Fractional Integration and Differentiation by new Transformation

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Abstract: In opposite to differentiation and integration of integer order, an important type of differentiation and integration is the so - called Fractional Calculus (FC) in which the differentiation and integration is of non-integer order. The idea of this work is to use a transformation known as the Extension AL-Zughair Transform (EZT) for fractional calculus, so it reviewed some basic properties and definitions of (FC) such as differentiation and integration with Riemann-Liouvial operator. This transformation for fractional differentiation and integration reinforced with some application examples at the end of the article for simplicity the Fractional Integrals and Fractional Derivatives

Key word*:* Fractional calculus; Extension AL-Zughair Transform; fractional integral, fractional derivative;

1. Introduction

Fractional calculus [1] has an important role in many applied sciences, especially applied mathematics[2]–[4]. It is known that calculus means integration and differentiation. Fractional calculus, as it is name suggests, refer to fractional integration and fractional differentiation[5].

The proposed transformation, which is an Extension AL-Zughair Transform (EZT)[6]–[9], has an important role in solving fractional differential equations[10], Therefore, in this work, This transformation have been applied to the fractional integrals and fractional derivatives [11], [12][13].

2. Fractional calculus

2.1 fractional integrals

Definition (2.1.1): let $\in R$. then the operator [1]

$$I_a^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}\,dt,$$

For $a \le x \le b$, is called the Riemann-liouville fractional integral operator of order α.

For $=0$, set $I^\alpha=1$, the identity operator. i.e., it mean $I^0 f(x)=f(x)$.

Theorem (2.1.2): Let $\alpha,\beta \ge 0$, and $f \in L_1[a,b]$ then,

$$I_a^\alpha I_a^\beta f(x)=I_a^{\alpha+\beta} f(x),$$

which is the defining property for a non-local operator of fractional type.

Corollary (2.1.3):

$$I_a^\alpha I_a^\beta f(x)=I_a^\beta I_a^\alpha f(x) .$$

2.2 The power function x^p(by Riemann-liouville fractional integral)

The function fractional degree which considers in this subsection is the important function $f=x^p$, where p is initially arbitrary. It shall see, however, that p must exceed –1 for integration to have the properties it demands of the operator.

From classical calculus, the first encounter with non-integer α will be restricted to negative α so that it may exploit the Riemann-Liouvial definition, thus:

$$I^\alpha f(x)=I^\alpha x^p=\frac{1}{\Gamma(\alpha)}\int_0^x t^p(x-t)^{\alpha-1}\,dt,\ \alpha>0\ and\ \ p>-1$$

Therefore,

$$I^\alpha f(x)=I^\alpha x^p=\frac{\Gamma(p+1)}{\Gamma(\alpha+p+1)}x^{\alpha+p}$$

2.3 Fractional Derivative

Definition (2.3.1): Let $\alpha \in R$ and $m \in N$, the Riemann-Liouville fractional differential operator defined as following that [10]

$$D^{\alpha}f = D^{m}D^{\alpha-m}f = \frac{d^{m}}{dx^{m}}I^{m-\alpha}f = D^{m}I^{m-\alpha}f$$

Definition (2.3.2): The inverted sequence of operators

$$D^{\alpha}f = D^{\alpha-m}D^{m}f = I^{m-\alpha}D^{m}f \quad ,m\epsilon N$$

Lead to an alternative decomposition of fractional derivative into an ordinary standard derivative followed by a fractional integral.

2.4 The power function x^p(by Riemann-liouville fractional derivative)

Let $f(x) = x^{p}, for\ some\ p > -1\ and\ \alpha > 0, then$

$$D^{\alpha}f(x) = I^{m-\alpha}D^{m}f(x) = I^{m-\alpha}D^{m}(x^{p}) = \frac{1}{\Gamma(m-\alpha)}\int_{0}^{x} D^{m}(t^{p})(x-t)^{m-\alpha-1}\,dt$$

$$= \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}x^{p-\alpha}\ ,p > -1, 0 < \alpha < 1, m = 1.$$

3. Expansion of Al-zughair transform

Definition (3.1): the al-zughair transform of a given function $f(x)$ is defined as [7]

$$\mathcal{Z}[f(x)] = \int_{1}^{e}\frac{(\ln x)^{\mathcal{P}}}{x}f(x)\,dx, \quad 1 \leq x \leq e.$$

Where $\mathcal{P}$ is positive constant

Definition (3.2):

From definition (3.1) and by transforming the limits of integration, will have

$$\mathcal{F}(\mathcal{P}) = \mathcal{S}[f(x)] = \int_{0}^{1} x^{\mathcal{P}}f(x)\,dx.$$

This is called expansion of al-zughair transform (EZT).

property (3.3): EZT is distinguished by the linear property, which is

$$\mathcal{S}[Af(x) \mp Bg(x)] = A\mathcal{S}[f(x)] \mp B\mathcal{S}[g(x)].$$

Where A and B are constants, $0 < x < 1$.

