{1}{3} & if\ \ y = 0 \\ 0 & if\ \ y \neq 0 \end{cases}$

   $F - Soc(x)(y) = \begin{cases} \frac{1}{2} & if\ \ y \in (\bar{7}) \\ 0 & o.w \end{cases}$

   $0_{\frac{1}{3}} + F - Soc(x)(y) = \begin{cases} \frac{1}{2} & if\ \ y \in (\bar{7}) \\ 0 & if\ \ y \notin (\bar{7}) \end{cases}$

   $(F - Soc(X):X)(y) = \begin{cases} 1 & if\ \ y \in 7Z \\ 0 & if\ \ y \notin 7Z \end{cases}$

   X is the F-Soc-semi T-ABSO module since $7^2_{\frac{1}{2}} \cdot \left(\bar{1}_{\frac{1}{3}}\right) \subseteq 0_{\frac{1}{3}}$ then $7_{\frac{1}{2}} \cdot 1_{\frac{1}{3}} = 7_{\frac{1}{3}} \subseteq 0_{\frac{1}{3}} + F - Soc(X), 0_{\frac{1}{3}} + F - Soc(X)(7) = \frac{1}{2} > \frac{1}{3}$ or $7^2_{\frac{1}{2}} \subseteq (F - Soc(X):X)$, since $F - Soc(X){:}X)\ (49_{\frac{1}{2}}) = 1 > 1/2$, but X is not F-semiprime submod, since $7_{\frac{1}{2}} \cdot \left(1_{\frac{1}{2}}\right) = 7_{\frac{1}{2}} \nsubseteq 0_{\frac{1}{3}}$ where $0_{\frac{1}{3}}(7) = 0 \ngeq \frac{1}{2}$

2) Every (F-T-ABSO) F-Soc-T-ABSO module is an F-Soc-semi T-ABSO module, but the converse is not true in general, for instance:

Let $X{:}Z_{12} \to I$ such that $X(y) = \begin{cases} 1 & if\ y \in Z_{12} \\ 0 & o.w \end{cases}$

It is clear that X is an F-module of a Z-module Z_{12}

$0_{\frac{1}{3}}(y) = \begin{cases} \frac{1}{3} & if\ y = 0 \\ 0 & if\ \ y \neq 0 \end{cases}$

$F - Soc(X)(y) = \begin{cases} \frac{1}{2} & if\ y \in (\bar{6}) \\ 0 & o.w \end{cases}$

$0_{\frac{1}{3}} + F - Soc(X)(y) = \begin{cases} \frac{1}{2} & if\ y \in (\bar{6}) \\ 0 & if\ \ y \notin (\bar{6}) \end{cases}$

$(F - Soc(X){:}X)(y) = \begin{cases} 1 & if\ y \in 6Z \\ 0 & if\ \ y \notin 6Z \end{cases}$

$(0_1{:}X)(y) = \begin{cases} 1 & if\ y \in 12Z \\ 0 & if\ \ y \notin 12Z \end{cases}$

X is $F - Soc - semi\ T - ABSO\ module$ since $4^{2}_{\frac{1}{2}} \cdot (\bar{3}_{\frac{1}{3}}) = 0_{\frac{1}{3}}$, then $4_{\frac{1}{2}} \cdot 3_{\frac{1}{3}} = 12_{\frac{1}{3}} \subseteq 0_{\frac{1}{3}} + F - Soc(X)$, since $\left(0_{\frac{1}{3}} + F - Soc(X)\right)(12) = \frac{1}{2} > \frac{1}{3}$, but X is not an F-T-ABSO module since $2_{\frac{1}{2}} \cdot 2_{\frac{1}{2}} \cdot \left(\bar{3}_{\frac{1}{3}}\right) = 0_{\frac{1}{3}}$, then $2_{\frac{1}{2}} \cdot 3_{\frac{1}{3}} = 6_{\frac{1}{3}} \neq 0_{\frac{1}{3}}$ where $0_{\frac{1}{3}}(6) = 0 \ngeq \frac{1}{3}$ and $2_{\frac{1}{2}} \cdot 2_{\frac{1}{2}} = 4_{\frac{1}{2}} \notin (0_1{:}X)$ where $(0_1{:}X)(4) = 0 \ngeq \frac{1}{2}$

3) Every F-quasi-prime module is an F-Soc-semi T-ABSO module, but the counter wise not true see the instance in part (1) where $X = Z_{49}$ as Z-module is an F-Soc-semi T-ABSO module, but X is not F-quasi-

prime module since $7_{\frac{1}{2}} \cdot 7_{\frac{1}{2}} \cdot \left(\bar{1}_{\frac{1}{3}}\right) = 0_{\frac{1}{3}}$ and $7_{\frac{1}{2}} \cdot \left(\bar{1}_{\frac{1}{4}}\right) \neq 0_{\frac{1}{3}}$.

