Study The Result Involution Graph for the Janko Group J_4

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Abstract— Consider a finite group G, and define I(G) as a collection of the involution elements in G. The simple undirected graph with vertex set being the elements of G with two vertices $x, y \in G$ are adjacent if $x \neq y$ and $xy \in I(G)$, is called the result involution graph and denoted by Γ_G^{RI}. In this work, we investigate the structure of the result involution graphs for the Janko groups J_4 to obtain certain number of graph attributes.

Keywords— Janko Groups, Result Involution Graph, Connectedness, Girth.

1 Introduction

Undoubtedly one of the most important techniques for examining group structure is yield by taking the action of a group on a graph. Many recent studies have demonstrated the efficacy of this strategy; for examples, see [1,4,6,8 and 9]. Having order 2 indicates that an element of a group is an involution. Devillers and Giudici initially presented an S_3-involution graph for a group G in [3] as a graph with a G-classes of involution as a vertex set, where two vertices are adjacent if they create an S_3-subgroup in a certain G-class. In their study, the structure and general properties of the S_3-involution graph for the group PSL (2,q) for q > 3 have been examined. Allow I(G) to be the whole collection of involution elements in the finite group G under discussion. The result involution graph, Γ_G^{RI}, is an undirected simple graph having the elements of G as a vertex set. Moreover, two vertices are connected by an edge if they are distinct and their product belong to I(G). The result involution graph and its properties were initially published by Jund and Salih [5]. In their investigation, they showed that for n>4, the graphs Γ_{Sn}^{RI} and Γ_{An}^{RI} are connected and have a diameter and radius at most 3 and girth 3. Also, they offer some really useful properties for the result involution graphs of the dihedral group and the quaternion group. Aubad and Salih, however, examined at the structure of the result involution graphs for the entire list of the Mathieu sporadic simple groups in [2]. The objective of this paper is to examine the result involution graph for the Janko group J_4. We present different graph features, such as the radius, diameter, clique number, and girth, along with the connectedness of the result involution graph. Assume that a graph Γ has vertex set $V(\Gamma)$ and edges set $E(\Gamma)$.Then Γ is connected if there is a path between any distinct vertices. Furthermore, the diameter of Γ is the length of the shortest path among the most distanced vertices. Also, the smallest of all maximum distances from a vertex to the other vertex is called the radius of Γ. Moreover, if there is an edge between any different vertices then the graph Γ is complete. The complete graph with *n* vertices is denoted by K_n. Finally, the girth of Γ is the length of the shortest cycle in the graph. For more information (see [10] and [11]).

The paper is structured as follows: In section 2, for the result involution graph, we provide some observations and graph notations. In section 3, certain result about Γ_{J4}^{RI} are furnished. Section 4 is where we come to our final conclusions and provide recommendations for further research.

2 Preliminary

We start the section by defining a few terms and presenting a number of facts that will later be crucial to the argument. We can proceed under the assumption that G is a Janko group J_4. Furthermore, let $s\mathbf{I}_G$=|I(G)| be denoted by the size of the set I(G). We first give the following formula to get the number of edges in the result involution graph:

Proposition 2.1. [5] Let *F* stand for the number of elements of order 4 in the finite group G. Then, the formula ½ $(s\mathbf{I}_G\ |G| - F)$ is used to compute the number of edges in the result involution graph.

For a finite group G, the result involution graph will have a large size of vertex set when the group G has a huge number of elements. In this case the graph Γ_G^{RI} is consequently quite challenging to handle. We employ the resize graph concept to address this problem, which seeks to compress the result involution graph by reducing the vertex set. The vertex set in the resize graph is made up of all of the G-conjugacy classes. The following is a definition of the resize graph:

Definition 2.2. [5] Suppose that G be a finite group, the resize graph of G has a vertex set being the complete set of the G-conjugacy classes, such that two vertices (G-conjugacy classes) X, Y $\subseteq$ G, are adjacent whenever they are distinct and their representatives are adjacent vertices in Γ_G^{RI}. The resized graph of the finite group G will be referred to as Γ_G^{RS}.

A relationship between a result involution graph and a resize graph could be seen in the findings that follow:

Proposition 2.3.[5] Assume that G be a finite group. Then the graph Γ_G^{RI} is connected if and only the graph Γ_G^{RS} is connected.