3.1 EZT for some selected function:

Expansion of Al-Zughair transformation for Some basic functions are given by [9]:

ID	$\boldsymbol{f(x)}$	$\boldsymbol{\mathcal{S}[f(x)]}$
1	1 (Unit function)	$\frac{1}{\mathcal{P}+1}$, $\mathcal{P} > -1$
2	k (k is constant)	$\frac{k}{\mathcal{P}+1}$, $\mathcal{P} > -1$
3	x^n , $n \in R$	$\frac{1}{(\mathcal{P}+1)+n}$, $\mathcal{P} > -(n+1)$
4	$(lnx)^n$, $n \in Z$	$\frac{(-1)^n n!}{(\mathcal{P}+1)^{n+1}}$, $\mathcal{P} > -1$
5	$sin(alnx)$	$\frac{-a}{(\mathcal{P}+1)^2+a^2}$, $\mathcal{P} > -1$ $and\ a\ is\ constant$
6	$cos(alnx)$	$\frac{\mathcal{P}+1}{(\mathcal{P}+1)^2+a^2}$, $\mathcal{P} > -1$ $and\ a\ is\ constant$

Theorem (3.1.1): $\mathcal{S}[x^{\pm r} f(x)] = \mathcal{F}(\mathcal{P} \pm r)$ if r is constant and $\mathcal{S}[f(x)] = \mathcal{F}(\mathcal{P})$.

Definition (3.1.2): The original function f(x) in definition 5 can be restored from $\mathcal{F}(\mathcal{P})$ with the help of the inverse EZT

$$f(x) = \mathcal{S}^{-1}[\mathcal{F}(\mathcal{P})].$$

Property (3.1.3): Let $f_1(x), f_2(x), \dots, f_n(x)$ defined when $0 \le x \le 1$, then $\mathcal{S}^{-1}$ has the linear property, i.e.

$$\begin{aligned}\mathcal{S}^{-1}[r_1\mathcal{F}_1(\mathcal{P}) \pm r_2\mathcal{F}_2(\mathcal{P}) \pm \cdots \pm r_n\mathcal{F}_n(\mathcal{P})] &= r_1\mathcal{S}^{-1}[\mathcal{F}_1(\mathcal{P})] \pm r_2\mathcal{S}^{-1}[\mathcal{F}_2(\mathcal{P})] \dots \pm r_n\mathcal{S}^{-1}[\mathcal{F}_n(\mathcal{P})] \\ &= r_1f_1(x) \pm r_2f_2(x) \dots \pm r_nf_n(x)\end{aligned}$$

where $r_1, r_2, \dots, r_n$ are constants.

Theorem (3.1.4): If $\mathcal{S}^{-1}[f(x)] = \mathcal{F}(\mathcal{P})$. Then

$\mathcal{S}^{-1}[\mathcal{F}(\mathcal{P} \pm r)] = x^{\pm r}f(x)$, where r is constant.

4. EZT for fractional calculus

This paper present two important properties that will be useful in obtaining the EZT of fractional integral and derivative operators.

Property (4.1): this property states that the EZT of the convolution

$$(f * g)(x) = \int_x^1 f(u)g(\frac{x}{u})\frac{du}{u}$$

is given as

$$\mathcal{S}[(f * g)(x)] = \mathcal{S}[f(x)]\mathcal{S}[g(x)]$$

Where $\mathcal{S}[f(x)]$ *and* $\mathcal{S}[g(x)]$ are EZT of $f(x)$ *and* $g(x)$ respectively.

Property (4.2): this property states that the EZT of $(x^n f^{(n)}(x))$ is given by:

$$\begin{aligned}\mathcal{S}\left[x^n f^{(n)}\right] = f^{(n-1)}(1) &+ (-1)^{(2n+1)}(\mathcal{P}+n)f^{(n-2)}(1) \\ &+ (-1)^{(2n+2)}(\mathcal{P}+n)\big(\mathcal{P}+(n-1)\big)f^{(n-3)} + \cdots \\ &+ (-1)^{3n-1}(\mathcal{P}+n)\big(\mathcal{P}+(n-1)\big)\dots(\mathcal{P}+2)f(1) \\ &+ (-1)^{3n}(\mathcal{P}+n)\big(\mathcal{P}+(n-1)\big)\dots(\mathcal{P}+2)(\mathcal{P}+1)\mathcal{S}[f]\end{aligned}$$

Where $f(x)$ is defined for $0 \le x \le 1$ and its derivatives $f^{(1)}(x), f^{(2)}(x), \dots, f^{(n)}(x)$ are exit.