4) Every F-prime module is $an\ F - Soc - semi\ T - ABSO\ module$, but the converse is not true to see the example in part (2) where X is $an\ F - Soc - semi\ T - ABSO\ module$, but X is not an F-prime module since $4_{\frac{1}{2}} \cdot \left(\bar{3}_{\frac{1}{3}}\right) = 0_{\frac{1}{3}}$, then $3_{\frac{1}{3}} \neq 0_{\frac{1}{3}}$ and $4_{\frac{1}{2}} \not\subseteq (0_1 : X)$

Proposition 3.5:

If an F-module X of a W-module G $is\ an\ F - Soc - semi\ T - ABSO\ module$, then $F - ann\ X\ is\ an\ F - Soc - semi\ T - ABSO$ ideal $of\ W$.

Proof: let $n_s^2 m_r \subseteq F - ann\ X$ for F-singletons n_s, m_r of W, then $n_s^2 m_r X \subseteq 0_1$, so that $n_s^2(m_r x_v) \subseteq 0_1$ for all $x_v \subseteq X$, but X is the F-Soc-semi T-ABSO module, then we have $n_s(m_r x_v) \subseteq 0_1 + F - Soc(X)$ or $n_s^2 \subseteq (0_1 + F - Soc(X) : X)$, hence $n_s m_r \subseteq (0_1 + F - Soc(X) : X)$ or $n_s^2 \subseteq (0_1 + F - Soc(X) : X)$ so that $n_s m_r \subseteq (0_1 : X) + F - Soc(W)$ or $n_s^2 \subseteq (0_1 : X) + F - Soc(W)$

Thus, F-annX is an $F - Soc - semi\ T - ABSO\ ideal$

Proposition 3.6:

An F-multiplication module X of a W-module G is an F-Soc-semi T-ABSO module iff F-annX is F-Soc-semi T-ABSO ideal of W.

Proof: ⇒) by proposition 3.5, we have the result.

⇐) let $n_s^2 x_v \subseteq 0_1$ for n_s F-singleton n_s of W and $x_v \subseteq X$, but X is an F-multiplication module, so that $< x_v >= HX$ for some F-ideal H of W, hence $n_s^2 HX \subseteq 0_1$, so that $n_s^2 H \subseteq (0_1 : X) = F - annX$, then either $n_s H \subseteq F - annX + F - soc(W)$ or $n_s^2 \subseteq F - annX + F - Soc(W)$ since F-annX is an F-Soc-semi T-ABSO ideal.

So that either $n_s H \subseteq (0_1 + F - Soc(X) : X)$ or $n_s^2 \subseteq (0_1 + F - Soc(X) : X)$, hence $n_s HX \subseteq 0_1 + F - Soc(X)$

or $n_s^2 \subseteq (0_1 + F - Soc(X){:}X)$, then either $n_s x_v \subseteq 0_1 + F - Soc(X)$ or $n_s^2 \subseteq (0_1 + F - Soc(X){:}X)$.

Thus, X is $an\ F - Soc - semi\ T - ABSO\ module$

Proposition 3.7:

Let W be a ring that has no F-zero divisors and $X \neq 0_1$ an F-divisible module of a W-module G such that $F - Soc(W) \subseteq F - ann(X)$. Then X is $an\ F - Soc - semi\ T - ABSO\ module$ iff X is an F-Soc-quasi prime module.

Proof: ⇒) let $r_n s_m x_v \subseteq 0_1$ for F-singletons $r_n, s_m\ of\ W\ and\ x_v \subseteq X$. If $r_n s_m \subseteq 0_1$, then $r_n \subseteq 0_1\ or\ s_m \subseteq 0_1$ so that $r_n x \subseteq 0_1\ or\ s_m x_v \subseteq 0_1$, hence $r_n x \subseteq 0_1 + F - soc(X)$ or $s_m x_v \subseteq 0_1 + F - Soc(X)$. If $r_n s_m \nsubseteq 0_1$, then $r_n \nsubseteq 0_1\ or\ s_m \nsubseteq 0_1$, since W that has no F-zero divisors

If $r_n x_v \subseteq 0_1 + F - Soc(X)$, then the proof is perfect.

