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Fermat Operational Matrix for Solving Nonlinear Fractional Differential Equations Using Spectral Galerkin Method

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ABSTRACT: This paper presents a truncated series of Fermat polynomials as a quick and effective way to solve fractional differential equations numerically. The suggested method converts the fractional differential equation with its beginning conditions into a set of algebraic equations. This transformation is achieved by using the Galerkin spectral technique in conjunction with the matrix of operations for fractional-order derivatives according to Caputo's definition of Fermat polynomials. The precision, efficiency, and stability of the suggested approach are illustrated through a series of numerical examples. The results also show excellent compatibility with exact solutions even when the exact solution is non-polynomial. In addition, comparisons with previously reported methods confirm the higher performance and reliability of the current strategy.

Keywords: Fermat polynomials; Nonlinear Fractional Equations; Fermat operational matrix; Galerkin method

1. INTRODUCTION

The capacity of fractional differential equations (FDEs) to simulate complicated processes that display memory effects and cumulative behaviors in a variety of disciplines, including as physics, engineering, biological systems, and control theory, has drawn a lot of interest in recent years. A particularly important class within this field is nonlinear fractional differential equations, where the system behavior is governed by nonlinear relationships involving fractional derivatives. The nonlinear nature of these equations significantly increases their complexity, making it difficult to obtain exact analytical solutions. Therefore, numerical methods are commonly employed to approximate their solutions. Spectral methods are very effective numerical techniques for solving FDEs, offering quick convergence and great accuracy in comparison to conventional techniques. These methods approximate the solution using a series of basis functions, with the expansion coefficients determined by spectroscopic techniques such as (Collocation), (Tau), and (Galerkin) methods. Among them, Galerkin's method is widely used, approximating the solution using a set of base functions while imposing the condition of orthogonality to the remainder with respect to the chosen domain of functions. For more details about spectral methods, you can refer to [1–4]. Fermat polynomials have also received attention in numerical analysis because of their mathematical properties and iterative relationships, which make them suitable for constructing spectral approximations and operational matrices of fractional derivatives. It is a special case of generalized Fibonacci polynomials of the type (p, q) [5]. Several studies [6–9] have shown that the use of Fermat polynomials within spectral methods leads to accurate and efficient numerical solutions to a variety of fractional equations. In order to solve Nonlinear FDEs, we provide a spectral approach based on Fermat polynomials in this paper. Fermat polynomials in a finite series are used to approximate the answer, a proposed Galerkin-type formulation is employed to convert the equation and its associated conditions into a set of algebraic equations that can be used to compute the expansion coefficients. Several numerical examples were given to illustrate the effectiveness of the suggested approach, comparing the results with other existing methods, showing high accuracy and rapid convergence using relatively few basis functions. The structure of this research is as the paper is organized as follows. First, the basic concepts of fractional differentiation are reviewed. Next, the definitions of Fermat polynomials are introduced. Then, the operational matrices for fractional derivatives in the Caputo sense are developed. After that, the proposed numerical method is described, followed by various numerical examples. Finally, concluding remarks are presented.

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2. PRELIMINARIES

The principles of fractional calculus theory, which will be applied in this investigation, are reviewed in this part.
Definition 1: [10][11] The Riemann-Liouville fractional integrals of order β of a function $\varphi(x)$ is defined as:

$$I^{\beta}\varphi(x) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\beta}}\,dt, & x > a, \\ \varphi(x) & \beta = 0 \end{cases} \tag{1}$$

These are the characteristics of the (R._L.) fractional integral [11]:

1. if $\beta > 0$ & $\delta > 0$ $I^{\beta}I^{\delta}\varphi(x) = I^{\beta+\delta}\varphi(x)$
2. let $n, m \in R$: $I^{\beta}\big(n\varphi(x) + m\psi(x)\big) = nI^{\beta}\varphi(x) + mI^{\beta}\psi(x)$
3. for $k > 0$:$I^{\beta}x^{k} = \frac{\Gamma(k+1)}{\Gamma(k+1+\beta)}x^{k+\beta}$
4. C is constant: $I^{\beta}C = \frac{C}{\Gamma(\beta+1)}x^{\beta}$.

