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Role of Fear and disease on the Dynamics of two prey-one predator model with extended Holling type II functional response

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ABSTRACT: This paper with the studding and modeling the dynamical interaction between two prey where one of the prey species behave stronger than other, and **one** predator species**,** The influence of fear is integrated into the development rate of vulnerable prey species owing to predation, with the spread of a SI illness among robust prey species. Here it is considered that predator both preys according to Holling type II functional response for two prey species. Then the formulation of the model is described and the boundness of solution of the system is discussed. The local stability as well as global stability for each model steady states is analyzed. Finally, with the help of MATLAB program, it is performed numerical simulations to support the evidence of our analytical results**.** Finally, a brief conclusion on the total work is given.

Keywords: Fear, disease, Functional response, local stability, Global stability.

1. INTRODUCTION

The dynamics of interaction between predator and their prey has been modeled by mathematician author [1-4]. Predators' functional response are the most important element prey-predator models, to represent the dynamics relationship between predators and their prey, and defined as the average number of eaten prey per predator per unit time [5]. There are many types of functions response; bilinear, prey dependent and predator dependent functional responses, Holling type II functional response is the most useful functional response and use by many authors [5], this type of functional response is characterized by decelerating intake rate of predation one prey species. In nature, many predator species consume more than one species of prey. For example, lions usually predate a number of large land-based animals, such as antelopes, buffaloes, crocodiles, giraffes, pigs, zebra, wild dogs and wildebeest. There are some works modeling one predator and multiple preys [2, 5, 6], Saiwan et al [5] extended Holling type II functional response is to more than one prey species [5].

Population dynamics may be affected by many factors like; fear, infectious illness, stage structure, delay, harvesting, cannibalism, anti-predator skills, refuge, and other population-affecting elements of the natural environment [8-11].

The reproduction of prey population may be reduced due to predation fears [12]. Wang et al [13] studied a predator–prey model with fear effect on prey reproduction, they noticed that the fear has no impact on the stability of the model when the system incorporate bilinear functional response, but the system become stable under fear effect, if it incorporate the Holling type II functional response, based on their system Pal et al[14] Studied the stability and bifurcation of a prey - predator that incorporate fear effect and harvesting cooperation. Zhang et al [15] showed that fear effect and prey refuge can stabilize an ecological model. For more results about fear effect, see [16-21].

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The parasite may change behavior of the prey so that infected prey is more vulnerable to predation. During the last four decades, many authors have been proposed and studied the infectious disease coupled with prey-predator interaction model [22-24].

Inspired by the aforementioned works, the aim of the current work is to consider an ecological model one predator species and two prey species, one of the prey species behave stronger than other. The effect of fear is incorporated in the growth rate of weak prey species due to the predator, and an SI disease spread among the strong prey. In next section, all assumption for proposing the model is given. In the third section, the existence conditions of all feasible equilibrium points are found. In section four and five, stability analysis (local as well as globally) of the model is studied. In section six, the model numerically solved. Finally in section seven, a brief conclusion on the total work is given.

2. THE MATHEMATICAL MODEL AND ITS SOLUTION PROPERTIES

Suppose a predator consume two species of prey namely; Strong prey and Weak prey. To represent the dynamic interaction between the species we derive a mathematical model based on the following assumptions:

i. At the instant time t, the weak prey individual numbers denote by $X(t)$. There is an SI disease among strong prey species that divided them in two classes, namely Susceptible strong prey and Infected strong prey and their individual numbers at instant time t, are denoted by $\mathrm{S}(\mathrm{t})$ and $\mathrm{I}(\mathrm{t})$. respectivley
ii. The weak prey and susceptible prey grows logistically wile the infected strong prey cannot reproduce.
iii. The effect of fear is incorporated in the growth rate of weak prey species due to the predator. The reproduction of weak prey species decreased by multiplied them by $\frac{1}{1+LP}$. where L is Level of fear due to weak prey response to anti-predators.
iv. The disease transmitted from Infected strong prey to Susceptible strong prey according to bilinear incidence rate λSI, where λ is infectious rate.
v. The predator individuals number at instant timet, represent by$\mathrm{Y}(\mathrm{t})$ and they can predate both.
vi. Weak prey and infected strong prey according to extended Holling type II functional response $\frac{\alpha_1 XP}{1+\alpha_1 T_1 X+\alpha_2 T_2 I}$ and $\frac{\alpha_2 IP}{1+\alpha_1 T_1 X+\alpha_2 T_2 I}$,respectively. Where α_1, α_2 are the predator's search efficiency for Weak prey and infected strong prey, respectively. T_1 , T_2 are the predator's average handling time of Weak prey and infected strong prey, respectively.
vii. The biomass of Weak prey and infected strong prey convers ate to biomass of predator with rate e_1 and e_1, respectively.

Then, such interaction dynamics can be modeled mathematically through the following system of differential equations

$$
\begin{aligned}
\frac{dX}{dt} &= \frac{bX}{1+lY} - d_1 X - cX^2 - \frac{\alpha_1 XY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} = f(X,S,I,Y) \\
\frac{dS}{dt} &= rS\left(1-\frac{S}{K}\right) - \lambda IS = g(X,S,I,Y) \\
\frac{dI}{dt} &= \lambda IS - \frac{\alpha_2 IY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} - (d_2+\gamma)I = h(X,S,I,Y) \\
\frac{dY}{dt} &= \frac{e_1\alpha_1 XY + e_2\alpha_2 IY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} - d_3 Y = Z(X,S,I,Y)
\end{aligned} \tag{1}
$$

Where the $X(0)>0$, $S(0)>0$, $I(0)>0$, $Y(0)>0$ and the pareameters b is reproduction rate of weak prey; $d_1, d_2 and\ d_3$ are Natural death rate of weak prey, infected strong prey and predators, respectively; c is Intraspecific competition rates of weak prey; r is intrinsic growth rate of susceptible strong prey; K is carrying capacity for strong prey.