If $r_n x_v \nsubseteq 0_1, r_n \nsubseteq 0_1$ and $X\ is\ an\ F - divisible\ module$, hence $r_n X = X$, then $x_v = r_n h_y$ for F-singletons $h_y \subseteq X$, thus $r_n s_m x_v = r_n s_m r_n h_y = r_n^2 s_m h_y \subseteq 0_1$, but 0_1 is an $F - Soc - semi\ T - ABSO\ submod$, then either $r_n s_m h_y \subseteq 0_1 + F - Soc(X)$ or $r_n^2 \subseteq F - annX + F - Soc(W)$, hence $r_n^2 \subseteq F - annX$ since $F - Soc(W) \subseteq F - ann(X)$, but $r_n \nsubseteq 0_1$, then $r_n^2 \nsubseteq 0_1$, so that $r_n^2 X = X \subseteq 0_1$

This is contradiction. Therefore $r_n^2 \nsubseteq F - ann(X)$, hence $r_n^2 \nsubseteq F - ann(X) + F - Soc(W)$, then only we have $r_n s_m h_y \subseteq 0_1 + F - Soc(X)$, so that $s_m x_v \subseteq 0_1 + F - Soc(X)$. Thus, 0_1 is $an\ F - Soc - quasi - prime\ submod$, that is X is an F-Soc-quasi-prime module.

⇐) It is obvious

Proposition 3.8:

$An\ F - module\ X\ of\ a\ W - module\ G$ is an $F - Soc - semi\ T - ABSO\ module$ iff either $F - ann\ r_s x_v + F - Soc(W) = F - ann\ r_k^2 x_v + F - Soc(W)$ for any F-singletons $r_s\ of\ W\ and\ x_v \subseteq X$ such that $r_s x_v \nsubseteq 0_1 + F - Soc(X)$ or $r_s^2 X \subseteq 0_1 + F - Soc(X)$

Proof:

($\Rightarrow$) let $y_b \subseteq F - ann\ r_s^2 x_v + F - Soc(W), r_s x_v \nsubseteq 0_1$ then $r_s^2 y_b x_v \subseteq 0_1 + F - Soc(X)$, but $X\ is\ an\ F - Soc - semi\ T - ABSO\ module$ and let $r_s^2 \nsubseteq (0_1 + F - Soc(X): X)$ hence $r_s y_b x_v \subseteq (0_1 + F - Soc(X): r_s x_v)$, $y_b \subseteq (0_1 + F - Soc(X): r_s x_v) = (0_1: r_s x_v) + F - Soc(W), y_b \subseteq F - ann r_s x_v + F - Soc(W)$.

($\Leftarrow$) It is obvious

Proposition 3.9:

Let an $F - module\ X\ of\ a\ W - module\ G$ is an $F - Soc - semi\ T - ABSO\ module, then\ F - ann_x\ J$ is an $F - Soc - semi\ T - ABSO\ submod$, for every F-ideal $J\ of\ W$ with $J \nsubseteq F - annX$;

Proof: since $JX \nsubseteq 0_1$ hence $(0_1 :_X J) \neq X$ let $s_a^2 x_v \subseteq (0_1 :_X J)$ for F-singleton s_a of R and $x_v \subseteq X$, then $s_a^2 J x_v \subseteq 0_1$ since X is an $F - Soc - semi\ T - ABSO\ module$ and by proposition (2.24), we get either $s_a x_v \subseteq 0_1 + F - Soc(X)$ or $s_a^2 \subseteq (0_1 + F - Soc(X): X)$, hence $s_a x_v \subseteq (0_1 + F - Soc(X) :_X J)$ or $s_a^2 \subseteq ((0_1 + F - Soc(X) :_X J: X)$. Thus $F - ann_x J$ is $an\ F - Soc - semi\ T - ABSO\ submod\ of\ X$.

Proposition 3.10:

Let X be an $F - module\ of\ a\ W - module\ G$ such that $(F - Soc(X): X)$ is an $F - Soc - semiprime\ ideal\ of\ W$. Then X is an $F - Soc - semi\ T - ABSO\ module$ iff X an F-Soc-semiprime module.