Definition 2: [13] The Riemann-Liouville fractional differential of order β of a function $\varphi(x)$ is defined as:

$$D^{\beta}\varphi(x) = \frac{1}{\Gamma(n-\beta)}\frac{d^{n}}{dx^{n}}\int_a^x \frac{\varphi(t)}{(x-t)^{\beta+1-n}}\,dt \tag{2}$$

Where n is an integer and $n-1 < \beta \le n$
Definition 3 [14]: The Caputo fractional derivative of order β of a function $\varphi(x)$ is defined as:

$${}^{C}D^{\beta}\varphi(x) = \frac{1}{\Gamma(n-\beta)}\int_a^x \frac{\varphi^{(n)}(t)}{(x-t)^{\beta+1-n}}\,dt \tag{3}$$

Where n is an integer and $n-1 < \beta \le n$
Some Properties of Fractional Derivative:

1. ${}^{C}D^{\beta}I^{\beta}\varphi(x) = \varphi(x)$
2. $I^{\beta}\ {}^{C}D^{\beta}\varphi(x) = \varphi(x) - \sum_{k=0}^{n-1}\frac{\varphi^{k}(0)}{k!}x^{k}, \quad x > 0$
3. $D^{\beta}x^{k} = \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}x^{k-\beta} \qquad , k \in \mathbb{N}, k \ge \lceil\beta\rceil$
4. ${}^{C}D^{\beta}\varphi(x) = D^{\beta}\varphi(x) - \sum_{k=0}^{n-1}\frac{(x-t)^{k-\beta}}{\Gamma(k+1-\beta)}D^{K}\varphi(t)$

3. A REVIEW OF FERMAT POLYNOMIALS

If we examine the (p,q)-Fibonacci polynomials' recurrence relation

$$u_n(x) = p(x)u_{n-1}(x) + q(x)u_{n-2}(x)$$

Considering the initial circumstances $u_0(x) = 0$ and $u_1(x) = 1$

by substituting $p(x) = 3x$ and $q(x) = -2$, we obtain the Fermat polynomials, which are considered A particular instance of the Fibonacci polynomials [15]. We can define the Fermat polynomials by means of the difference equation[16].

$$F_i(x) = 3xF_{i-1}(x) - 2F_{i-2}(x) \qquad F_0(x) = 0, F_1(x) = 1, \quad i \ge 2 \tag{4}$$

we can write the Binet formula for the Fermat polynomials. First, we find the two roots:

$$\alpha(x) = \frac{3x + \sqrt{9x^2 - 8}}{2}, \quad \beta(x) = \frac{3x - \sqrt{9x^2 - 8}}{2}$$

$$F_i = \frac{\alpha^{i}(x) - \beta^{i}(x)}{\alpha(x) - \beta(x)} \tag{5}$$

A Fermat polynomial can be written explicitly over the domain $x \in (0,1)$ as follows:

$$F_k(x) = \begin{cases} \dfrac{\left(3x+\sqrt{9x^2-8}\right)^k - \left(3x-\sqrt{9x^2-8}\right)^k}{2^k\sqrt{9x^2-8}}, & x \neq \dfrac{2}{3} \\ 2^{\frac{k}{2}} sin\left(\dfrac{\pi}{4}k\right), & x = \dfrac{2}{3} \end{cases} \quad k = 1,2,\cdots \tag{6}$$

The polynomial $F_k(x)$, of degree k−1, is characterized by integer coefficients, which is an important point

The analytic version of Fermat polynomials can be expressed as.$F_{i+1}(x), i \geq 1$ as follows:

$$F_{i+1} = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-2)^m \, 3^{i-2m} \binom{i-m}{m} x^{i-2m} \tag{7}$$

Theorem1. [6] The inversion formula presented below is applicable for every nonnegative integer k

$$x^k = \frac{1}{3^k} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{2^i(k+1-2i)}{k+1-i} \binom{k}{i} F_{k+1-2i}(x) \tag{8}$$

