New Type of Functions with Strongly Closed Graphs

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Abstract

In this paper, new type of functions with strongly closed graph called strongly semi-pre closed graph based on semi-pre closed sets. If the codomain of an almost semi pre continuous (or semi pre continuous or semi pre irresolute) function f is semi-pre T_2. Then the graph $G(f)$ is strongly semi-pre closed.

Keywords

Closed graph, semi-closed graph, pre-closed graph, strongly semi-pre closed graph

1. Introduction

Mashhour [2] and Levine [7] et al. presented the notion of pre - open, semi - open, semi – pre open sets and semi-pre continuity in a topological space and studied their different properties. Recently the authors [4] studied with the assist of pre- open sets, the concept of The functions with pre- closed graph utilizing preopen sets of Mashhour. In that paper, functions having pre closed graph have been studied in some detail. In 1996, Paul and Bhattacharyya [9] introduced the notion of functions with strongly pre closed graph in a different context. They did not carry their investigation concerning this new concept except one result. In this paper we study this notion in some detail and establish that functions with strongly semi-pre closed graph is a stronger notion than that of pre closed graph but weaker than strongly closed graph defined by Long and Herrington [8]. Some basic properties of functions with strongly semi- pre closed graph have been obtained.

2.Preliminaries

Definition 2.1

1. Let X is a topological space and $A \subseteq X$. The letter A° denotes the interior of A is defined by:-

$$A^{\circ} = \cup \{G \subseteq X: G \text{ is open set and } G \subseteq A\}.$$

2. Assume (X, τ) be a topological space , and let $A \subset X$ then it is called semi-open (briefly so) if $A \subseteq \overline{(A^{\circ})}$. The family of so sets is written as SO, while the family of all so sets containing x is denoted as $SO\ (X, x)$.
And the complement of A in X is a semi-closed set (briefly sc). The family of sc sets is written as SC, while the family of all sc sets containing x is denoted as $SC\ (X, x)$. **[7]**.
3. Assume (X, τ) be a topological space, and let $A \subset X$ then it is called pre-open (briefly po) if $A \subset (\overline{A})^{\circ}$. The family of po sets is written as PO, while the family of all po sets containing x is denoted as $PO\ (X, x)$.
And the complement of A in X is a pre-closed set (briefly pc). The family of pc sets is written as PC, while the family of all pc sets containing x is denoted as $PC\ (X, x)$. **[2]**.
4. Assume that X is a topological space, and let $A \subseteq X$. The letter $\overline{A}^{p}$ denotes the pre- closure of a set A **[5]** is defined by

$$\overline{A}^{p} = \cap \{F \subseteq X \,;\, F \text{ is pre-closed set and } A \subseteq F\}$$

5. Let $A \subset X$. Then A is regular open set (briefly ro) iff $A = (\overline{A})^{\circ}$ **[10]**.

Example 2.1

Assume (X, τ) be a topological space such that $X = \{1, 2, 3, 4\}$ and

$$\tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Assume $A = \{1, 2, 3\}$, then $A^{\circ} = \{1, 2\}$ and $\overline{(A^{\circ})} = \{1, 2, 3, 4\}$.
Since $A \subset \overline{(A^{\circ})}$. Then A is a semi-open set. $A^{c} = \{3, 4\}$ is a semi-closed set.

Example 2.2

Assume that (X,τ) be a topological space. $X = \{a,b,c,d,e\}$, $\tau = \{x,\emptyset, \{a\},\{c,d\},\{a,c,d\}\}$.
And let A, B are a subsets of X such that $A = \{b,e\}$ and $B = \{c,d\}$.
The closure of these sets are $\overline{A} = \{b,e\}$, $\overline{B} = \{b,c,d,e\}$.
A is not a pre-open set since $(\overline{A})^{\circ} = \emptyset$, $A \nsubseteq (\overline{A})^{\circ}$.
B is a pre-open set since $(\overline{B})^{\circ}$ = {c, d}, $B \subseteq (\overline{B})^{\circ}$.
$B^{c} = \{a,b,e\}$ is a pre-closed set.

Proposition 2.1 [7]

Let $A \subseteq X$.Then A is so set if there is an open set U such that $U \subset A \subset \overline{U}$.

Proposition 2.2 [6]

A set $B \subseteq X$ is sc set if there is a closed set F such that $F^{\circ} \subset B \subset F$.

Definition 2.2 [1]

Assume that (X,τ) is be a topological space, then the set $A \subset X$ is called semi-pre-open (briefly spo) if $A \subset \overline{(\overline{A})^{\circ}}$. The family of spo sets is written as SPO, while the family of all spo sets containing x is denoted as $SPO\ (X,x)$.
And the complement of A in X is a semi-pre-closed set (briefly spc). The family of spc sets is written as SPC, while the family of all spc sets containing x is denoted as $SPC\ (X,x)$.