Property (4.3): this property states that the EZT of $(f^{(n)}(x))$ is given by:

$$\mathcal{S}\left[f^{(n)}\right] = f^{(n-1)}(1) + (-1)^{(2n+1)}(\mathcal{P})f^{(n-2)}(1) + (-1)^{(2n+2)}(\mathcal{P})(\mathcal{P}-1)f^{(n-3)} + \cdots + (-1)^{3n-1}(\mathcal{P})(\mathcal{P}-1)\ldots(\mathcal{P}-n+2)f(1) + (-1)^{3n}(\mathcal{P})(\mathcal{P}-1)\ldots(\mathcal{P}-n+2)(\mathcal{P}-n+1)\mathcal{S}[x^{-n}f]$$

Where $\mathcal{S}[x^{-n}f] = \boldsymbol{\mathcal{F}}[\boldsymbol{\mathcal{P}-n}]$ **,** $f(x)$ is defined for $0 \le x \le 1$ and its derivatives $f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x)$ are exit.

4.1 EZT for fractional integration

Let's start with Expansion Al-zughair transform (EZT) of fractional integral. According to the following proposition.

Proposition (4.1.1): let $f(x) = x^n$

$$\mathcal{S}[I^{\alpha}f(x)] = \frac{\Gamma(n+1)}{(p+\alpha+1)\Gamma(\alpha+n+1)}\mathcal{S}[f(x)]$$

Proof:

let $\alpha > 0$ and $\mathrm{g(x)} = x^{\alpha}$. Then the fractional integral $I^{\alpha}f(x)$ in definition (2.1.1) and the convolution in Property (4.1) can be rewritten as:

$$I^{\alpha}f(x) = I^{\alpha}x^{n} = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}g(x)f(x) = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}x^{\alpha}x^{n}$$

It is calculated using the definition (3.2)

$$\mathrm{G}(\mathcal{P}) = \mathcal{S}[\mathrm{g(x)}] = \mathcal{S}[\mathrm{x}^{\alpha}] = \frac{1}{\mathcal{P}+\alpha+1}.$$

And

$$\mathcal{F}(\mathcal{P}) = \mathcal{S}[\mathrm{f(x)}] = \mathcal{S}[\mathrm{x^n}] = \frac{1}{\mathcal{P}+\mathrm{n}+1}.$$

Hence

$$\mathcal{S}[I^{\alpha}f(x)] = \frac{\Gamma(n+1)}{(p+\alpha+1)\Gamma(\alpha+n+1)}\frac{1}{\mathcal{P}+\mathrm{n}+1}$$

$$\mathcal{S}[I^{\alpha}f(x)] = \frac{\Gamma(n+1)}{(p+\alpha+1)\Gamma(\alpha+n+1)}\mathcal{S}[f(x)]$$

4.2 EZT for fractional differentiation

In this subsection, let's turn to Expansion Al-zughair transform (EZT) of Riemann-liouville fractional derivative operator with α , according to the following proposition.

Proposition (4.2.1):

Let $\alpha \geq 0,\ m-1 \leq \alpha < m\ , f(x) = x^n\ and\ m \in N,$ then:

$$\mathcal{S}[\mathrm{D}^{\alpha}\mathrm{f(x)}] = \frac{\Gamma(\mathrm{n}+1)}{(\mathcal{P}+1-\alpha)\Gamma(\mathrm{n}-\alpha+1)}\mathcal{S}[\mathrm{f(x)}]$$

Proof:

Using definition (2.3.2), will have

$$\mathcal{S}[\mathrm{D}^{\alpha}\mathrm{f(x)}] = \mathcal{S}[\mathrm{I}^{\mathrm{m}-\alpha}\mathrm{f}^{(\mathrm{m})}(\mathrm{x})]$$

by Proposition (4.1.1) will get:

$$\mathcal{S}[D^{\alpha}f(x)] = \mathcal{S}\big[\mathrm{I}^{\mathrm{m}-\alpha}\mathrm{f}^{(\mathrm{m})}(\mathrm{x})\big] = \frac{1}{(\mathcal{P}+1-\alpha)\Gamma(n-\alpha+1)}\mathcal{S}[x^n]$$

$$\mathcal{S}[D^{\alpha}f(x)] = \frac{1}{(\mathcal{P}+1-\alpha)\Gamma(n-\alpha+1)}\mathcal{S}[f(x)].$$

5. Examples

5.1 The Unit Function of Fractional Order:

Considering first the integration to order α of the function $f \equiv 1$, for which it is convenient to reserve the special notion, It shall refer to this function as the unit function. In order to find a Compute $\mathcal{S}[I^{\alpha}f(x)]$, where $f(x)$ is unit function and $0 < \alpha < 1,$ using Proposition (4.1.1) , then

$$\mathcal{S}[I^{\alpha}f(x)] = \frac{1}{(\mathcal{P}+\alpha+1)\Gamma(\alpha+1)}\mathcal{S}[f(x)] = \frac{1}{(\mathcal{P}+\alpha+1)\Gamma(\alpha+1)}\mathcal{S}[1]$$
$$= \frac{1}{(\mathcal{P}+\alpha+1)\Gamma(\alpha+1)}\frac{1}{(\mathcal{P}+1)}.$$