Theorem2.[6] The formula below establishes the connection between Fermat polynomials and their first derivatives:

$$DF_{i+1}(x) = 3 \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} 2^k \, (i-2k) F_{i-2k}(x) \tag{9}$$

4. THE CAPUTO FRACTIONAL DERIVATIVE'S FERMAT OPERATIONAL MATRIX

Let φ(x) denote a function on (0,1) that is square-integrable under the Lebesgue measure. It is assumed that φ(x) can be expanded in terms of a collection of linearly independent Fermat polynomials as follows:

$$\varphi(x) = \sum_{m=1}^{\infty} r_j \, F_j(x)$$

The following is applicable if the series is shortened.

$$\varphi(x) \approx \varphi_N(x) = \sum_{m=1}^{N+1} r_m \, F_m(x) = \mathrm{R}^T \Phi(x) \tag{10}$$

Where

$$\mathrm{R}^T = [r_1, r_2, r_3, \ldots\ldots, r_{N+1}] \tag{11}$$

And

$$\Phi(x) = [F_1(X), F_2(x), F_3(x), \ldots\ldots, F_{N+1}(x)]^T \tag{12}$$

Assuming $\frac{d\Phi(x)}{dx}$, we may write it as:

$$\frac{d\Phi(x)}{dx} = \mathrm{G}^1 \Phi(x) \tag{13}$$

$\varphi(x) = R^T \mathrm{G}^1 \Phi(x)$ where the non-zero entries of the matrix G^1,can be obtained directly from relation (9). which is the $(N+1) \times (N+1)$ (OMD).

Remark1. The overall structure of the (OMD) where $\mathrm{G}^1 = g_{ij}^{(1)}$ then can be written as follows:

$$g_{ij}^{(1)} = \begin{cases} 3(j+1)2^{\frac{i-j-1}{2}} & \text{if } i > j, (i+j) odd \\ 0 & , otherwise \end{cases} \tag{14}$$

4.1 Building of Fermat OMFD

In this section, we aim to formulate the operational matrix of fractional derivatives as an extension of the corresponding matrix for integer-order derivatives. It follows directly from Relation (9) that for any positive integer:

$$\frac{d^s\Phi(x)}{dx^s} = \mathrm{G}^{(s)}\Phi(x) = \left(\mathrm{G}^{(1)}\right)^s\Phi(x) \tag{15}$$

Theorem3:[6] Let the vector be Φ(x) of Fermat polynomials described by Equation (12). For $t \in (0,1)$ and any $\beta > 0$, one has:

$$D^\beta\Phi(x) = x^{-\beta}\mathrm{G}^{(\beta)}\Phi(x) \tag{16}$$

where $\mathrm{G}^{(\beta)} = g_{i,j}^{(\beta)}$, which is explicitly written as follows, represents the $(N+1)\times(N+1)$ Fermat (OMFE)of order β in the Caputo sense.

$$G^{(\beta)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_\beta(\lceil\beta\rceil,1) & \gamma_\beta(\lceil\beta\rceil,\lceil\beta\rceil) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_\beta(i,1) & \cdots & \gamma_\beta(i,i) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_\beta(n+1,1) & \gamma_\beta(n+1,2) & \gamma_\beta(n+1,3) & \cdots & \gamma_\beta(N+1,N+1) \end{pmatrix} \tag{17}$$

You may write the entries $g_{i,j}^{(\beta)}$ like this:

$$g_{ij}^{(\beta)} = \begin{cases} \gamma_\beta(i,j) & \text{if } i \geq \lceil\beta\rceil, i \geq j \\ 0 & , otherwise \end{cases} \tag{18}$$

Where

$$\gamma_\beta(i,j) = j \sum_{\substack{m=\lceil\beta\rceil \\ (i+j)odd,(j+m)odd}}^{i} \frac{m!\,(-1)^{\frac{1}{2}(i-2j+m+1)}2^{\frac{i-j}{2}}\left(\frac{i+m-1}{2}\right)!}{\left(\frac{i-m-1}{2}\right)!\left(\frac{m-j+1}{2}\right)!\left(\frac{m+j+1}{2}\right)!\,\Gamma(-\beta+m+1)} \tag{19}$$