Example 2.3

Let (X,τ) be a topological space. $X = \{x,y,z\}$, $\tau = \{X,\emptyset, \{x\},\{z\},\{x,y\},\{x,z\}\}$
$SPO(X,x) = \{X,\emptyset, \{x\},\{z\},\{x,y\},\{x,z\},\{y,z\}\}$,
$SPC(X,x) = \{X,\emptyset, \{x\}, \{z\}, \{x,y\}, \{y\},\{y,z\}\}$

Definition 2.3

Assume that X be a topological space and let $A \subseteq X$. The letter $\overline{A}^{sp}$ denotes the semi-pre closure of A and it is defined as follows:

$$\overline{A}^{sp} = \cap \{F \subseteq X\ ; F \text{ is semi- pre closed set and } A \subseteq F\}$$

Definition 2.4

1. Assume that X and Y are be a topological spaces, the function $f : X \to Y$ is called pre- closed function (pre-open) **[3]** if each pre-closed (pre-open) set $O \subseteq X$ then $f(O)$ be a pre-closed (pre-open) set in Y.
2. Assume that X and Y are be a topological spaces, the function $f : X \to Y$ is called semi-pre closed function (semi-pre open) if each semi-pre closed (semi-pre open) set $O \subseteq X$ then $f(O)$ be a semi-pre closed (semi-pre open) set in Y.
3. A function $f : (X,\tau) \to (Y,\delta)$ is called pre-continuous (briefly pco) **[2]** if and only if for each $x \in X$ and each $V \in \sum(f(x))$ there exists a $U \in PO(X,x)$ such that $f[U] \subset V$.
4. A function $f : (X,\tau) \to (Y,\delta)$ is called semi – pre continuous (briefly $spco$) if and only if for each $x \in X$ and each $V \in \sum(f(x))$ there exists a $U \in SPO(X,x)$ such that $f[U] \subset V$.
5. A function $f : (X,\tau) \to (Y,\delta)$ is called semi-pre irresolute (briefly spi) if for each $x \in X$ and for each $V \in PO\big(Y,f(x)\big)$ there exists $U \in PO(X,x)$ such that $f[U] \subset V$.

Definition 2.5 [11]

Assume $f : (X,\tau) \to (Y,\delta)$ be a function . Then the subset $G(f) = \{(x,f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \delta)$ is called the graph of f.

Definition 2.6 [11]

Take any topological space for X and Y. If the graph of a function f in the product space $X \times Y$ is closed, then the function f is said to have a closed graph.

Lemma 2.1 [11]

Assume that $f : X \to Y$ be assumed . Then $G(f)$ is closed if and only if for each $(x,y) \in (X \times Y) - G(f)$ There are $U \in \sum(x)$ in X and $V \in \sum(y)$ in Y such that
$f[U] \cap G(f) = \emptyset$.

3.Strongly semi pre-closed graph

Theorem 3.1

If $f : X \rightarrow Y$ is a function with strongly semi-pre closed graph, then for each $x \in X$, $f(x) = \cap \left\{ \overline{(f(U))}^{sp} : U \in SPO(X,x) \right\}$.

Proof. Let's say the theorem is incorrect. There is a point then. $y \in Y$ such that $y \neq f(x)$ and $y \in \cap \left\{ \overline{(f(U))}^{sp} : U \in SPO(X,x) \right\}$. This means that $y \in \overline{(f(U))}^{sp}$ for every $U \in SPO(X,x)$. So $V \cap f(U) \neq \emptyset$ for every $V \in SPO(Y,y)$. This, in its turn, indicates that $\overline{V}^{sp} \cap f(U) \supset V \cap f(U) \neq \emptyset$ This is in opposition to the assertion that f is a function with a strongly semi-pre closed graph.

Definition 3.1

Let $f : X \rightarrow Y$, then the graph $G(f)$ is said to be strongly semi-pre closed if for each
$(x,y) \in (X \times Y) - G(f)$ there are $U \in SPO(X, x)$, $V \in SPO(Y,y)$, such that$[U \times \overline{V}^{sp}] \cap G(f) = \emptyset$.

Lemma 3.1

The function $f : X \rightarrow Y$ has a strongly semi-pre closed graph if and only if for each $(x,y) \in (X \times Y) - G(f)$ there are $U \in SPO(X,x), V \in SPO(Y,y)$, such that $f[U] \cap \overline{V}^{sp} = \emptyset$.

Definition 3.2

A space (X,τ) is called semi-pre T_1 if and only if for $x,y \in X$ such that $x \neq y$ there exists $a\ spo$ set containing x but not y and a spo set containing y but $not\ x$.

Definition 3.3

A space (X,τ) is called semi-pre T_2 if and only if for $x,y \in X$, $x \neq y$ there exists $U \in SPO(X,x)$, $V \in SPO(Y,y)$ such that $U \cap V = \emptyset$.

Remark 3.1

Every semi-pre T_2-space is semi-pre T_1-space.