Clearly the right side of system (1), are continuous and has partial derivatives on the space R^4, and hence, system (1) satisfies the Lipschitzian condition. Therefore, by uniqueness Theorem, it has unique solution. Further, the time derivative of X, S, I and Z are zero when $X=0, S=0, I=0\ and\ Y=0$, respectively. Therefore, if the solution initiates at a non-negative point, then the component X, S, I and Y of the solution points of system (1), cannot cross any coordinates of the solution points. Hence components X, S, I and Y of solution points is always non negative. Further some properties of solutions of system (1) are proved in the following lemma and theorems.

Lemma 1: In system (1), the following inequalities are hold:

a. If, $b \leq d_1$, then $\lim_{t\to\infty} X = 0$. Otherwise, $\lim_{t\to\infty} Sup\ X \leq \frac{b-d_1}{c}$

b. $\lim_{t\to\infty} S \leq K$

c. $\lim_{t\to\infty} Sup\ (X+S+I+Y) \leq M$ where, $M = \frac{b|b-d_1|}{cm} + \frac{rK}{m}$ with

$m = Min\{d_1, r, d_2+\gamma, d_3\}$

Proof (a) From the first equation of system (1), it gets

$$\frac{dX}{dt} \leq (b-d_1)X\left(1-\frac{cX}{b-d_1}\right)$$

So, If, $b \leq d_1$, then $\lim_{t\to\infty} X = 0$ otherwise, $\lim_{t\to\infty} Sup\ X \leq \frac{b-d_1}{c}$

Proof (b) From the second equation of system (1), it gets

$$\frac{dS}{dt} \leq rS\left(1-\frac{S}{K}\right)$$

So, $\lim_{t\to\infty} Sup\ S \leq K$

Proof (c) Applying part (a) and part(b) in system(1), then as $t \to \infty$, it gets

$$\frac{d(X+S+I+Y)}{dt} \leq b\frac{|b-d_1|}{c} + rK - d_1X - rS - (d_2+\gamma)I - d_3Y$$

Accordingly

$$\frac{d(X+S+I+Y)}{dt} \leq b\frac{|b-d_1|}{c} + rK - m(X+S+I+Y)$$

And hence $\lim_{t\to\infty} Sup\ (X+S+I+Y) \leq M$

From above lemma the following theorem can be derived.

Theorem 1: All solutions of system (1) are bounded.

3. STABILITY ANALYSIS

This section including three subsections. In the first subsection, the existence conditions of all feasible and possible steady stat points of system 1 are determined and their local and globally stability are investigated in the second and third subsection, respectively.

3.1 Existence criteria of steady states

System (1) has at most the following nine steady states.

i. The trivial steady state $P_0(0,0,0,0)$ is always exist.

ii. The axial steady state are $P_1\left(\frac{b-d_1}{c},0,0,0\right)$, and $P_2 = (0,K,0,0)$are .

So, P_2 is always exist, While P_1 is exist, if

$$b > d_1 \tag{2}$$

iii. The planer steady states are $P_3\left(\frac{b-d_1}{c},K,0,0\right)$, $P_4\left(0,\frac{d_2+\gamma}{\lambda},\frac{r(\lambda K-d_2-\gamma)}{K\lambda^2},0\right)$ and $P_5(X_5,0,0,Y_5)$ where, $X_5 = \frac{d_3}{\alpha_1(e_1-T_1d_3)}$ and

$$Y_5 = \frac{e_1X_5\left(-l(d_1+cX_5)-\frac{d_3}{e_1X_4}+\sqrt{\left(l(d_1+cX_4)+\frac{d_3}{e_1X_5}\right)^2+4\frac{d_3l}{e_1X_4}(b-(d_1+cX_5))}\right)}{2d_3l}$$

Therefore, P_3 is exist if condition (2) holds, P_4 and P_5 are exist if the following conditions holds, respectively.

$$\lambda K > d_2 + \gamma \quad (3)$$
$$b > (d_1 + cX_5) \quad (4)$$

iv. Weak prey-free steady state is $P_6(0, S_6, I_6, Y_6)$ where,
$S_6 = K - \frac{K\lambda d_3}{r\alpha_2(e_2 - d_3T_2)}$, $I_6 = \frac{d_3}{\alpha_2(e_2 - d_3T_2)}$ and $Y_6 = \frac{e_2\lambda}{\alpha_2} S_6 I_6$.
Therefore, P_6 is exist if,
$$r\alpha_2(e_2 - d_3T_2) > d_3\lambda \quad (5)$$

v. Predator-free steady state is $P_7\left(\frac{b-d_1}{c}, \frac{d_2+\gamma}{\lambda}, \frac{r(\lambda K - d_2 - \gamma)}{K\lambda^2}, 0\right)$ .exist if conditions (2) and (3) hold.

vi. Infected strong prey-free steady state is $P_8(X_5, K, 0, Y_5)$. So, P_8 if and only if P_5 is exist.

vii. Coexistence steady state is $P_9(X_9, S_9, I_9, Y_9)$ where is a solution to the system

$$\frac{b}{1+lY} - cX - \frac{\alpha_1 Y}{1+\alpha_1T_1X+\alpha_2T_2I} = d_1$$
$$r\left(1 - \frac{S}{K}\right) - \lambda I = 0$$
$$\lambda S - \frac{\alpha_2 Y}{1+\alpha_1T_1X+\alpha_2T_2I} = d_2 + \gamma$$
$$\frac{e_1\alpha_1X + e_2\alpha_2I}{1+\alpha_1T_1X+\alpha_2T_2I} = d_3$$

3.2 Local stability

To study the topological structure (local asympticaly stabile LAS) near an steady state P of system (1), the following transformation used

$$V_1(t) = X(t) - X\ , V_1(t) = S(t) - S, V_3(t) = I(t) - I\ \ V_4(t) = Y(t) - Y.$$