For instance, when N=5, the operational matrix $\mathrm{G}^{\frac{3}{2}}$ is defined as:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{8}{\sqrt{\pi}} & 0 & \frac{4}{\sqrt{\pi}} & 0 & 0 & 0 \\ 0 & \frac{32}{\sqrt{\pi}} & 0 & \frac{8}{\sqrt{\pi}} & 0 & 0 \\ \frac{272}{5\sqrt{\pi}} & 0 & \frac{264}{5\sqrt{\pi}} & 0 & \frac{64}{5\sqrt{\pi}} & 0 \\ 0 & \frac{768}{7\sqrt{\pi}} & 0 & \frac{576}{7\sqrt{\pi}} & 0 & \frac{128}{7\sqrt{\pi}} \end{pmatrix}$$

5. METHOD OF SOLUTION

The recommended approach for creating a numerical solution to the Non Liner equation is described in this section. Examine the Nonlinear fractional differential equation that follows:

$$D^s\varphi(x) + D^\beta\varphi(x) = H[\varphi(x), z(x)] \quad , \ x \in (0,1) \tag{20}$$

with initial conditions

$$\varphi^{(i)}(0) = v_i \quad , \quad i = 0,1,\dots,m \tag{21}$$

where $m < \beta \leq m+1$, β Largest fractional power $H\big(\varphi(x), z(x)\big)$ is Nonlinear function in $\varphi(x), z(x)$, $D^s\varphi(x)$ is the derivative of integers,$D^\beta\varphi(x)$ is the Caputo derivative.
Now we can approximate $\varphi(x)$

$$\varphi(x) \approx \varphi_N(x) = \mathrm{R}^T\Phi(x) \tag{22}$$

$$D^s\varphi(x) \approx \mathrm{R}^T\mathrm{G}^{(s)}\Phi(x) \tag{23}$$

The following approximation can be produced by virtue of Theorem 3:

$$D^\beta\Phi(x) \approx x^{-\beta}\mathrm{R}^T\mathrm{G}^{(\beta)}\Phi(x) \tag{24}$$

The formula for the residual of (20) is obtained by applying the approximations in (23) and (24).

$$x^\beta R(x) = x^\beta \mathrm{R}^T\mathrm{G}^{(s)}\Phi(x) + \mathrm{R}^T\mathrm{G}^{(\beta)}\Phi(x) - x^\beta H[\varphi(x), z(x)] \tag{25}$$

and hence the application of Our proposed Galerkin-type approach

$$\big\langle R(x), F_j(x)\big\rangle = \int_0^1 x^\beta\, R(x)F_j(x)dx = 0 \quad ,j = 1,2,3,\dots,N-m \tag{26}$$

Also, by inserting equation (22) into equation (21) ,we get :

$$\begin{aligned} \varphi(0) &= R^T\,\Phi(0) = v_0 \\ \varphi^1(0) &= R^T G^1\,\Phi(0) = v_1 \\ &\vdots \\ \varphi^m(0) &= R^T\,G^m\Phi(0) = v_m \end{aligned} \tag{27}$$

Finally, using Eqs. (26) and (27) of dimension (N + 1), a system of algebraic equations is obtained. It is possible to solve these linear eq. given the unknown expansion factors.r_i. As a result, Equation (22) can estimate φ(x).

6. NUMERICAL EXAMPLES

Example1: Consider the following Nonlinear FDE [17][18]

$$D^3\varphi(x) + D^{\frac{5}{2}}\varphi(x) + \varphi^2(x) = x^4 \qquad \varphi(0) = 0 \ \ ,\varphi'(0) = 0 \ \ ,\varphi''(0) = 2 \qquad x \in (0,1) \tag{28}$$

The precise solution to equation (28) is $\varphi(x) = x^2$ by applying the method from Section 5 with $N = 3$, Now we can approximate $\varphi(x)$

$$\varphi(x) = R^T\Phi(x) = r_1F_1(x) + r_2F_2(x) + r_3F_3(x) + r_4F_4(x)$$

The formula provides the residual of Eq. (28).