Proof: ($\Rightarrow$) let $r_s^2 x_v \subseteq 0_1$ for F-singletons $r_s\ of\ W\ and\ x_v \subseteq X$ since X is an $F - Soc - semi\ T - ABSO\ module$, then $r_s x_v \subseteq 0_1 + F - Soc(X)$ or $r_s^2 \subseteq (0_1 + F - Soc(X) :_W X)$. Hence $r_s x_v \subseteq 0 + F - Soc(X)$ or $r_s^2 x_v \subseteq F - Soc(X): X)$, since $(F - Soc(X): X)$ is an F-Soc-semiprime ideal of W. Thus $r_s x_v \subseteq 0_1 + F - Soc(X), \forall x_v \subseteq X$.

Then 0_1 is an F-Soc-semiprime submod, so that X is an F-Soc-semiprime module.

(⇐)It's evident

Proposition 3.11:

let X be an $F-module\ of\ a\ W-module\ G$. if X is an $F-Soc-semi\ T-ABSO\ module$, then $F-annK$ is an $F-Soc-semi\ T-ABSO\ ideal$ for each non-empty F-submod $K\ of\ X$.

Proof: let K be a non-empty F-submodule of $X\ and\ F-ann_wK \neq \lambda_w$ where $\lambda_w(r)=1\ \forall r \in W$ because if $F-ann_wK=\lambda_w$ then $K=0_1$ which is discrepancy. Now, assume that $r_s^2 b_y \subseteq F-ann_wK$ for F-singletons $r_s, b_y\ of\ W$. Hence $r_s^2 b_y K \subseteq 0_1$. Since X is an $F-Soc-semi\ T-ABSO\ module$, then either $r_s b_y K \subseteq 0_1+F-Soc(X)$ or $r_s^2 \subseteq (0_1+F-Soc(X):_w X)$ by proposition (2.24). Hence $r_s b_y \subseteq (0_1:K)+F-Soc(W)$ or $r_s^2 \subseteq (0_1:X)+F-Soc(W) \subseteq (0_1:K)+F-Soc(W)$. Thus, $F-annK$ is an F-Soc-semi T-ABSO ideal.

Proposition 3.12:

Let $X=X_1 \oplus X_2\ be\ an\ F-module\ of\ a\ W-module\ G=G_1 \oplus G_2$ then X is $an\ F-Soc-semi\ T-ABSO\ module$ iff $X_1\ and\ X_2$ are F-Soc-semi T-ABSO modules

Proof: (⇒) let $r_n^2 x_v \subseteq 0_1$ and $r_n^2 y_h \subseteq 0_1$ for F-singleton $r_n\ of\ W$ and $x_v \subseteq X_1, y_h \subseteq X_2$, hence $r_n^2(x_v, y_h) \subseteq (0_1, 0_1)$ but X is an F-Soc-semi T-ABSO module, then either $r_n(x_v, y_h) \subseteq (0_1, 0_1)+F-Soc(X)=X_1 \oplus X_2)$ or $r_n^2 \subseteq F-annX+F-Soc(W)=F-ann(X_1) \cap F-ann(X_2)+F-Soc(W)$ so that $r_n x_v \subseteq 0_1+F-Soc(X_1)$ or $r_n^2 \subseteq F-ann(X_1)+F-Soc(W)$ and $r_n y_h \subseteq 0_1+F-Soc(X_2)$ or $r_n^2 \subseteq F-ann(X_2)+F-Soc(W)$. Thus $X_1\ and\ X_2$ are F-Soc-semi T-ABSO modules.

⇐) let $r_n^2(x_v, y_h) \subseteq (0_1, 0_1)$ for F-singleton $r_n\ of\ W$ and $(x_v, y_h) \subseteq X=X_1 \oplus X_2$, hence $r_n^2 x_v \subseteq 0_1$ and $r_n^2 y_h \subseteq 0_1$ but $X_1\ and\ X_2$ are F-Soc-semi T-ABSO modules, then either $r_n x_v \subseteq 0_1+F-Soc(X_1)$ or $r_n^2 \subseteq F-ann(X_1)+$

$F - Soc(W)$ and $r_n y_h \subseteq 0_1 + F - Soc(X_2)$ or $r_n^2 \subseteq F - ann(X_2) + F - Soc(W)$, so that $r_n(x_v, y_h) \subseteq (0_1 + F - Soc(X_1), 0_1 + F - Soc(X_2)$ or $r_n^2 \subseteq F - ann(X_1) + F - Soc(W) \cap F - ann(X_2) + F - Soc(W)$, hence $r_n(x_v, y_h) \subseteq (0_1, 0_1) + F - Soc(X_1 \oplus X_2) = (0_1, 0_1) + F - Soc(X)$ or $r_n^2 \subseteq F - ann(X_1) \cap F - ann(X_2) + F - Soc(W) = F - annX + F - Soc(W)$.