Lemma 3.2

For any $A \subset X$, $\overline{A}^{sp} \subset A$.

Definition 3.4

Let (X,τ) be a topological space and $A \subset X$, the set A is called semi-pre neighborhood of a point x in X, if there exists semi-pre open set U in X such that $x \in U \subset A$.

Theorem 3.2

Assume that $f : X \to Y$ be $aspc$ and let Y is semi-pre T_2. Then $G(f)$ is strongly semi-pre closed.

Proof. Let $(x,y) \in (X \times Y) - G(f)$, $y \neq f(x)$. Since Y is semi-pre T_2 there exists a set $V \in SPO(y)$ in Y such that $f(x) \notin \overline{V}^{sp}$. Now $\overline{V}^{sp}$ is a regular closed set in Y. So, $Y - \overline{V}^{sp} \in RO(Y,f(x))$. Therefore, by the almost semi-pre continuity of f there exists $U \in SPO(X,x)$ such that $f[U] \subset Y - \overline{V}^{sp}$ whence $f[U] \cap V = \emptyset$. By Lemma 3.2, one obtains $f[U] \cap \overline{V}^{sp} = \emptyset$.

Corollary 3.1

If $f : X \to Y$ is spc and Y is semi-pre T_2, then $G(f)$ is strongly semi-pre closed.

Proof. Since semi-pre continuity implies almost semi-pre continuity, the result follows .

Remark 3.2

As seen by the following example, the inverse of the aforementioned conclusion is often false.

Example 3.1

Let $X = \{1, 2, 3, 4\}$ be equipped with topologies $\tau_1 = \{\emptyset, X, \{3, 4\}\}$ and τ_d is the discrete topology. Then (X, τ_d) is T_2 and the graph of the identity function $i : (X, \tau_1) \to (X, \tau_d)$ is strongly semi-pre closed but i is not $aspc$.

Theorem 3.3

Let $f : X \to Y$ be spi and let Y is semi-pre T_2. Then $G(f)$ is strongly semi-pre closed .

Proof. Assume $(x, y) \in (X \times Y) - G(f)$, $f(x) \neq y$. Since Y is semi-pre T_2, $\exists\ U \in SPO(Y, f(x))$, $V \in SPO(Y, y)$ such that $U \cap V = \emptyset$ since $V \in SPO(Y, y)$ we have $f(x) \notin \overline{V}^{sp}$. Then $Y - \overline{V}^{sp} \in SPO(Y, f(x))$ since f have spi. Then $U \in SPO(X, x)$ such that $[U] \subset Y - \overline{V}^{sp}$. Hence $f[U] \cap \overline{V}^{sp} = \emptyset$. That mane $G(f)$ is a strongly semi-pre closed graph.

Theorem 3.4

If $f : X \to Y$ is spi and Y is semi-pre T_2. Then $G(f)$ is strongly semi-pre closed.

Proof. Let $(x, y) \in (X \times Y) - G(f)$. since Y is semi-preT_2. Then there exists $V \in SPO(Y, y)$ such that $f(x) \notin \overline{V}^{sp}$. Then $(Y - \overline{V}^{sp}) \in SPO(Y, f(x))$. The spi of f gives a $U \in SPO(X, x)$ such that $f[U] \subset (Y - \overline{V}^{sp})$.
Consequently, $f[U] \cap \overline{V}^{sp} = \emptyset$. Then $G(f)$ is strongly semi-pre closed.

Remark 3.3

In general, the opposite of Theorem 3.1 is untrue. Example 3.1 makes this plain.

Definition 3.5

A function $f : X \rightarrow Y$ is almost semi-pre irresolute (briefly $aspi$) if and only if for each $x \in X$ and for each semi- pre nbd V of $f(x)$, $\overline{f^{-1}[V]}^{sp}$ is a semi-pre nbd of x.

Remark 3.4

Obviously every spi function is $aspi$ but the converse is not true.

Remark 3.5

An $aspi$ function may fail to have a strongly semi-pre closed graph as the following example shows.

Example 3.2

Assume that $X = Y = \{1, 2\}$ be equipped with the topologies τ_d and τ where τ_d is the discrete topology and $\tau = \{\emptyset, X, \{1\}\}$. Let $i : (X, \tau_d) \rightarrow (Y, \tau)$ be the identity function.
Then i is $aspi$ but $G(i)$ is not strongly semi-pre closed.

Remark 3.6

In a view of Example 3.1, a function having a strongly semi-pre closed graph need not be spi.

4.Conclusion

1. Let $f : X \rightarrow Y$ be $aspc$ and let Y is semi-pre T_2. Then $G(f)$ is strongly semi-pre closed.
2. If $f : X \rightarrow Y$ is spc and Y is semi-pre T_2, then $G(f)$ is strongly semi-pre closed .
3. Let $f : X \rightarrow Y$ be spi and let Y is semi-pre T_2. Then $G(f)$ is strongly semi-pre closed.

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