Then the following linear system is obtained
$$\frac{dV(t)}{dt} = J(P)V(t)$$
Where, $V(t) = \begin{pmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \\ V_4(t) \end{pmatrix}$ and $J(P) = \begin{pmatrix} f_X & f_S & f_I & f_Y \\ g_X & g_S & g_I & g_Y \\ h_X & h_S & h_I & h_Y \\ Z_X & Z_S & Z_I & Z_Y \end{pmatrix}_P$

Suppose λ_{iX}, λ_{iS}, λ_{iI} and λ_{iY} represent the eigenvalues of the Variational matrix at P_i, in the X-, S-,I- and Y-directions, respectively; $i = 0,2,3,\ldots,8$. Then,

i. $\lambda_{0X} = b - d_1$, $\lambda_{0S} = r > 0$, $\lambda_{0I} = -(d_2 + \gamma)$ and $\lambda_{0Y} = -d_3$.

ii. $\lambda_{1x} = d_1 - b$, $\lambda_{1S} = r > 0$, $\lambda_{1I} = -(d_2 + \gamma)$ and $\lambda_{1Y} = \frac{e_1\alpha_1(b-d_1)}{c+\alpha_1T_1(b-d_1)} - d_3$.

iii. $\lambda_{2X} = b - d_1$, $\lambda_{2S} = -r$, $\lambda_{2I} = \lambda K - (d_2 + \gamma)$ and $\lambda_{2X} = -d_3$.

iv. $\lambda_{3X} = d_1 - b$, $\lambda_{3S} = -r$, $\lambda_{3I} = \lambda K - (d_2 + \gamma)$ and $\lambda_{3Y} = \frac{e_1\alpha_1(b-d_1)}{C+\alpha_1T_1(b-d_1)} - d_3$.

v. $\lambda_{4X} = b - d_1$, $\lambda_{4S}, \lambda_{4I} = \frac{-\frac{rS_4}{K} \pm \sqrt{(\frac{rS_4}{K})^2 - 4\lambda I_4(d_2+\gamma)}}{2}$ and $\lambda_{4Y} = \frac{e_2\alpha_2I_4}{1+\alpha_2T_2I_4} - d_3$.

vi. $\lambda_{5S} = \mathrm{r} > 0$, $\lambda_{5I} = -\frac{\alpha_2Y_5}{(1+\alpha_1T_1X_5)} - (d_2 + \gamma)$ and $\lambda_{5S}, \lambda_{5Y} = \frac{-A_1 \pm \sqrt{A_1^2 - 4A_1B_1}}{2}$ Where,

$$A_1 = d_1 + 2cX_5 + \frac{\alpha_1Y_5}{(1+\alpha_1T_1X_5)^2} - \frac{b}{1+lY_5} \text{ and } B_7 = \frac{e_1\alpha_1Y_5}{(1+\alpha_1T_1X_5)^2}\left[\frac{X_5bl}{(1+lY_5)^2} + \frac{\alpha_1X_5}{(1+\alpha_1T_1X_5)}\right].$$

vii. $\lambda_{6X} = \frac{b}{1+lY_6} - d_1 - \frac{(1+\alpha_2T_2I_6)\alpha_1Y_6}{(1+\alpha_2T_2I_6)^2}$ and λ_{6S}, λ_{6I} and λ_{6Y} satisfy the equation $V^3 + A_2V^2 + B_2V + C$,
where $A_2 = \frac{rS_6}{k} - \lambda S_6 + \frac{\alpha_2Y_6}{(1+\alpha_2T_2I_6)^2} + (d_2 + \gamma)$, $B_2 = -\frac{r\lambda S_6^2}{k} + \frac{rS_6\alpha_2Y_6}{k(1+\alpha_2T_2I_6)^2} + \frac{rS_6(d_2+\gamma)}{k}$ and
$$C = \frac{rS_6}{k}\frac{\alpha_2I_6e_2\alpha_2Y_6}{(1+\alpha_2T_2I_6)^3}$$

viii. $\lambda_{7X} = d_1 - b,\ \lambda_{7Y} = \frac{e_1\alpha_1(b-d_1)}{c\lambda K^2+\alpha_1 T_1 \lambda K^2(b-d_1)+\alpha_2 T_2 cr(\lambda K-d_2-\gamma)} - d_3$ and $\lambda_{7S}\ ,\ \lambda_{7I} = \frac{-A_3 \pm \sqrt{A_3^2-4A_3B_3}}{2}$ Where,

$A_3 = 2r\frac{(d_2+\gamma)}{K\lambda} - \frac{r\lambda K^2-(\lambda K-d_2-\gamma)}{\lambda K^2}$ and $B_3 = (d_2+\gamma)\frac{r(\lambda K-d_2-\gamma)}{K^2}$.

ix. $\lambda_{8S} = -\mathrm{r},\ \lambda_{8I} = \lambda K - \frac{\alpha_2 Y_5}{(1+\alpha_1 T_1 X_5)} - (d_2+\gamma),\ \ \lambda_{8X}\ ,\ \lambda_{8Y} = \frac{-A_4 \pm \sqrt{A_4^2-4A_4B_4}}{2}$ Where,

$A_7 = d_1 + 2cX_5 + \frac{\alpha_1 Y_5}{(1+\alpha_1 T_1 X_5)^2} - \frac{b}{1+lY_5}$ and $B_7 = \frac{e_1\alpha_1 Y_5}{(1+\alpha_1 T_1 X_5)^2}\left[\frac{X_5 bl}{(1+lY_5)^2} + \frac{\alpha_1 X_5}{(1+\alpha_1 T_1 X_5)}\right]$.

From determining above Eigenvalues, the following theorem can be derived.