5.2 The Constant Function:

From a function $f = k$, where k is any constant and $0 < \alpha < 1$, one has

$$\mathcal{S}[I^{\alpha}f(x)] = \boldsymbol{\mathcal{S}[I^{\alpha}(k)]} = \frac{1}{(\mathcal{P}+\alpha+1)\Gamma(\alpha+1)}\mathcal{S}[f(x)] = \frac{1}{(\mathcal{P}+\alpha+1)\Gamma(\alpha+1)}\mathcal{S}[k]$$
$$= \frac{1}{(\mathcal{P}+\alpha+1)\Gamma(\alpha+1)}\frac{k}{(\mathcal{P}+1)}.$$

If k any constant including zero and $\mathcal{S}[I^{\alpha}f(x)]$ is finite for $x > 0$, It may conclude that by setting $k = 0$, will get:
$\mathcal{S}[I^{\alpha}f(x)] = \mathcal{S}[I^{\alpha}(0)] = 0,$ for all α.

5.3 The power function x^q :

The function fractional degree that considering in this subsection is the important function $f = x^q$, where q is initially arbitrary.
Then to find EZT for fractional integration for this function, by using Proposition (4.1.1), will have

$$\mathcal{S}[\mathrm{I}^{\alpha}\mathrm{f(x)}] = \mathcal{S}[\mathrm{I}^{\alpha}\mathrm{x}^{\mathrm{q}}] = \frac{\Gamma(q+1)}{(p+\alpha+1)\Gamma(\alpha+q+1)}\mathcal{S}[\mathrm{x}^{\mathrm{q}}]$$
$$= \frac{\Gamma(n+1)}{(p+\alpha+1)\Gamma(\alpha+q+1)}\frac{1}{\mathcal{P}+\mathrm{q}+1}$$

While, the EZT for fractional differentiation for $f = x^q$, $0 < \alpha < 1, m = 1$, by Proposition (4.2.1), it gets

$$\mathcal{S}[\mathrm{D}^{\alpha}\mathrm{f(x)}] = \mathcal{S}[\mathrm{D}^{\alpha}\mathrm{x}^{\mathrm{q}}] = \frac{1}{(\mathcal{P}+1-\alpha)\Gamma(1-\alpha)}\mathcal{S}[\mathrm{f}'(\mathrm{x})]$$
$$= \frac{1}{(\mathcal{P}+1-\alpha)\Gamma(1-\alpha)}\frac{q}{(\mathcal{P}+q)}.$$

5.4 Power Natural logarithm function $(lnx)^n$:

With k arbitrary constant, the EZT of fractional integration for the power natural logarithm function $f(x) = (lnx)^n$ by using Proposition (4.1.1), as following:

$$\mathcal{S}[I^{\alpha}f(x)] = \mathcal{S}[I^{\alpha}(lnx)^n] = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\mathcal{S}[(lnx)^{n+\alpha}]$$
$$= \frac{(-1)^{n+\alpha}(n+\alpha)!\,\Gamma(n+1)}{\Gamma(\alpha+n+1)(\mathcal{P}+1)^{\alpha+n+1}}.$$

As well as for EZT of fractional differentiation of the power natural logarithm function $f(x) = (lnx)^n$ by using Proposition (4.2.1), as following:

$$\mathcal{S}[D^{\alpha}f(x)] = \mathcal{S}[D^{\alpha}(lnx)^n] = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\frac{(-1)^{-\alpha+n}(-\alpha+n)!}{(\mathcal{P}+1)^{-\alpha+n+1}}$$

If $0 < \alpha < 1$, $then\ m = 1\ and\ n = 1$ will have:

$$\mathcal{S}[I^{\alpha}(lnx)\,] = \frac{(-1)^{\alpha+1}(\alpha+1)!}{\Gamma(\alpha+2)(\mathcal{P}+1)^{\alpha+2}}.$$

and

$$\mathcal{S}[D^{\alpha}(lnx)\,] = \frac{1}{\Gamma(2-\alpha)}\frac{(-1)^{-\alpha+1}(-\alpha+1)!}{(\mathcal{P}+1)^{-\alpha+2}}$$

6. Conclusion

The subject of fractional calculus is extremely challenging and has many illdefined concepts to deal with. In any case, the requirement for new methods of solving differential equations when fractional order is necessary in physical and engineering applications. Therefore, it used a new transformation which is the Expansion Al-Zughair transformation of fractional calculus, to be able to get the fractional derivatives and fractional integrals that are an extension of solving fractional differential equations.

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