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Applications of Semi-pre-c-Open Sets

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ABSTRACT: Semi-pre-c-open sets are a subtle development in topology that has a substantial impact on both theoretical investigations and real-world applications. This study explores the several uses of semi-pre-c-open sets in diverse fields. First, we look into how they may be used to improve the structure of generalized topological spaces and give a more comprehensive framework for studying compactness, continuity, and convergence. A more precise categorization of functions is made possible by semi-pre-c-open sets, opening up new function spaces and expanding on previously established findings. Furthermore, their significant applications in digital topology help to advance digital continuity notions and more efficient picture processing methods. We develop and study semi-pre-c-open sets, a novel class of generalized open sets in a topological space. The semi-pre-open sets class includes this class. It is demonstrated that the topology generated by semi-pre-c-open sets is the same. Semi-pre-c-derived set, semi-pre-c-interior points, semi-pre-c-closure, semi-pre-c-closure, semi-pre-c-frontier, and semi-pre-c-exterior are among the topological aspects of these sets that we present and examine. Using examples and counterexamples, the existence of their relationship is also examined.

Keywords: semi − pre − c − open set, semi-pre-c-interior points, semi-pre-c-closure, semi-pre-c-neighborhood, semi-pre-c-derived set

1. INTRODUCTION

According to Levinein (1963), Semi-open sets and semi-continuity in topological spaces Njastad (1965), Mashhour (1982), and Andrijevic (1986), the concepts of semi-open sets, pre-open sets, and semi-preopen sets have all been presented and investigated. A new class of generalized open sets was introduced by Andrijevic (1996); it included pre-open and semi-open sets, but it was smaller than semi-pre-open sets. A unique class of c-open sets was presented by Alqahtani (2023), who also examined its basic characteristics and contrasted it with existing set types. This work aims to provide a novel definition of semi-pre-c-open sets, examine their topological characteristics, and explore the topology that arises from their family. Finally, we will draw comparisons between this topology and other established topologies.

We study the basic characteristics of semi-pre-c-interior points, semi-pre-c-frontier, semi-pre-c-interior points, semi-pre-c-derived sets, semi-pre-c-closure of a set, and semi-pre-c-outside with many examples in this work. Additionally, a study is conducted on the relationship between the properties and available properties. Quite recently, Marwa and Al-Hachami**[15]** in 2022 presented the notions of Semi pre-generalized-closed sets. Jassim and Al-Hachami [16]in 2022 also present his idea about of Semi Feebly Separation Axioms**.** Recently, Mohammed and Al-Hachami [17] in 2022 introduced independently the notions of normal space: OR, Og.

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Unless otherwise indicated, X is used in this study to represent a topological space without any presumed separation axioms. If $A \subseteq X$, then $int(A)$ and $cl(A)$ represent A's interior and closure, respectively. The complement of semi-closed sets, pre-closed sets, semi-pre-c-closed sets, b-closed sets, and regular closed sets, respectively, is semi-open sets, pre-open sets, semi-pre-c-open sets, b-open sets, and regular open sets. Semi-open, pre-open, semi-pre-c, b-open, regular open, regular closed, and semi-open sets are the families represented by P-O(X, T), S-P-C-O(X, T), B-O(X, T), R-O(X, T), and R-C(X, T), respectively.

1. 1 Definition:

Let (X, T) be a topological space. A subset F of X is called:

2. semi − open[3] if $F \subseteq \overline{F^{\circ}}$.
3. pre − open[2] if $F \subseteq \overline{F}^{\circ}$
4. Semi − pre − open[1] if $F \subseteq \overline{F^{\circ}}^{\circ}$
5. Regular − open[4] if $F = \overline{F}^{\circ}$
6. C − open [5] if $\overline{F} \setminus F$ is countable set. That is, F is an open and its frontier is a countable.set.

Definition1.2.[6]: A b-open subset of a space X is called bc-open if for each $x \in A$, there exists a closed set F such that $x \in F \subseteq A$.

Definition 1.3. [14]: A preopen subset A of a space X is called pc-open if for each $x \in A$, there exists a closed set F such that $x \in F \subseteq A$.