Theorem 2

a. The steady states $P_0(0,0,0,0), P_1\left(\frac{b-d_1}{c},0,0,0\right)$ and $P_5(X_5,0,0,Y_5)$ are unstable.

b. $P_2(0,K,0,0)$ is LAS if and only if $b < d_1\ \ and\ \ \lambda K < d_2+\gamma$.

c. If $P_3\left(\frac{b-d_1}{c},K,0,0\right)$ exists, then it is LAS if and only if $\lambda K < d_2+\gamma\ \ and\ \ \frac{e_1\alpha_1 X_3}{1+\alpha_1 T_1 X_3} < d_3$.

d. If $P_4(0,S_4,I_4,0)$ exists, then it is LAS if and only if $b < d_1\ \ \ and\ \ \frac{e_2\alpha_2 I_4}{1+\alpha_2 T_2 I_4} < d_3$.

e. If $P_6(0,S_6,I_6,Y_6)$ exists, then it is LAS if and only if $\frac{b}{1+lY_6} < d_1 + \frac{\alpha_1 Y_6}{(1+\alpha_2 T_2 I_6)}$, $A_2 > 0$ and $A_2B_2 > C$.

f. If $P_7\left(\frac{b-d_1}{c},\frac{d_2+\gamma}{\lambda},\frac{r(\lambda K-d_2-\gamma)}{K\lambda^2},0\right)$ exists, then it is LAS if and only if $\lambda_{7Y} < 0$, $A_3 > 0$and $B_3 > 0$.

g. If $P_8(X_5,K,0,Y_5)$ exists, then it is LAS if and only if $\lambda K < \frac{\alpha_2 Y_5}{(1+\alpha_1 T_1 X_5)} + (d_2+\gamma)$ and $A_7 > 0$.

3.3 Global stability

Global stability (or globally asymptotically stable GAS) means that any trajectories finally tend to the attractor of the system, regardless of initial conditions. Therefore, most of biological systems, especially prey predator system, are needed to be globally stable. Since $P_0(0,0,0,0), P_1\left(\frac{b-d_1}{c},0,0,0\right)$ and $P_5(X_5,0,0,Y_5)$ are not LAS, so they cannot be GAS. However, GAS for the other steady states of system (1) is established the following theorem.
If P_0 LAS, then it also become GAS.

Theorem 3. Suppose $P_2(0,K,0,0)$ is LAS, then it is also GAS

Proof. Consider the function

$$L_1(X,S,I,Y) = X + S - K - K\ln\left(\frac{X}{k}\right) + I + Y$$

Then $L_1(X,S,I,Y) > 0\ \ and\ L_1(X,S,I,Y) = 0$ if and only if, $(X,S,I,Y) = (0,k,0,0)$. Further,

$$\frac{dL_1}{dt} = \left[\frac{\mathrm{bX}}{1+\mathrm{lY}} - \mathrm{d_1X} - \mathrm{cX^2} - \frac{\alpha_1\mathrm{XY}}{1+\alpha_1\mathrm{T_1X}+\alpha_2\mathrm{T_2I}}\right] + \frac{(S-k)}{S}\left[\mathrm{rS}\left(1-\frac{\mathrm{S}}{\mathrm{K}}\right) - \lambda\mathrm{SI}\right] + \left[\lambda\mathrm{IS} - \frac{\alpha_2\mathrm{IY}}{1+\alpha_1\mathrm{T_1X}+\alpha_2\mathrm{T_2I}} - (\mathrm{d_2}+\gamma)I + \left[\frac{\mathrm{e_1}\alpha_1\mathrm{XY}+\mathrm{e_2}\alpha_2\mathrm{IY}}{1+\alpha_1\mathrm{T_1X}+\alpha_2\mathrm{T_2I}} - \mathrm{d_3}Y\right]\right]$$

Since $\mathrm{e_1} < 1$ and $\mathrm{e_2} < 1$ and $P_2(0,K,0,0)$ is LAS so Theorem 2(b) u that $b < d_1\ \ and\ \ \lambda K < d_2+\gamma$ and hence $\frac{dL_1}{dt} < 0$.
Consequently $P_2(0,K,0,0)$ is GAS.

Theorem 4. Suppose $P_3\left(\frac{b-d_1}{c},k,0,0\right)$ is exist then it is GAS if

$$\lambda K < d_2+\gamma\ \ and\ \ (b-d_1)(bl+e_1\alpha_1) < \mathrm{c}d_3 \qquad (6)$$

Proof. Consider the function

$$L_2(X,S,I,Y) = X - \frac{b-d_1}{c} - \frac{b-d_1}{c}\ln\left(\frac{cX}{b-d_1}\right) + S - K - K\ln\left(\frac{X}{k}\right) + I + Y$$

Since $P_3\left(\frac{b-d_1}{c}, k, 0, 0\right)$ so $b > d_1$. Therefore. $L_2(X,S,I,Y) > 0$ *and* $L_2(X,S,I,Y) = 0$ if and only if, $(X,S,I,Y) = \left(\frac{b-d_1}{c}, k, 0, 0\right)$. Further

Then

$$\frac{dL_2}{dt} = \left(X - \frac{b-d_1}{c}\right)\left[\frac{b}{1+lY} - d_1 - cX - \frac{\alpha_1 Y}{1+\alpha_1 T_1 X + \alpha_2 T_2 I}\right] + (S-k)\left[r\left(1-\frac{S}{K}\right) - \lambda I\right]$$

$$+\lambda IS - \frac{\alpha_2 IY}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} - (d_2+\gamma)I + \frac{e_1\alpha_1 XY + e_2\alpha_2 IY}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} - d_3 Y$$

$$\leq \frac{-blXY}{1+lY} - c\left(X - \frac{b-d_1}{c}\right)^2 - \frac{r}{k}(S-k)^2 + (\lambda k - (d_2+\gamma)I + \left(\frac{bl(b-d_1)}{c(1+lY)} + \frac{\alpha_1(b-d_1)}{c(1+\alpha_1 T_1 X + \alpha_2 T_2 I)} - d_3\right)Y$$

So under condition 6 $\frac{dL_2}{dt} < 0$. Consequently $P_3\left(\frac{b-d_1}{c}, k, 0, 0\right)$ is GAS

Theorem 5 Suppose$P_4(0, S_4, I_4, 0)$ is exists, then $P_4(0, S_4, I_4, 0)$ is GAS

$$b < d_1 \quad and \quad e_2 < d_3 T_2$$

Proof. Since $P_4(0, S_4, I_4, 0)$ is LAS, so Theorem2 (d) guarantees the following conditions

.