$$x^{\frac{5}{2}}R(x) = x^{\frac{5}{2}}\mathrm{R}^T\mathrm{G}^{(3)}\Phi(x) + \mathrm{R}^T\mathrm{G}^{\left(\frac{5}{2}\right)}\Phi(x) + x^{\frac{5}{2}}\varphi^2(x) - x^{\frac{5}{2}}z(x)$$

where

$$\mathrm{G}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 162 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathrm{G}^{\left(\frac{5}{2}\right)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{48}{\sqrt{\pi}} & 0 & \frac{12}{\sqrt{\pi}} \end{pmatrix}$$

we obtain

$$\frac{2}{7}\mathrm{r}_1^2 + \frac{4}{3}\mathrm{r}_1\mathrm{r}_2 + \frac{164}{77}\mathrm{r}_1\mathrm{r}_3 + \frac{116}{39}\mathrm{r}_1\mathrm{r}_4 + \frac{18}{11\mathrm{r}_2^2} + \frac{220}{39}\mathrm{r}_2\mathrm{r}_3 + \frac{468}{55}\mathrm{r}_2\mathrm{r}_4 + \frac{416}{77\mathrm{r}_3^2} + \frac{1190}{66}\mathrm{r}_3\mathrm{r}_4 + \frac{2308}{139\mathrm{r}_4^2} + \frac{915}{10}\mathrm{r}_4 = \frac{2}{15} \tag{29}$$

by initial conditions we have

$$r_1 - 2r_3 = 0 \tag{30}$$
$$3r_2 - 12r_4 = 0 \tag{31}$$
$$18r_3 = 2 \tag{32}$$

Finally, by solving Eqs. (29)-(32) we obtain

$$r_1 = \frac{2}{9} \quad , \quad r_2 = 0 \quad , \quad r_3 = \frac{1}{9} \quad , \quad r_4 = 0$$

consequently $\varphi(x) = \begin{pmatrix} \frac{2}{9} & 0 & \frac{1}{9} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3x \\ 9x^2 - 2 \\ 27x^3 - 12x \end{pmatrix} = x^2$

Example2: Consider the following Nonlinear FDE [19][20]

$$D^4\varphi(x) + D^{\frac{7}{2}}\varphi(x) + \varphi^3(x) = x^9 \qquad \varphi(0) = 0 \ \ ,\varphi'(0) = 0 \ \ ,\varphi''(0) = 0 \ \ ,\varphi^3(0) = 6 \qquad x \in (0,1) \tag{33}$$

The precise solution to equation (33) is $\varphi(x) = x^3$ by applying the method from Section 5 with $N = 5$, by applying the method, we obtain six algebraic equations: two obtained from the residual of (33) and four obtained from the initial conditions. This system is then solved, and we obtain expansion factors.

$$r_1 = 0 \quad , \quad r_2 = \frac{4}{27} \quad , \quad r_3 = 0 \quad , \quad r_4 = \frac{1}{27} \quad , \quad r_5 = 0 \quad , \quad r_6 = 0$$

consequently $\varphi(x) = \begin{pmatrix} 0 & \frac{4}{27} & 0 & \frac{1}{27} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3x \\ 9x^2 - 2 \\ 27x^3 - 12x \\ 81x^4 - 54x^2 + 4 \\ 243x^5 - 216x^3 + 36x \end{pmatrix} = x^3$

Example3: Consider the following Nonlinear FDE [21]

$$D^{\beta}\varphi(x) = \frac{2 - x^{\beta}}{1 + \varphi(x)} \qquad \varphi(0) = 1, \quad x \in (0,1), \qquad \beta \in (0,1] \tag{34}$$

The precise solution to equation (34) is $\varphi(x) = -1 + \sqrt{-x^2 + 4x + 4}$, at $\beta = 1$ by applying the method from Section 5, In Fig. 1 We show how the approximate solution behaves, and Fig. 2 shows the absolute errors obtained using the current approach. In Table 1, The method is based on a power series of fractional order published in reference [21] with the attained absolute error acquired by the provided FGM for $N = 5$.