Definition 1.4. [7]: A semi open subset A of a space X is called sc-open if for each $x \in A$, there exists a closed set F such that $x \in F \subseteq A$.

Definition 1.5. [11]: A set A of a topological space is called θ-open if for each $x \in A$ there exists an open set G such that, $x \in G \subseteq \overline{G} \subseteq A$.

Definition1.6. [10]: A subset A of a space X is θ -semi-open if for each $x \in A$, there exists a semi-open set G such that $x \in G \subseteq \overline{G} \subseteq A$.

Definition1.7. [8]: Let (X, T) be a topological space. and $A \subseteq X$ is called b-open if $A \subseteq \overline{A^{\circ}} \cup \overline{A}^{\circ}$.

Definition1.8 .[22]**:** Let (X, T) be a topological space. and $A \subseteq X$ is called Regular-open if $A = \overline{A}^{\circ}$.

Definition 1.9. [12]: A topological space X is called an Alexandroff space, if any arbitrary intersection of open sets is open.

Definition 1.10. [9] [13] [9]: A topological space X is called:

1. A locally indiscrete if and only if every open set is closed.
2. A regular space if for each $x \in X$ and for each open set G containing x, there exist an open set H such that $x \in H \subseteq \overline{H} \subseteq G$.
3. An extremely disconnected if the closure of any open set is open.

2. SEMI-PRE-C-OPEN SET

Definition2.1: Consider the topological space (X, T) , where X is the underlying set and T is the topology defined on it. We refer to a subset $F \subseteq X$ as a $\text{semi} - \text{pre} - c - \text{open}$ set if it satisfies the following criteria: (1) F is $\text{semi.pre} - \text{open}$, and (2) for every a belonging to F, there exists a closed set K such that a is an element of K and K is a subset of F.

A subset F of X is termed a $\text{semi} - \text{pre} - c - \text{closed}$ set if and only if the complement of F in X, denoted as F^c, is $\text{semi} - \text{pre} - c -. \text{open}$. We denote the families of $\text{semi} - \text{pre} - c - \text{open}$ and $\text{semi} - \text{pre} - c - \text{closed}$ in a the topological space (X, T)as $SP - C - O(X, T)$and $SP - C - C(X, T)$respectively.

Proposition 2.2: Let (X, T)be a topological space and $K \subseteq X$ is $sp - c - \text{open}$ if and only if it is $sp - \text{open}$ and can be expressed as a union of closed sets.

Proof: Let K be a $sp - c - \text{open}$.
Then, K is $sp - \text{open}$ and for each $a \in K$ there exists a closed set M_a in X such that $a \in M_a \subseteq K$.
This implies that $\bigcup_{a \in K} M_a \subseteq K \subseteq \bigcup_{a \in K} M_a$.
Thus, $K = \bigcup_{a \in K} M_a$ where F_a is closed set for each $a \in K$.
The reverse implication follows directly from the definition of $sp - c - \text{open}$ sets.

Remark 2.3: Every $sp - c - \text{open}$ is $sp - \text{open}$ but the converse need not be true in general, consider the following example.

Example 2.4: Let $X = \{a, b, c\}$ with a topology $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, then:
i. $C(X, T) = \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$.
ii. $SP - O(X, T) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
iii. $SP - C - O(X, T) = \{\emptyset, X, \{a, c\}, \{b, c\}\}$. Note that $\{a\} \in SP - O(X, T)$ but $\{a\} \notin SP - C - O(X, T)$.

Proposition 2.5: Arbitrary union of $sp - c - \text{open}$ sets is $sp - c - \text{open}$ set.

Proof: Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of $sp - c - \text{open}$ sets in a topological space (X, T). Then F_α is $sp - \text{open}$ set for each $\alpha \in \Delta$ and $\bigcup_{\alpha \in \Delta} F_\alpha$ is $sp - \text{open}$ set. If $a \in \bigcup_{\alpha \in \Delta} F_\alpha$, then their exists $\lambda \in \Delta$ such that $a \in F_\lambda$. Since F_λ is $sp - c - \text{open}$ set, there exists a closed set K such that $a \in K \subseteq F_\lambda \subseteq \bigcup_{\alpha \in \Delta} F_\alpha$.
Hence, $\bigcup_{\alpha \in \Delta} F_\alpha$ is $sp - c - \text{open}$ set.