From first equation of system (1), we have

$$\frac{dX}{dt} \leq (b-d_1)X$$

So, under condition $b < d_1$, it gets $\lim_{t\to\infty} X(t) = 0$

Thus as $t \to \infty$, From fourth equation of system (1), we get

$$\frac{dY}{dt} \leq Y\left[\frac{e_2\alpha_2 I}{1+\alpha_2 T_2 I} - d_3\right] \leq \left[\frac{e_2}{T_2} - d_3\right]Y$$

So $\frac{dY}{dt}$ is negative due to $e_2 < d_3 T_2$, consequently $\lim_{t\to\infty} Y(t) = 0$.

Therefore, as time approaches infinity, system 1 reduced to the following subsystem

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1-\frac{S}{K}\right) - rIS = F(S,I) \\ \frac{dI}{dt} &= \lambda IS - (d_2+\gamma)I = G(S,I) \end{aligned} \qquad (7)$$

Consider now the function $H(S,I) = 1/SI$, clearly H is a continuously differentiable function. Further,

$$\frac{\partial(HF)}{\partial S} + \frac{\partial(HG)}{\partial I} = -\frac{r}{KI}$$

it is clear that $\frac{\partial(HF)}{\partial X} + \frac{\partial(HG)}{\partial Z}$does not change sign and is not identically zero. So, by Bendixson-Dulac criterion, there is no periodic curve in of the SI-plane. Since (S_4, I_4) represent unique positive equilibrium point of the subsystem (7), so

$\lim_{t\to\infty} S(t) = S_4$ and $\lim_{t\to\infty} I(t) = I_4$, This completes the proof.

Theorem6. Suppose $P_6(0, S_6, I_6, Y_6)$ is exist then it is GAS if

$$b < d_1 + \frac{e_1\alpha_1 Y_6}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} \qquad (8)$$

$$\lambda S_6 < d_2 + \frac{e_2\alpha_2 Y_6}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} \qquad (9)$$

$$\alpha_2 I_6 < d_3 \qquad (10)$$

Proof**.** Let $L_3 = X - S - S_6 - S_6\ln(\frac{S}{S_6}) + I - I_6 - I_6\ln(\frac{I}{I_6}) + Y - Y_6 - Y_6\ln(\frac{Y}{Y_6})$

Clearly $L_3(X,S,I,Y) > 0$ *and* $L_3(X,S,I,Y) = 0$ if and only if, $(X,S,I,Y) = (0, S_6, I_6, Y_6)$. Further,

$$\frac{dL_3}{dt} = \left(\frac{b}{1+lY} - d_1 - \frac{e_1\alpha_1 Y_6}{1+\alpha_1 T_1 X + \alpha_2 T_2 I}\right)X - cX^2 + \frac{\alpha_1(e_1-1)XY + \alpha_2(e_2-1)IY}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} - \frac{r}{K}(S-S_6)^2$$

$$+\left(\lambda S_6 - d_2 - \frac{e_2\alpha_2 Y_6}{1+\alpha_1 T_1 X+\alpha_2 T_2 I}\right) I + (d_2+\gamma) I_6 + d_3 Y_6 - \lambda I_6 S + \left(\frac{\alpha_2 I_6}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} - d_3\right) Y$$

Therefore $\frac{dL_3}{dt}$ is negative under conditions (9-10) and hence $P_6(0, S_6, I_6, Y_6)$ is GAS.

Theorem7 Suppose $P_7\left(\frac{b-d_1}{c}, \frac{d_2+\gamma}{\lambda}, \frac{r(\lambda K-d_2-\gamma)}{K\lambda^2}, 0\right)$ is exist then it is GAS if

$$b + cX_7 < d_1 \tag{11}$$
$$\lambda S_7 < d_2 + \gamma \tag{12}$$
$$\alpha_1 X_7 + \alpha_2 I_7 < d_3 \tag{13}$$
$$r + \frac{r}{K} S_7 < \lambda I_7 \tag{14}$$

Proof. Consider the function

$$l_4(X,S,I,Y) = X - X_7 - X_7 \ln\left(\frac{X}{X_7}\right) - S - S_7 - S_7 \ln\left(\frac{S}{S_7}\right) + I - I_7 - I_7 \ln\left(\frac{I}{I_7}\right) + Y.$$ Where,

$X_7 = \frac{b-d_1}{c}$, $S_7 = \frac{d_2+\gamma}{\lambda}$ and $I_7 = \frac{r(\lambda K-d_2-\gamma)}{K\lambda^2}$.

Since $P_7(X_7, S_7, I_7, 0)$ exists, so $L_4(X,S,I,Y) > 0$ and $L_4(X,S,I,Y) = 0$ if and only if, $(X,S,I,Y) = (X_7, S_7, I_7, 0)$. Further,

$$\frac{dL_4}{dt} = (X - X_7)\left(\frac{b}{1+Ly} - d_1 - cx - \frac{\alpha_1 Y}{1+\alpha_1 T_1 X+\alpha_2 T_2 I}\right) + (S - S_7)\left(r\left(1-\frac{S}{K}\right) - \lambda I\right)$$
$$+(I - I_7)\left(\lambda S - \frac{\alpha_2 Y}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} - (d_2+\gamma)\right) + \frac{e_1\alpha_1 XY + e_2\alpha_2 IY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} - d_3 Y$$

$$= \frac{\alpha_1(e_1-1)XY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} + \frac{\alpha_2(e_2-1)IY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} + \left(\frac{b}{1+Ly} - d_1 + cX_7\right) X - cX^2 + (\lambda S_7 - d_2 + \gamma) I + \left(r + \frac{r}{K} S_7 - \lambda I_7\right) S$$
$$\left(\frac{\alpha_1 X_7}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} + \frac{\alpha_2 I_7}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} - d_3\right) Y - \frac{r}{K} S^2 - \frac{bX_7}{1+Ly} - d_1 X_7 - (d_2+\gamma) I_7 - r S_7$$

Therefore $\frac{dL_4}{dt}$ is negative under conditions (11-14) and hence $P_7(X_7, S_7, I_7, 0)$ is GAS.

