Computational Investigation of The Result Involution Graphs for The Conway Group Co_3

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Abstract— The result involution graph, Γ_G^{RI}, for a finite group G, is an simple graph has the group G elements as a vertex set and two vertices are adjacent if they are distinct and their product is an involution element. In this paper, the result involution graphs for the Conway group Co_3 are investigated. The connectivity of Γ_{Co3}^{RI} and particular features are computing such as the dimeter, the girth and the clique number.

Keywords— Conway Groups, Result Involution Graph, Connectedness, Girth.

1 Introduction

The action of a group on a graph is probably one of the most significant methods for analyzing group structure. The extensive citations in [1,2,4 and 7] highlight the numerous and varied graphs that have been examined for different groups. An element of a group is said to be involution if it has order 2. The involutions of a group have a substantial impact on its structure. As an illustration, the study of involutions plays a significant role in the classification of finite simple groups. An S3-involution graph for a group G was first described by Devillers and Giudici in [3] as a graph has a G- conjugacy classes of involution as a vertex set, and two vertices are adjacent if they produce an S3-subgroup in a particular G-class. The structure and general characteristics of the S3-involution graph for the group PSL (2,q) for $q > 3$ have been investigated in that paper. Let I(G) represent the collection of all involution elements in the finite group G that we are considering. The result involution graph, Γ_G^{RI}, is an undirected simple graph, having G as a vertex set, and two vertices are joined by an edge if they are distinct, and their product lie in I(G). Jund and Salih first published the result involution graph and its characteristics [5]. In their study, they demonstrated that for n>4, graphs Γ_{Sn}^{RI} and Γ_{An}^{RI} are connected having diameter and radius at most 3, also, girth is 3. The structure of the result involution graphs for the Mathieu sporadic simple groups was also examined by Aubad and Salih in [2]. The results in involution graphs for Conway group Co_3, Γ_{Co3}^{RI} are investigated in this paper. The radius, diameter, and girth of Γ_{Co3}^{RI}, among other properties, are specifically determined.

Consider a graph Γ and let $V(\Gamma)$ and $E(\Gamma)$, respectively be the vertex set and edges set of Γ. If a path exists between any two different vertices in a graph, then graph is said to be connected. Also, the length of the shortest path among the most distanced vertices is the diameter of Γ. The radius of Γ is the smallest of all maximum distances from a vertex to every other vertex. Furthermore, the graph Γ is complete when there is an edge between any distinct vertices. The n vertices complete graph is signified by K_n. The clique of size k is a complete subgraph in Γ, and the maximum clique size is called the clique number and distinguished by $\omega(\Gamma)$. Finally, the length of the shortest cycle in Γ is considered as the girth of Γ.

The paper is arranged as follows: In section 2, we offer some fundamental findings and notations for the result involution graphs. In section 3, for the Conway group Co_3 result involution graph, we establish specific findings. Finally, in section 4, we draw conclusions and offer suggestions for more studies.

2 Preliminary

To begin, we define a few concepts and describe a few observations that will be helpful later on in the work. We should allow G to be isomorphic to the Conway group Co_3 going forward. Additionally, define $s\mathbf{I}_G$=|I(G)| as the size of the set I(G) in the group G. In order to calculate the number of edges set in the result involution graph, we initially provide the following formula:

Proposition 2.1. [5] Let G be a finite group, and let F represent the number of elements of order 4 in G. Then the number of edges in the result involution graph is calculated using the formula ½ $(s\mathbf{I}_G\,|G| - F)$.

Note that a resize graph of a group G is a simple graph including all of the conjugacy classes for the group G as a vertex set. Also, two G-conjugacy classes X, Y are adjacent if and their representatives are adjacent in Γ_G^{RI} [5]. The resize graph of the group G may be depicted by the symbol Γ_G^{RS}. The relation among a result involution graph and resize graph is demonstrated in the following findings:

Proposition 2.2. [5] Let G be a finite group. Then the connectivity of the graphs Γ_G^{RI} and Γ_G^{RS} are equivalent.

These observations will be shown by the example immediately follows.