Remark 2.6: The intersection of even two $sp - c - \text{open}$ sets need not be $sp - c - \text{open}$ set. Consider the following example.

Example2.7: In Example 1.5, $F = \{a, c\}$ and $K = \{b, c\}$ are sp-c-open sets but $F \cap K = \{c\}$ is not $sp - c - \text{open}$ set.

Corollary 2.8: Arbitrary intersection of $sp - c - \text{closed}$ sets is $sp - c - \text{closed}$ set.
Proof: Using Proposition 2.5 and De Morgan's Law.

Proposition 2.9: let (X, T) be a topological space and $F \subseteq X$ is $sp - c - open$ if and only if for each $a \in F$, there exists $sp - c - open$ set K such that $a \in K \subseteq F$.

Proof: Assume that F is $sp - c - open$ set in the (X, T) , then for each $a \in A$, put $K = F$ is $sp - c - open$ set containing a such that $a \in K \subseteq F$
Conversely, suppose that for each $a \in F$, there exists a $sp - c - open$ set K such that $a \in K \subseteq F$, thus $F = \cup K_a$ where $K_a \in SP - C - O(X, T)$ for each a, therefore F is $sp - c - open$ set.

Proposition 2.10: Every regular closed set is sp-c-open set.

Proof: Assume that F be regular closed set, then $F = \overline{F^\circ}$ but $\overline{F^\circ} \subseteq \overline{\overline{F}^\circ}$ and so, $F \in SP - O(X, T)$. Since $F \in C(X, T)$, by Definition 2.1.1 then , $F \in SP - C - O(X, T)$.

Proposition 2.11: Let (X, T) be a topological space is locally indiscrete, then each s-open sets are $sp - c - open$ set.

Proof: Consider F be a $s - open$. Thus $F \subseteq \overline{F^\circ} \subseteq \overline{\overline{F}^\circ}$. Hence F is $sp - open$. Since (X, T) is locally indiscrete, $F^\circ \in C(X, T)$ and $F \subseteq \overline{F^\circ} = F^\circ$. Consequently,$F \in O(X, T)$ and for any $a \in A$, $a \in F^\circ \subseteq F$ .Thus, by Definition 2.1.1, $F \in SP - C - O(X, T)$.

Proposition 2.12: Let (X, T) be a topological space and F ⊆ X. If F is $\theta - s - open$ set. Then $F \in SP - C - O(X, T)$.
Proof: Assume that F be a $\theta - semi$ open set. For each $a \in F$ there exists a $s - open$ set G_a such that $a \in G_a \subseteq \overline{G_a} \subseteq F$. This implies that $F = \bigcup_{a \in F} G_a$ which means F is a union of $s - open$ sets. Therefore, $F \in S - O(X, T)$ and so, $F \in SP - O(X, T)$. Also, $F = \bigcup_{a \in F} \overline{G_a}$ which is a union of closed sets. Therefore, $F \in SP - C - O(X, T)$.

Corollary 2.13: Let (X, T) be a topological space and F ⊆ X. If F is $\theta - open$ set, then $F \in SP - C - O(X, T)$.
Proof: Since every $\theta - open$ set is $\theta - s - open$ set, then by Proposition 1.13, we get the result.

Remark 2.14: Since $s - open$ sets ⇒ $b - open$ sets ⇒ $sp - open$ sets, then $sc - open$ sets ⇒ $bc - open$ sets ⇒ $sp - c - open$ sets.

Remark 2.15: Since $p - open$ sets ⇒ $b - open$ sets ⇒ $sp - open$ sets, then $pc - open$ sets ⇒ $bc - open$ sets ⇒ $sp - c - open$ sets.