Theorem8 Suppose $P_8(X_8, S_8, 0, Y_8)$ is exist then it is GAS if .

$$b < d_1 + cX_8 + \frac{e_1\alpha_1 Y_8}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} \tag{15}$$
$$\lambda S_8 < (d_2+\gamma) + \frac{e_2\alpha_2 Y_8}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} \tag{16}$$
$$d_3 Y_8 < \frac{bX_8}{1+Ly} + d_1 X_8 + r S_8 \tag{17}$$

Proof. Consider the function

$$L_5(X,S,I,Y) = X - X_8 - X_8 \ln\left(\frac{X}{X_8}\right) - S - S_8 - S_8 \ln\left(\frac{S}{S_8}\right) + I + Y - Y_8 - Y_8 \ln\left(\frac{Y}{Y_8}\right)$$

Clearly, $L_5(X,S,I,Y) > 0$ and $L_5(X,S,I,Y) = 0$ if and only if, $(X,S,I,Y) = (X_8, S_8, 0, Y_8)$. Further,

$$\frac{dL_5}{dt} = (X - X_8)\left(\frac{b}{1+Ly} - d_1 - cX - \frac{\alpha_1 Y}{1+\alpha_1 T_1 X+\alpha_2 T_2 I}\right) + (S - S_8)\left(-\frac{1}{K}(S - S_8) - \lambda I\right) + \lambda IS - \frac{\alpha_2 IY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} - (d_2+\gamma) I +$$

$$(Y - Y_8)\left(\frac{e_1\alpha_1 X + e_2\alpha_2 I}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} - d_3\right)$$

$$= \frac{\alpha_1(e_1-1)XY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} + \frac{\alpha_2(e_2-1)IY}{1+\alpha_1 T_1 X+\alpha_2 T_2 I} + \left(\frac{b}{1+Ly} - d_1 - cX_8 - \frac{e_1\alpha_1 Y_8}{1+\alpha_1 T_1 X+\alpha_2 T_2 I}\right) X - cX^2 - \frac{1}{K}(S - S_8)^2 + \left(\lambda S_8 - (d_2+\gamma) - \right.$$

$$\left. \frac{e_2\alpha_2 Y_8}{1+\alpha_1 T_1 X+\alpha_2 T_2 I}\right) I - d_3 Y - \frac{bX_8}{1+Ly} - d_1 X_8 - r S_8 + d_3 Y_8$$

Since $e_1 < 1 \wedge e_2 < 1$, therefore $\frac{dL_5}{dt}$ is negative under conditions (15-17) and hence $P_8(X_8, S_8, 0, Y_8)$ is GAS.

Theorem9 Suppose $P_9(X_9\,, S_9\,, I_9, Y_9)$ is exist then it is GAS if.

$$b + cX_9 < d_1 + \frac{\alpha_1 Y_9}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} \quad (18)$$
$$r + \frac{r}{K} S_9 < \lambda I_9 \quad (19)$$
$$\lambda S_9 < d_2 + \gamma + \frac{e_2 \alpha_2 Y_9}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} \quad (20)$$
$$\alpha I_9 + \alpha_1 X_9 < d_3 \quad (21)$$
$$d_1 X_9 + (d_2 + \gamma) I_9 + d_3 Y_9 < r S_9 + \frac{b X_9}{1+Ly} \quad (22)$$

Proof. Consider the function

$$L_6(X,S,I,Y) = X - X_9 - X_9 \ln\left(\frac{X}{X_9}\right) + S - S_9 - S_9 \ln\left(\frac{S}{S_9}\right) + I - I_9 - I_9 \ln\left(\frac{I}{I_9}\right) + Y - Y_9 - Y_9 \ln\left(\frac{Y}{Y_9}\right)$$

Clearly, $L_6(X,S,I,Y) > 0 \;\; and\; L_6(X,S,I,Y) = 0$ if and only if, $(X,S,I,Y) = (X_9\,, S_9\,, I_9, Y_9)$. Further,

$$\frac{dL_6}{dt} = (X - X_9)\left(\frac{b}{1+Ly} - d_1 - cX - \frac{\alpha_1 Y}{1+\alpha_1 T_1 X + \alpha_2 T_2 I}\right) + (S - S_9)\left(r\left(1 - \frac{S}{K}\right) - \lambda I\right) + (I - I_9)\left(\lambda S - \frac{\alpha_2 Y}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} - (d_2 + \gamma)\right) + (Y - Y_9)\left(\frac{e_1 \alpha_1 X + e_2 \alpha_2 I}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} - d_3\right)$$

$$= \left(\frac{b}{1+Ly} - d_1 + cX_9 - \frac{e_1 \alpha_1 Y_9}{1+\alpha_1 T_1 X + \alpha_2 T_2 I}\right) X - cX^2 + \frac{\alpha_1 (e_1 - 1) XY + \alpha_2 (e_2 - 1) IY}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} + \left(r + \frac{r}{K} S_9 - \lambda I_9\right) S - \frac{r}{K} S^2$$
$$+ \left(\lambda S_9 - d_2 - \gamma - \frac{e_2 \alpha_2 Y_9}{1+\alpha_1 T_1 X + \alpha_2 T_2 I}\right) I + d_1 X_9 + (d_2 + \gamma) I_9 + d_3 Y_9 - r S_9 - \frac{b X_9}{1+Ly}$$
$$+ \left(\frac{\alpha I_9 + \alpha_1 X_9}{1+\alpha_1 T_1 X + \alpha_2 T_2 I} - d_3\right) Y$$

Since $e_1 < 1 \wedge e_2 < 1$,therefore $\frac{dL_6}{dt}$ is negative under conditions (18-22) and hence $P_9(X_9\,, S_9\,, I_9, Y_9)$ is GAS.