We will mostly employ GAP [13] and YAGS [7] as a computational approach in the following example:

Example 2.4. Let G≅D_{14} be the dihedral group of order 14. The vertex set of the result involution graph Γ_{D14}^{RI} is V={e, (2,7)(3,6)(4,5), (1,2)(3,7)(4,6), (1,2,3,4,5,6,7), (1,3)(4,7)(5,6), (1,3,5,7,2,4,6), (1,4)(2,3)(5,7), (1,4,7,3,6,2,5), (1,5)(2,4)(6,7), (1,5,2,6,3,7,4), (1,6)(2,5)(3,4), (1,6,4,2,7,5,3), (1,7,6,5,4,3,2), (1,7)(2,6)(3,5)}. We should note that we will label the vertex set in next figure as follows: The first element of the vertex set that is e, is labeled by 1, and the second element (2,7)(3,6)(4,5) is labeled by 2, and so on. This will utilize in all figures of this paper.

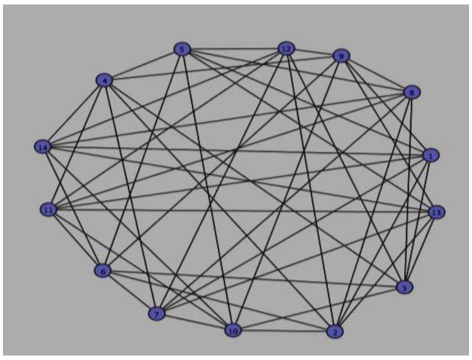

Fig. 1. The Result Involution Graph Γ_G^{RI}, G≅ D_{14}

Hence, the computational approach yields that the graph Γ_{D14}^{RS} is connected has the following properties:

Table 1. Properties of the Result Involution Graph Γ_G^{RI}, G≅D_{14}

Γ_G^{RI}	$E(\Gamma_G^{RI})$	Girth	Radius	Diameter
G≅D_{14}	49	4	2	2

On the other hand, the vertex set of Γ_{D14}^{RS} are the G- conjugacy class{1A, 7A, 7B,7C, 2A }. The next figure is the resize graph of G is presented:

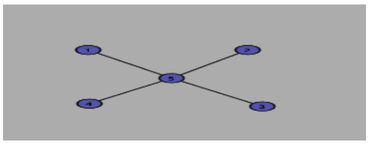

Figure 2. The Resize Graph Γ_G^{RS}, G≅D_{14}

From **Figure 2.** we see that Γ_{D14}^{RS} is (1,1)-biregular graph such that the disjoint set of vertices are presented by the labels X={5} and Y={1,2,3,4}. Also, the graph is connected with clique number and radius of size 1, also the diameter of size 2.

3 The Main Results

In this work, the structure of the J_4 result involution graph will be examined. We will employ GAP and the Online Atlas [12] to complete the aim of the study. We should be aware that J_4 has 51747149311 involution elements, which are divided into two classes 2A and 2B with sizes of 3980549947 and 47766599364, respectively. As a result, there are several involutions to consider. So, we will analyze the resizing graph Γ_{J4}^{RS} to reach our observations.

3.1 The Structures of Γ_{J4}^{RS}

One can see that from the Online Atlas that the Janko group has 62 distinct J_4-conjugacy classes. Therefore, the resize graph Γ_{J4}^{RS} with order 62. The graph Γ_{J4}^{RS} vertex set could be described in the below : { 1A , 2A , 2B , 3A , 4A , 4B , 4C , 5A , 6A , 6B , 6C , 7A , 7B , 8A , 8B , 8C , 10A , 10B , 11A , 11B ,12A , 12B , 12C , 14A , 14B , 14C , 14D , 15A , 16A , 20A , 20B , 21A , 21B , 22A , 22B , 23A , 24A , 24B , 28A , 28B , 29A , 30A , 31A , 31B , 31C , 33A , 33B ,35A , 35B , 37A , 37B , 37C , 40A , 40B , 42A , 42B , 43A , 43B , 43C , 44A , 66A , 66B }.
Now, by employing computational technology, we have discovered the connectedness of Γ_{J4}^{RS} as well as the following characteristics:

Table 2. Properties of the Resize Graph Γ_{J4}^{RS}

$E(\Gamma_{J4}^{RS})$	Girth	Radius	Diameter
1782	3	2	3

Besides that, the vertex set degree sequence is provided in the next:
{2, 25, 47, 60, 60, 60, 60, 60, 60, 60, 60, 58, 58, 60, 60, 60,60, 60, 59, 60, 60, 60, 60, 58, 58, 58, 58, 59, 60, 60, 60, 58,58, 59, 60, 59, 60, 60, 58, 58, 59, 59, 59, 59, 59, 59, 59, 58,58, 59, 59, 59, 59, 59, 58, 58, 59, 59, 59, 59, 59, 59}.

3.2 The Edges Set of Result Involution Graphs

In this section, for the Janko group J_4, we give a full detail about the edges set of the result involution graph Γ_{G}^{RI}. The information relates to the number of edges joining any two G-Conjugacy parts together. The findings will play a vital role in analyzing the structures of the result involution graphs for the Janko groups. In order to get the required result, we will utilize GAP and the Online Atlas.