Proposition2.16: Let (X, T) be a topological space, then $SC - O(X, T) \cup PC - O(X, T) \subseteq BC - O(X, T) \subseteq SP - C - O(X, T)$.

Proposition 1.18: Let (X, T) be an Alexanderoff space, then $SC - O(X, T) = BC - O(X, T) = SP - C - O(X, T)$.

Proof: By Proposition 2.16, we have $SC - O(X, T) \subseteq BC - O(X, T) \subseteq SP - C - O(X, T)$. If $F \in SP - C - O(X, T)$, then F is a union of closed sets in an Alexandroff space, implies that F is closed. Therefore, $F \subseteq \overline{\overline{F}^\circ} = \overline{F^\circ}$ and thus, $F \in S - O(X, T)$. Hence, $F \in SC - O(X, T)$. Thus, $SC - O(X, T) \subseteq BC - O(X, T) \subseteq SP - C - O(X, T) \subseteq SC - O(X, T)$ and we get that $SC - O(X, T) = BC - O(X, T) = SP - C - O(X, T)$.

3. NEW OPERATORS IN TOPOLOGICAL SPACE

In this section, we define and study semi-pre-c-operators named, semi-pre-c-closure, semi-pre-c-interior, semi-pre-c-frontier. Also, we extend our study to semi-pre-c-neighborhood, semi-pre-c-limit points and semi-pre-c-derived sets which are based on the notation of semi-pre-c-open sets. At the end of this section, we give new equivalent definitions for semi-pre-c-open sets and semi-pre-c-closed sets using the concept of semi-pre-c-frontier.

Definition 3.1: Let (X, T) be a topological space and $A \subseteq X$. The semi-pre-c-closure of A in X is the set,

$$\overline{A_{s.pc}} = \cap \{ K \subseteq X : K \text{ is } sp-c-closed \text{ set and } A \subseteq K \}$$

Remark 3.2: The following two examples show that there is no relations between $\overline{A}$ and $\overline{A_{s.p.c}}$.

Example 3.3: Consider $X = \{a, b, c\}$ with a topology $T = \{\emptyset, X, \{a\}\}$. Then, $\overline{\{b, c\}_{s.p.c}} = X$ and $\overline{\{b, c\}} = \{b, c\}$. But $X \nsubseteq \{b, c\}$, which implies, $\overline{A_{s.p.c}} \nsubseteq \overline{A}$ where $A \subseteq X$.

Example3.4: Let $X = \{a, b, c\}$ with a topology $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.
Then, $\overline{\{b\}} = \{b, c\}$ and $\overline{\{b\}_{s.p.c}} = \{b\}$. But $\{b, c\} \nsubseteq \{b\}$, which implies, $\overline{A} \nsubseteq \overline{A_{s.p.c}}$. Hence, in general we have, $\overline{A} \neq \overline{A_{s.p.c}}$ where $A \subseteq X$.

Theorem 3.5: Let (X, T) be a topology space and let F and K are subsets of X, If F $sp-c-$ open set, then $F \cap \overline{K_{s.p.c}} \subseteq \overline{(F \cap K)_{s.p.c}}$

Proof: Let $b \in F \cap \overline{K_{s.p.c}}$. This implies $x \in F$ and $x \in \overline{K_{s.p.c}} = K \cup K'_{s.p.c}$. If $x \in K$, then $x \in F \cap K \subseteq \overline{(F \cap K)_{s.p.c}}$. If $b \notin K$, then $x \in K'_{s.p.c}$, meaning for all sp-c-open set G containing x, $M \cap K = \emptyset$. Since $F \in SP-C-O(X, T)$, so $G \cap f$ is also a sp-c-open set containing x. Therefore, $G \cap (F \cap K) = (M \cap F) \cap K = \emptyset$,hence $x \in (F \cap K)'_{s.p.c} \subseteq \overline{(F \cap K)_{s.p.c}}$. Thus, $F \cap \overline{K_{s.p.c}} \subseteq \overline{(F \cap K)_{s.p.c}}$

Theorem3.6: Let (X, T) be a topological space and $A \subseteq X$. Then, $\overline{A_{s.p.c}}$ is the smallest sp-c-closed set containing A.