Thus X is an $F - Soc - semi\ T - ABSO\ module$.

4 Acknowledgment

The writher would wishes to express his profound gratitude for the referee his for attentive reading and for giving them time to correct our article.

5 References

[1] L. A. Zadeh, "Fuzzy Sets" Information and Control, vol. 8, pp. 338-353, 1965 . https://doi.org/10.1016/S0019-9958(65)90241-X
[2] A. Rosenfeld, "Fuzzy groups," Journal of Mathematical Analysis and Applications, vol. 35, pp. 512-517, 1971. https://doi.org/10.1016/0022-247X(79)90182-3
[3] C. V. Naegoita and D. A. "Ralescu, Application of Fuzzy Sets in System Analysis", Birkhauser, Basel, Switzerland, 1975.
[4] F. Z. Pan, "Fuzzy finitely generated modules," Fuzzy Sets and Systems, vol. 21, no. 1, pp. 105-113, 1987. https://doi.org/10.1016/0165-0114(87)90156-4
[5] H. K. Saikia and M. C. Kalita, "On fuzzy essential submodules,"Journal of Fuzzy Mathematics, vol.17, no. 1, pp. 109-117, 2009. https://www.researchgate.net/publication/328355742
[6] I. M. Hadi, (2004), "Semiprime Fuzzy Submodules of Fuzzy Modules", Ibn-Al-Haitham J. for Pure and Appl. Sci., vol.17, no.3, pp.112-123, 2004.
[7] S. S. Marie, H.Y. Khalaf, "Fuzzy Soc-T-ABSO Submodules", Iraqi Journal for Computer and Science and Mathematics, vol. 3, no.1, pp. 124-134, 2022.

[8] H.Y. Kalaf. and W. H. Hanoon , "Semi-T-ABSO Fuzzy Submodules and Semi-T-ABSO Fuzzy Modules", Second International Conference For Applied and Pure Mathematics, 2019.
[9] M. M. Ali Radman, "On Fuzzy Semiprime Submodules", International Journal of Innovation and Applied Studies, vol. 13, no.4, pp. 929-934, 2015.
[10] L. Martinez, "Fuzzy Modules Over Fuzzy Rings in Connection with Fuzzy Ideals of Rings", J.Fuzzy Math., vol. 4, pp. 843-857, 1996. https://doi.org/10.1155/2014/730932
[11] M. M. Zahedi. "A characterization of L-Fuzzy Prime Ideals", Fuzzy Sets and Systems, vol. 44, pp. 147-160, 1991. https://doi.org/10.1016/0165-0114(91)90040-W
[12] M. M. Zahedi. "On L-Fuzzy Residual Quotient Modules and P. Primary Sub-modules", Fuzzy Sets and Systems, vol. 51, pp.333-344; 1992. https://doi.org/10.1016/0165-0114(92)90024-X
[13] R. H. Jaber, "Prime Fuzzy Submodand Prime Fuzzy Modules", M. Sc. Thesis, University of Baghdad, 2001.
[14] M. Mashinchi and M.M. Zahedi, "On L-Fuzzy Primary Sub-modules", Fuzzy Sets Systems, vol. 4, no. 9, pp. 843-857; 1996. https://doi.org/10.1016/0165-0114(92)90328-2
[15] M. C. Kalita,"A study of fuzzy algebraic structures: some special types", Ph.D Thesis, Gauhati University, Gauhati, India, 2007.
[16] A. Maysoun. "F-regular fuzzy modules", M. Sc. Thesis, University of Baghdad, 2002.
[17] A. A Qaid "Some Results on Fuzzy Modules", M .Sc., Thesis, Unversity of Baghdad, 1999.
[18] M, B Salman,. and W. H. Hanoon, " Fuzzy Soc–Semi T–ABSO Sub-modules and Related Concepts", Journal of Interdisciplinary Mathematics, (to appear)
[19] M. A. Hamil, "Semiprime Fuzzy Modules", Ibn Al-Haitham Journal for Pure and Applied Science, vol. 25, no. 1, 2012

Article submitted 1 March 2023. Accepted at 11 April 2023

Published at 30 Jun 2023.