Table1. The findings concerning the approximation solutions' absolute errors Example 3

x	Results of [19]	Present method
0.118	0.4×10^{-4}	3.85×10^{-5}
0.216	2.2×10^{-3}	2.46×10^{-5}
0.314	6.3×10^{-3}	1.95×10^{-5}
0.412	1.34×10^{-2}	2.43×10^{-5}
0.510	2.42×10^{-2}	2.92×10^{-5}
0.608	3.90×10^{-2}	2.82×10^{-5}
0.706	5.83×10^{-2}	2.30×10^{-5}
0.804	8.23×10^{-2}	2.01×10^{-5}
0.902	1.114×10^{-1}	2.32×10^{-5}
1	1.458×10^{-1}	2.27×10^{-5}

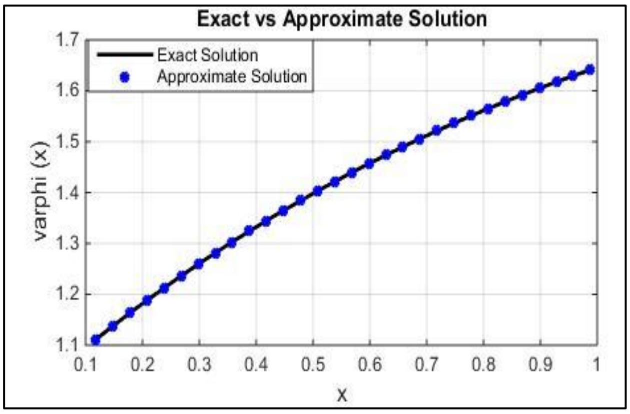

Figure1. Graphical representations of the exact and approx. sol. for Example 3

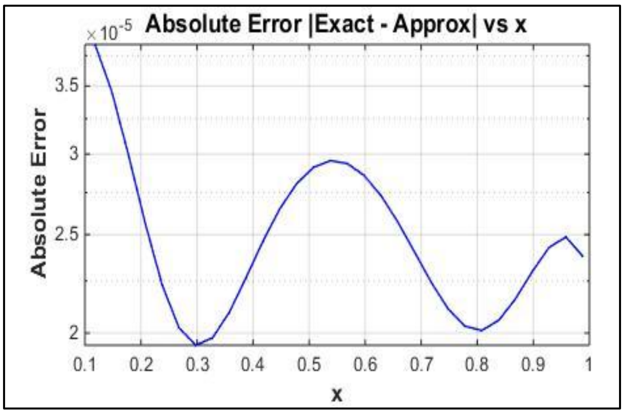

Figure 2. Absolute errors of Example 3

Example4: Consider the following Nonlinear FDE [22][23]

$$D^{\beta}\varphi(x)+\varphi(x)-\varphi^{2}(x)=0 \qquad \varphi(0)=\tfrac{1}{2}, \quad x\in(0,1), \qquad \beta\in(0,1] \tag{35}$$

The precise solution to equation (35) is $\varphi(x)=\frac{e^{-x}}{1+e^{-x}}$ at $\beta=1$ by applying the method from Section 5, In Table 2, We have compared the findings of the homotopy analysis sumudu transform method (HASTM) published in reference [22] with the attained absolute error acquired by the provided FGM for $N=6$ and the findings of the Jacobi collocation method (JCM) described in reference [23] are contrasted with the achievable absolute error produced by the provided (FGM) for $N=11$ in Table 3 Fig.3 and 4 show the approximate answer and the absolute accuracy that our technique produced when $N=6$, respectively

Table2. The findings concerning the approximation solutions' absolute errors Example 4

x	HASTM	Present method
0	0	0
0.2	6.640×10^{-7}	2.48×10^{-8}
0.4	2.099×10^{-5}	3.84×10^{-8}
0.6	1.563×10^{-4}	2.70×10^{-8}
0.8	6.411×10^{-4}	3.12×10^{-8}
1	1.892×10^{-3}	2.64×10^{-8}

Table3. The findings concerning the approximation solutions' absolute errors Example 4

x	JCM	Present method
0	0	5.25×10^{-16}
0.2	6.63×10^{-11}	5.19×10^{-13}
0.4	6.15×10^{-11}	5.01×10^{-13}
0.6	5.77×10^{-11}	4.17×10^{-13}
0.8	5.55×10^{-11}	3.74×10^{-13}
1	5.09×10^{-11}	5.04×10^{-13}