Example 2.3. Let G≅Q_{16} the Quaternion group of order 16. Then the vertex set of Γ_{Q16}^{RI} describe in the following set:{ e , (1,2,3,4,5,6,7,8)(9,10,11,12,13,14,15,16), (1,3,5,7)(2,4,6,8)(9,11,13,15)(10,12,14,16),(1,4,7,2,5,8,3,6)(9,12,15,10,13,16,11,14),(1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16),(1,6,3,8,5,2,7,4)(9,14,11,16,13,10,15,1

2),(1,7,5,3)(2,8,6,4)(9,15,13,11)(10,16,14,12),(1,8,7,6,5,4,3,2)(9,16,15,14,13,12,11,10),(1,9,5,13)(2,16,6,12)(3,15,7,11)(4,14,8,10),(1,10,5,14)(2,9,6,13)(3,16,7,12)(4,15,8,11),(1,11,5,15)(2,10,6,14)(3,9,7,13)(4,16,8,12),(1,12,5,16)(2,11,6,15)(3,10,7,14)(4,9,8,13),(1,13,5,9)(2,12,6,16)(3,11,7,15)(4,10,8,14),(1,14,5,10)(2,13,6,9)(3,12,7,16)(4,11,8,15),(1,15,5,11)(2,14,6,10)(3,13,7,9)(4,12,8,16),(1,16,5,12)(2,15,6,11)(3,14,7,10)(4,13,8,9)}.

Then, employing principally Gap [9] and Yags [6] soft computing techniques. We may deduce that the result involution graph $\Gamma_{\mathrm{Q}16}^{RI}$ is disconnected with 13 components 10 of them has one vertex while the remaining 3 is with two vertices. The result involution graph for $\Gamma_{\mathrm{Q}16}^{RI}$ is shown in the following figure. Keep in mind that we will modify each vertex according to its location in the vertex set. For instance, the first member of the vertex set, which is *e*, is labeled 1, the second element, (1,2,3,4,5,6,7,8)(9,10,11,12,13,14,15,16), is labeled 2, and so on. We should note that this numbered will apply to all figures in this paper.

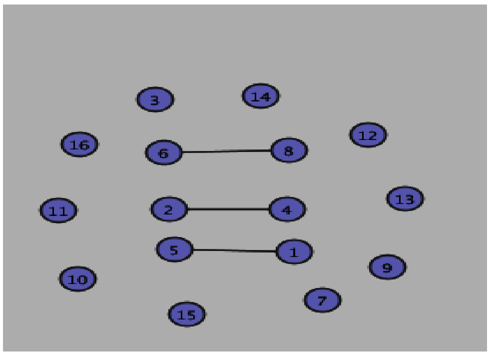

Fig. 1. The Result Involution Graph Γ_{G}^{RI}, G≅ Q16

Observe the resize graph Γ_{Q16}^{RS} at this instant. The vertex set for this graph is provided as the set of G-conjugacy classes: 1A, 2A, 4A, 8A, 8B, 4B and 4C. The graph Γ_{Q16}^{RS} describe in below figure:

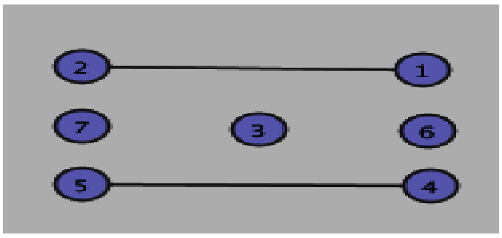

Fig. 2. The Resize Graph Γ_{G}^{RS}, G≅Q16

We deduce from **Figure 2** that Γ_{Q16}^{RS} is disconnected holding five components, three of which have a single vertex and the others with two linked vertices.

3 The Main Results

The structure of the result involution graph for Co_3 will be analyzed in this. To achieve the goal of the study, we shall use Gap and the Online Atlas [8] for this purpose. We should note that the number of the involution elements in Co_3 are $\boldsymbol{sI}_{Co3}$= 2778975 which splits into two classes 2A and 2B with sizes respectively equal 170775 and 2608200. There are therefore several involutions to take into account. Consequently, in order to arrive at our observations, we will investigate the resize graph Γ_{Co3}^{RS}.

Lemma 3.1: The resize graph of Co_3, Γ_{Co3}^{RS} is connected with dimeter 3, radius 2, girth 3 and clique number 34.