Proof: Let $\{ : \alpha \in \Delta \}$ be the collection of all sp-c-closed sets in X containing A. Then, by Corollary 2.1.12, $\overline{A_{s.p.c}} = \bigcap_{\alpha \in \Delta} F_\alpha$ is sp-c-closed set and $A \subseteq \bigcap_{\alpha \in \Delta} F_\alpha$. Since $\bigcap_{\alpha \in \Delta} F_\alpha \subseteq F_\alpha$ for each $\alpha \in \Delta$, then $\overline{A_{s.p.c}} = \bigcap_{\alpha \in \Delta} F_\alpha$ is the smallest sp-c-closed set containing A.

Theorem3.7: Let (X, T) be a topological space and $A \subseteq X$ is sp-c-closed if and only if $\overline{A_{s.p.c}} = A$.

Proof: Let A be sp-c-closed set. By Theorem 2.3.5, $\overline{A_{s.p.c}} \subseteq A$. But from Definition 2.3.1, $A \subseteq \overline{A_{s.p.c}}$. Hence, $A = \overline{A_{s.p.c}}$.
Conversely, let $\overline{A_{s.p.c}} = A$. Since $\overline{A_{s.p.c}} \in SP-C-C(X, T)$, Therefore $A \in SP-C-C(X, T)$.

Example3.8: In the usual topology R, $\overline{(0,1]}_{s.p.c} = (0,1]$ because $(0,1] \in SP-C-C(X,T)$and if Q is the set of rationales, then $\overline{Q}_{s.p.c} = Q$.

Theorem3.9: Let (X,T) be a topological space and $A \subseteq X$ and $z \in X$. Then $z \in \overline{A}_{s.p.c}$ if and only if for any $K \in SP-C-O(X,T)$ such that $z \in K$ we get, $A \cap K \neq \emptyset$.

Proof: If $z \notin \overline{A}_{s.p.c}$, then there exists $B \in SP-C-C(X,T)$ such that $A \subseteq B$ and $z \notin B$. But $B^c \in SP-C-O(X,T)$ containing z and therefore, $A \cap B^c \subseteq A \cap A^c = \emptyset$ which is a contradiction and hence, $z \notin \overline{A}_{s.p.c}$.

Conversely, suppose that there exists a sp-c–open set K where $z \in K$ and $A \cap K = \emptyset$., then $A \subseteq K^c$ where K^c is sp-c–closed set, hence $z \notin \overline{A}_{s.p.c}$.

Theorem3.10: Suppose we have a topological space (X, T) and consider two subsets A and B of X.

1. $A \subseteq \overline{A}_{s.p.c}$.
2. $\overline{\emptyset}_{s.p.c} = \emptyset$ and $\overline{X}_{s.p.c} = X$.
3. If $A \subseteq B$, then $\overline{A}_{s.p.c} \subseteq \overline{B}_{s.p.c}$.
4. $\overline{(\overline{A}_{s.p.c})}_{s.p.c} = \overline{A}_{s.p.c}$.
5. If $\overline{A}_{s.p.c} \cap \overline{B}_{s.p.c} = \emptyset$, then $A \cap B = \emptyset$.
6. $\overline{A}_{s.p.c} \cup \overline{B}_{s.p.c} \subseteq \overline{(A \cup B)}_{s.p.c}$.
7. $\overline{(A \cap B)}_{s.p.c} \subseteq \overline{A}_{s.p.c} \cap \overline{B}_{s.p.c}$.

Proof: (1) From Theorem3.6.

(2) From Definition 3.1.

(3) Let $A \subseteq B$ and assume $x \in \overline{A}_{s.p.c}$.

Then by Theorem3.9, for any $G \in SP-C-O(X,T)$ containing x, we have $A \cap G \neq \emptyset$. But $A \subseteq B$ which implies, $B \cap G \neq \emptyset$ where B is any sp-c-open set containing x. Hence, $x \in \overline{B}_{s.p.c}$ and therefore, $\overline{A}_{s.p.c} \subseteq \overline{B}_{s.p.c}$ hence $x \in \overline{A}_{s.p.c}$.