4. NUMERICAL SIMULATION

In order to support the analytical finding in this paper, some numerical simulations are performed; all the simulations are carried out through runga kutta method of order six method, using MATLAB. First, let choose the set of parameter values.

$$\text{b} = 2.5;\; r = 1.3;\; l = 0.02\,;\; c = 0.01;\; \lambda = 0.01; k = 50\,; d_1 = 0.05;\; d_2 = 0.05;\; d_3 = 0.05; \alpha_1 = 0.001;\; \alpha_2 = 0.001; T_1 = 3;\; T_2 = 2; e_1 = 0.7;\; e_2 = 0.7; \quad (23)$$

Note the above parameter values satisfy the condition for global stability of the coexistence steady state. Therefore Fig.1 confirm analytical result in Theorem 9

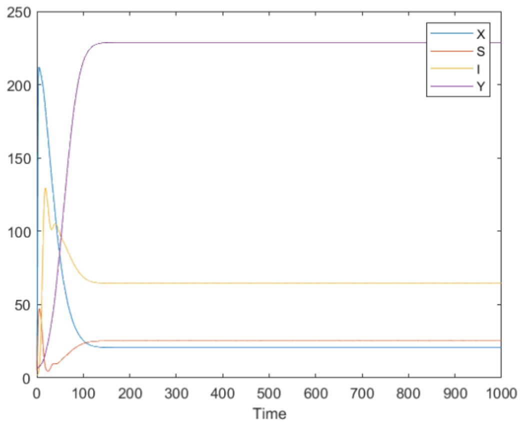

Figure1: The figure shows that the model trajectory approaches P_9

Again let we choose the following set of parameter values.

$$b = 1.5;\ r = 4.3;\ l = 0.02\ ;\ c = 0.01;\ \lambda = 0.01; k = 50\ ; d_1 = 0.05;\ d_2 = 0.05; \\ d_3 = 0.05; \alpha_1 = 0.001;\ \alpha_2 = 0.001; T_1 = 3;\ T_2 = 2; e_1 = 0.5;\ e_2 = 0.5; \tag{24}$$

The parameter values in (24) satisfy the condition for global stability of the weak prey-free steady states, and Fig.2 shows that the model approaches weak prey Steady state, when the parameter values are as given in (**24**), and confirm analytical result in Theorem 6

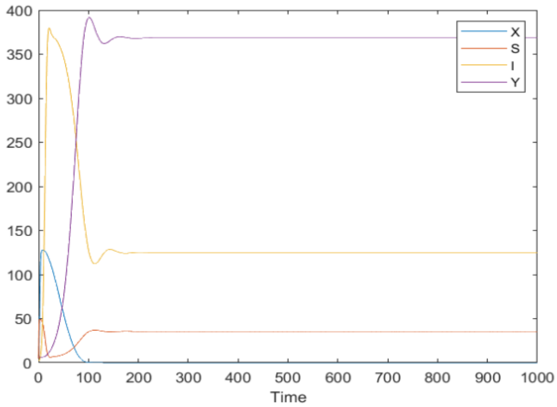

Figure 2: The figure shows that the model trajectory approaches P_6

Again let we choose the following set of parameter values.

$$b = 2.5;\ r = 1.3;\ l = 0.02\ ;\ c = 0.01;\ \lambda = 0.01; k = 50\ ; d_1 = 0.05;\ d_2 = 0.05; \\ d_3 = 0.05; \alpha_1 = 0.001;\ \alpha_2 = 0.001; T_1 = 3;\ T_2 = 2; e_1 = 0.2;\ e_2 = 0.2; \tag{25}$$

The parameter values in (25) satisfy the condition for global stability of the predator-free steady states, it shows that the model approaches predator Steady state, when the parameter values are as given in **(25),** and confirm analytical result in Theorem 7.

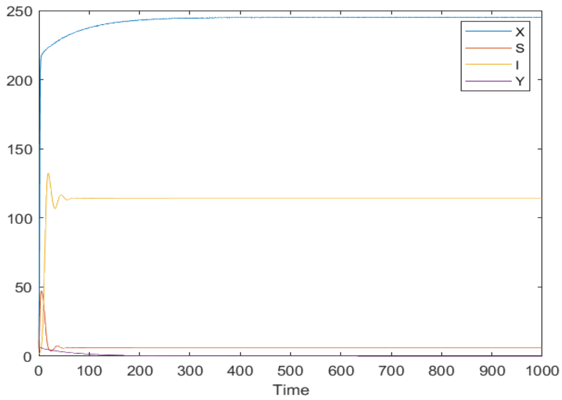

Figure 3: The figure shows that the model trajectory approaches P_7

Again let we choose the following set of parameter values.

$$b = 2.5;\ r = 1.3;\ l = 0.02\ ;\ c = 0.01;\ \lambda = 0.001; k = 50\ ; d_1 = 0.05;\ d_2 = 0.05; \\ d_3 = 0.05; \alpha_1 = 0.001;\ \alpha_2 = 0.001; T_1 = 3;\ T_2 = 2; e_1 = 0.7;\ e_2 = 0.7; \tag{26}$$

The parameter values in (26)satisfy the condition for global stability of the strong infected prey-free steady states, it shows that the model approaches strong infected prey -free Steady state, when the parameter values are as given in (**26**), and confirm analytical result in Theorem 8