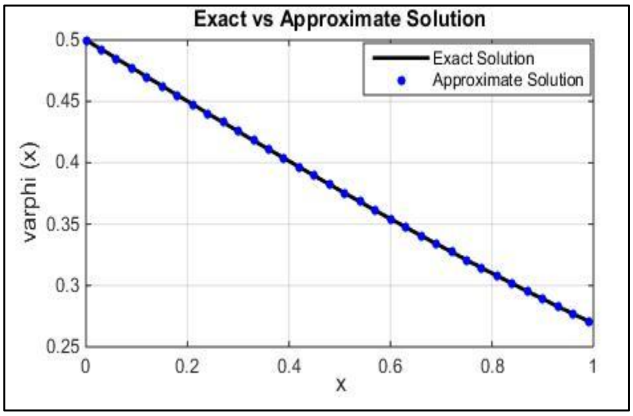

Figure 3. Graphical representations of the exact and approx. sol. for Example 4

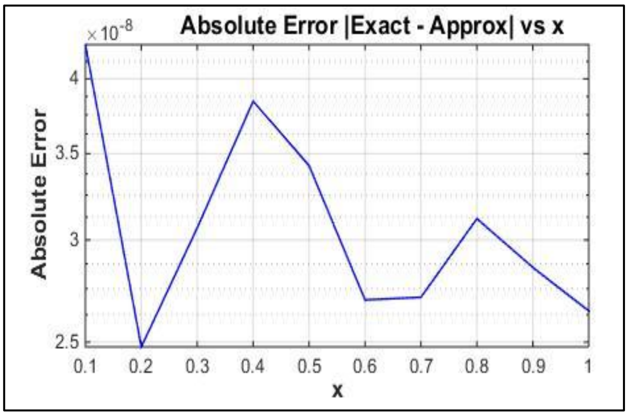

Figure 4. Absolute errors of Example 4

Example5: Consider the following Nonlinear FDE [17]

$$D^{1.3}\varphi(x) + \varphi^2(x) = \frac{20}{7\Gamma(0.7)}x^{0.4} + x^4 \qquad \varphi(0) = 0 \ , \varphi'(0) = 0 \qquad x \in (0,1) \tag{36}$$

The precise solution to equation (36) is $\varphi(x) = x^2$ by applying the method from Section 5, The estimated solution is displayed beside the corresponding precise real solution and the absolute error in Fig.5,Fig. 6. In Table 3, We have compared the Legendre wavelets optimization matrix (LWOM) published in reference [17] with the attained absolute error acquired by the provided FGM for $N = 2$,The comparison indicates that the maximum absolute error produced by the proposed FGM is of order (1.14×10^{-16}), while the LWOM method yields an error of order (9.3×10^{-3}), highlighting the effectiveness and high precision of the proposed method.

Table 4. The findings concerning the approximation solutions' absolute errors Example 5

x	LWOM	Present method
0	0.1×10^{-9}	0
0.1	9.30×10^{-5}	1.14×10^{-18}
0.2	3.72×10^{-4}	4.56×10^{-18}
0.3	8.37×10^{-4}	1.03×10^{-17}
0.4	1.49×10^{-3}	1.82×10^{-17}
0.5	2.33×10^{-3}	2.85×10^{-17}
0.6	3.35×10^{-3}	4.10×10^{-17}
0.7	4.56×10^{-3}	5.58×10^{-17}
0.8	5.95×10^{-3}	7.29×10^{-17}
0.9	7.54×10^{-3}	9.23×10^{-17}
1	9.30×10^{-3}	1.14×10^{-16}