Proof: From the Online Atlas we can see that the vertex set of Γ_{Co3}^{RS} are {1A, 2A, 2B, 3A, 3B, 3C, 4A, 4B, 5A, 5B, 6A, 6B, 6C, 6D, 6E, 7A, 8A, 8B, 8C, 9A,9B, 10A, 10B, 11A, 11B, 12A, 12B, 12C, 14A,15A, 15B, 18A, 20A, 20B, 21A, 22A, 22B, 23A, 23B, 24A, 24B, 30A}. Thus Γ_{Co3}^{RS} has 42 vertices. Now we create the resize graph Γ_{Co3}^{RS} with above vertices inside Gap by using the Yags packages code **GraphByEdges**. The first thing to note about Γ_{Co3}^{RS} is that the vertex degrees sequence of vertices are respectively:

{ 2, 15, 33, 22, 36, 36, 36, 39, 35, 39, 36, 39, 40, 40, 38, 39, 40, 40, 40, 39, 38, 37, 40, 37, 37, 39, 39, 40, 39, 39, 39, 39, 38, 38, 37, 37, 37, 37, 37, 39, 38, 39}. The graph Γ_{Co3}^{RS} is connected because of the degree of vertex 8A is 40 so it has an edge with all the vertices of the graph expect for 1A which connected with 8A by 2A or 2B, so that the radius is 2. Also the following vertices {11A, 11B, 20A, 20B,22A,22B,23A,23B} has distance 3 with 1A, the other vertex has distance either 1 or 2, so that the dimeter is 3. Finally, the class 1A,2A and 2B are complete subgraph thus the girth is 3.

In the following theorem we provide the connectivty and detreminate the girth, dimeter and the raduis of graph Γ_{Co3}^{RI}:

Theorem 3.2: The result involution graph Γ_{Co3}^{RI} is connected with dimeter 3, radius 2, girth 3.

Proof: Lemma 3.1 provides the connectivity of the Γ_{Co3}^{RS} and by **Propostion 2.2** the connectivity of the Γ_{Co3}^{RI} is follow. Furthermore, Gap calculation can be employed to find 53794125 edges between the elements in the class 2A and those elements are all linked with the identity element. Therefore, there are a cycle of length 3, which means that $\omega(\Gamma_{Co3}^{RI})$ =3.

Moreover, there are an edges from the elements of the class 6C and the other Co_3-conjugacy classes except the identity element. As we see in the below set:

{2A(13771296000),2B(165255552000),3A(9180864000),3B(720697824000),3C(716107392000),4A(1301387472000),4B(7002704016000),5A(5536060992000),5B(30076510464000),6A(3029685120000),6B(4709783232000),6C(12927804120000),6D(70977259584000),6E(89733764736000),7A(173270446272000),8A(43090385184000),8B(67259009664000),8C(196199654112000),9A(42415591680000),9B(77725194

624000),10A(136418458176000),10B(344557825920000),11A(281595460608000),11B(281595460608000),12A(61061926464000),12B(114274214208000),12C(203429584512000),14A(385954341696000),15A(198967684608000),15B(385871713920000),18A(366536814336000),20A(317538543168000),20B(317538543168000),21A(288536193792000),22A(294981160320000),22B(294981160320000),23A(254824061184000),23B(254824061184000),24A(338525998272000),24B(231440400576000),30A(185086218240000)}. Thus length of the shortest path between all the vertices rather than the identity element is at most 3. Hence, the graph Γ_{Co3}^{RI} has diameter 3. Now, the connected with the identity element can come from the class 2A or 2B which both linked with identity element. Thus the Γ_{Co3}^{RI} has radius 2.

The next corollary follow immediately from the consequences of **Theorem 3.2**:

Corollary 3.3 : For any distinct elements *x,y* in the Conway group Co_3. One of the following are holds:

i- Their product are involution element.

ii- There exist an certain $z \in$ G, satisfy his product with *x* and *y* produce an involution elements.

iii- There are an elments $z,w \in$ G, satisfy (*xz*), (*zw*) and (*wy*) are involution elements.

4 Conclusion

In this work the results involution graph for the Conway group Co_3 have been investigated. The method of computing used to ascertain certain features of graphs. For example, the radius, the diameter and the girth, also comprehensive details on the resizing graph. This research may be used to analyze more advanced simple groups, such as Monster groups, Pariahs groups, or exceptional Lie type groups.

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6 References

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