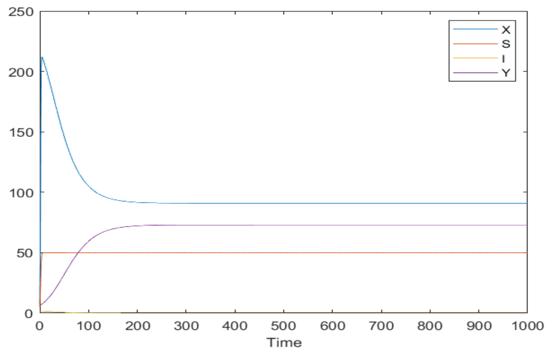

Figure 4: The figure shows that the model trajectory approaches P_8

Again let we choose the following set of parameter values.

$$\begin{gathered} b = 0.05;\ r = 1.3;\ l = 0.2\ ;\ c = 0.01;\ \lambda = 0.001; k = 50\ ; d_1 = 0.6;\ d_2 = 0.05; \\ d_3 = 0.04; \alpha_1 = 0.001;\ \alpha_2 = 0.001; T_1 = 7;\ T_2 = 8; e_1 = 0.5;\ e_2 = 0.5; \end{gathered} \tag{27}$$

Above parameter values satisfy the condition for global stability of the strong infected prey-free steady states, it shows that the model approaches strong infected prey -free Steady state, when the parameter values are as given in (**27**). and confirm analytical result in Theorem 3

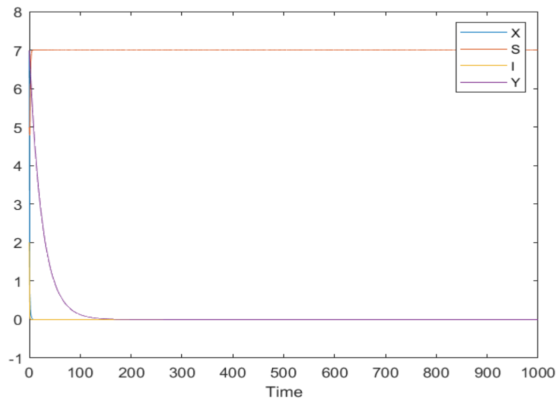

Figure 5: The figure shows that the model trajectory approaches P_2

Again let we choose the following set of parameter values.

$$\begin{gathered} b = 0.05;\ r = 1.3;\ l = 0.6\ ;\ c = 0.9;\ \lambda = 0.9; k = 7\ ; d_1 = 0.6;\ d_2 = 0.05; \\ d_3 = 0.04; \alpha_1 = 0.001;\ \alpha_2 = 0.001; T_1 = 7;\ T_2 = 8; e_1 = 0.5;\ e_2 = 0.5; \end{gathered} \tag{28}$$

Above parameter values the condition for global stability of the weak prey and predator free steady states, it shows that the model approaches weak prey and predator-free Steady state, when the parameter values are as given in (**28**), and confirm analytical result in Theorem 5.

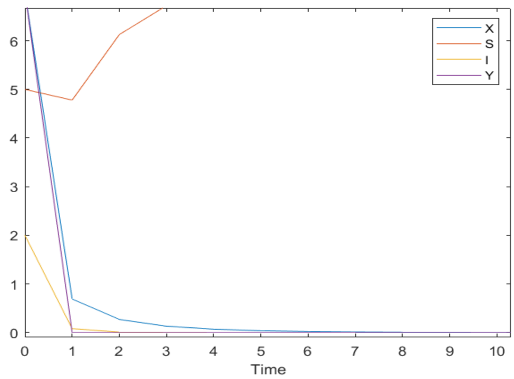

Figure 6: The figure shows that the model trajectory approaches P_4

Again let we choose the following set of parameter values

$$\begin{aligned} b &= 2.5;\ r = 1.3;\ l = 0.02\ ;\ c = 0.01;\ \lambda = 0.001;k = 10\ ;d_1 = 0.05;\ d_2 = 0.05; \\ d_3 &= 0.05;\alpha_1 = 0.001;\ \alpha_2 = 0.001;T_1 = 3;\ T_2 = 2;e_1 = 0.1;\ e_2 = 0.2; \end{aligned} \tag{29}$$

Above parameter values satisfy the condition for global stability of $\boldsymbol{P_3}$, Fig.7. shows that the model approaches $\boldsymbol{P_3}$, when the parameter values are as given in $(\mathbf{28})$, and confirm analytical result in Theorem 5

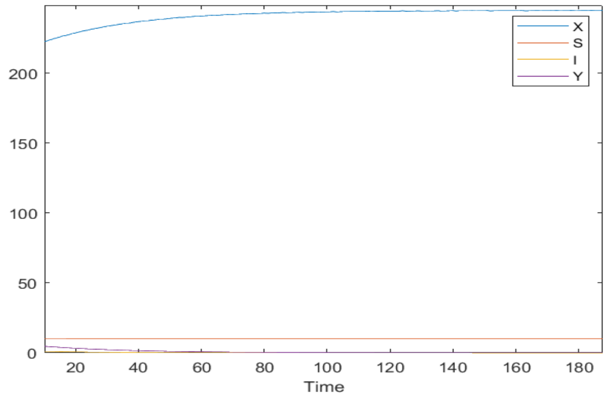

Figure 7: These parameter values satisfy the condition for global stability of $\boldsymbol{P_3}$

5. CONCLUSION

In this paper, the dynamical interaction between two prey (strong and weak) and one predator species under the influence of fear is integrated into the development rate of vulnerable prey species owing to predation, with the spread of a SI illness among robust prey species **is** modeled mathematically through set of differential equations. It has been shown that the model is bounded and it has at most ten steady states that three of the them are always unstable, while for the other steady states the criteria for local asymptotically stability are determined in Theorem 2 and globally asymptotically stable are determined in Theorems in section 3.3. Finally with the help of MATLAB program, it is performed numerical simulations to support the evidence of analytical results regarding stability to each steady state**.**

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