Apart from the identity element, there are edges from the elements of class 3A and the other J_4-conjugacy classes. As shown in the set below:

{2A(32282578514165760),2B(303456238033158144),3A(5263674426734727168),4A(2530954155510595584),4B(87679483244474204160),4C(96331214286270627840),5A(242764990426526515200),6A(610173342582898688),6B(1221787988166126796800),6C(1395339130258281922560),7A(1686338597328773971968),7B(1686338597328773971968),8A(961039449335309008896),8B(3565029710476353208320),8C(3666267876696777031680),10A(1308864863278336573440),10B(27117365951899238400000),11A(101238166220423823360),11B(8677557104607756288000),12A(13414057024206156595200),12B(8381073903533657948160),12C(33987098659713712128000),14A(20594735528269074923520),14B(20594735528269074923520),14C(30675164364788418478080),14D(30675164364788418478080),15A(5403225557135762915328 0),16A(560570188957661056 20480),20A(14317969222602797875200),20B(14317969222602797875200),21A(41883675624906770350080),21B(41883675624906770350080),22A(89523464129203352 37120),22B(80874832214944288604160),23A(70461763689414981058560),24A(36474665029701268930560),24B(36474665029701268930560),28A(56837999035180803686400),28B(56837999035180803686400),29A(58833837169240587632640),30A(58544585265753662423040),31A(53453751764383778734080),31B(53453751764383778734080),31C(53453751764383778734080),33A(27045052976027507097600),33B(27045052976027507097600),35A(51371138059277917224960),35B(51371138059277917224960),37A(43040683238854471188480),37B(43040683238854471188480),37C(43040683238854471188480),40A(34189574992154559774720),40B(34189574992154559774720),42A(36156487935865651200000),42B(36156487935865651200000),43A(41825825244209385308160),43B(41825825244209385308160),43C(41825825244209385308160),44A(33235043710647706583040),66A(22374949516093150986240),66B(22374949516093150986240)}.

Moreover, we could a locate 5038110418018854 edges connecting the elements in class 2B.

3.3 The Structures of Γ_{J4}^{RI}

In the following theorem, a full details about the structures of the result involution graph for the Janko group J_4 :

Theorem 3.1: The result involution graph for J_4 is connected has diameter 3, radius 2, girh 3.

Proof:

The connectedness of the Γ_{J4}^{RS} is shown in **Table 2.** thus the connectivity of the Γ_{J4}^{RI} is demonstrated by **Proposition 2.3**. Furhrermore, there are 5038110418018854 edges between the elements of the class 2B, which are all connected by the identity element. Due to the existence of a cycle with length 3, Γ_{J4}^{RI} has girth 3. Furthermore, each vertices of the class 3A have an edges with each other classes. Hence, rather than being the identity element, the shortest path connecting all vertices has a maximum length of 3. Hence, the diameter of the graph Γ_{J4}^{RI} is 3. Now, the identity element is linked with 3A, the connection with it can come from either class 2B. Therefore, the radius of the Γ_{J4}^{RI} is 2.

Given the earlier finding, we offer the following crucial corollaries:

Corollary 3.2: Assume that G isomorphic to the Janko group J_4. Thus for a random elements $x,y \in$ G, we must have one of the following:

i- *xy* is an involution element.

ii- The elements wx and *wy* are an involution for a specific $w \in$ G.

iii- For a particular elments $z,w \in$ G, we have (*xz*), (*zw*) and (*wy*) are an involution element.

iv- There are three different elements in G, such that each one of them produce an involution if product with the others.

Proof:

Theorem 3.1 emphasizes that the diameter and the radius of the result involution graph Γ_G^{RI} are at most 2 and 3 respectively. Thus if the distance between *x* and *y* equal to 1, their product is an involution element and the statement (**i**) is followed. Moreover, one could a locate *w* in G satisfy the condtions in (**ii**) if the distance between *x* and *y* is 2 (ii). Furthermore, the diameter of the graph Γ_G^{RI} is 3, thus we can have the case that the distance between *x* and *y* is equal to 3. Therefore, there must be such $z,w \in$ G that fulfill the requirements of (**iii**).Finally, we have the girth is 3 for the result involution graphs of G. Thus there are three different elements *x*,*y* and *z* in G such that (*xy*), (*xz*) and (*yz*) have order 2, and (**iv**) is proved.

4 Conclusions

The result involution graph for the Janko group J_4 has been analyzed in this paper. the process of computation used to determine certain characteristics of graphs. For instance, the radius, diameter, and girth, as well as in-depth information on the resize graph. The results of this study may be used to investigate more complex simple groups, such as Monster groups, Pariahs groups, and exceptional Lie type groups.

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