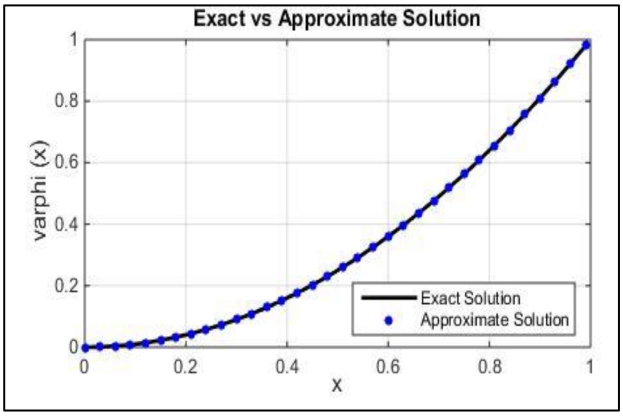

Figure 5. Graphical representations of the exact and approx. sol. for Example 5

Figure 6. Absolute errors of Example 5

Example6: Consider the following Nonlinear FDE [24]

$$D^{\beta}\varphi(x) - \varphi^2(x) = 1 \qquad \varphi(0) = 0, \quad x \in (0,1), \qquad \beta \in (0,1] \tag{37}$$

The precise solution to equation (37) is $\varphi(x) = \tan(x)$ at $\beta = 1$ by applying the method from Section 5, In Table 5, The results of the reproducing kernel discretization method (RKDM) reported in reference [24] have been compared with the achievable absolute error acquired by the provided (FGM) for $N = 10$. The estimated solution is displayed beside the corresponding precise real solution and the absolute error in Fig. 2, Fig. 3.

Table5. The findings concerning the approximation solutions' absolute errors Example 6

x	HASTM	Present method
0.2	2.824×10^{-6}	4.869×10^{-6}
0.4	1.447×10^{-5}	4.339×10^{-6}
0.6	5.158×10^{-5}	5.260×10^{-6}
0.8	1.794×10^{-4}	8.200×10^{-6}
1	7.668×10^{-4}	1.331×10^{-5}

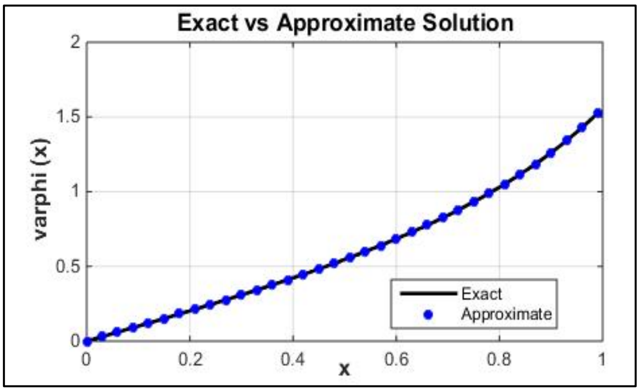

Figure 7. Graphical representations of the exact and approx. sol. for Example 6

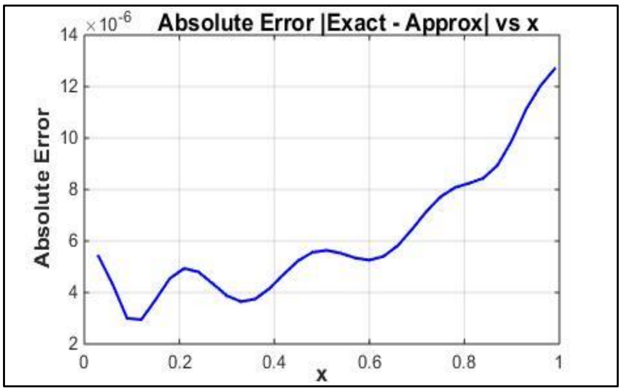

Figure 8. Absolute errors of Example 6

7. CONCLUSIONS

In this research, it was demonstrated how to use Fermat polynomials to solve Nonlinear fractional differential problems numerically. The suggested approach is based on combining the spectral Galerkin technique with the (OMFD) of Fermat polynomials. By using a small number of basic functions, we were able to achieve accurate solutions compared to the exact solutions. Comparisons were also made between the obtained numerical results and previously published results using different methods, which clearly demonstrated the effectiveness of the current method and its efficiency in producing accurate and reliable results. This study demonstrates that the suggested method is an effective and trustworthy tool for resolving Nonlinear fractional differential equations.

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