(4) Since $\overline{A}_{s.p.c} \in SP-C-C(X,T)$, $\overline{(\overline{A}_{s.p.c})}_{s.p.c} = \overline{A}_{s.p.c}$.

(5) Let $A \cap B \neq \emptyset$, then there is $x \in A \cap B$ which implies $x \in A$ and $x \in B$. By Part (1), $x \in \overline{A}_{s.p.c}$ and $x \in \overline{B}_{s.p.c}$ and so $\overline{A}_{s.p.c} \cap \overline{B}_{s.p.c} \neq \emptyset$.

(6) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by Part (3), $\overline{A}_{s.p.c} \subseteq \overline{(A \cup B)}_{s.p.c}$ and $\overline{B}_{s.p.c} \subseteq \overline{(A \cup B)}_{s.p.c}$. Hence, $\overline{A}_{s.p.c} \cup \overline{B}_{s.p.c} \subseteq \overline{(A \cup B)}_{s.p.c}$.

(7) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Part (3), $\overline{(A \cap B)}_{s.p.c} \subseteq \overline{A}_{s.p.c}$ and $\overline{(A \cap B)}_{s.p.c} \subseteq \overline{B}_{s.p.c}$. Hence, $\overline{(A \cap B)}_{s.p.c} \subseteq \overline{A}_{s.p.c} \cap \overline{B}_{s.p.c}$.

Definition 3.11: Let (X,T) be a topological space. A set $U_x \subseteq X$ is said to be semi pre –c-neighborhood of a point $x \in X$ if and only if there exists sp-c-open set V in X such that $x \in V \subseteq U_x$.

Proposition 3.12: In a topological space(X,T), a subset A of X is sp-c-open if and only if it is a sp-c- neighbourhood of each of its points.

Proof: Let $A \subseteq X$ be a sp-c-open set, since for every $x \in A$,$x \in A \subseteq A$ and A is sp-c-open. This shows A is a sp-c-neighborhood of each of its points.
Conversely, suppose that A is a sp-c-neighborhood of each of its points. Then for each $x \in$, there exists $B_x \in SP - C - O(X, T)$such that $B_x \in A$. Then $A = \{B_x : x \in A\}$. Since each B_x is sp-c-open. It follows that A is sp-c-open set.

Definition3.15: Let (X, T) be a topological space. A point $x \in X$ is said to be semi pre-c-interior point of a set $A \subseteq X$ if and only if there exists sp-c-open set V in X such that $x \in V \subseteq A$.
i.e., $A^{\circ}_{s.p.c}$ is the largest semi - open set of X contained in A.

Example3.16: Consider (X, T) defined in Example 2.4. Then, if $A = \{a, c\}$, then $A^{\circ} = \{a\}$ and $(A_{s.p.c})^{\circ}. = \{a, c\}$ which implies, $(A_{s.p.c})^{\circ} \nsubseteq A^{\circ}$. On the other hand, if $B = \{a\}$, then $B^{\circ} = \{a\}$ and $(B_{s.p.c})^{\circ} = \emptyset$ that is, $B^{\circ} \nsubseteq (B_{s.p.c})^{\circ}$.. So that, in general, $B^{\circ} \neq (B_{s.p.c})^{\circ}$.

Proposition 3.17: Let (X, T) be a topological space and $A \subseteq X$. Then, $(A_{s.p.c})^{\circ}$. is the largest sp-c-open subset of X contained in A.
Proof: Let $U = \{U_\alpha : \alpha \in \Delta\}$ be a collection of all sp-c-open sets contained in A. Then, by Definition 2.3.16, $\bigcup_{\alpha\in\Delta} U_\alpha = (A_{s.p.c})^{\circ}$. and so, by Proposition 2.1.9, $(A_{s.p.c})^{\circ}$. is sp-c-open set. Let V be sp-c-open set such that $V \subseteq A$, then for any $y \in V$ we have, $y \in V \subseteq A$ and so, $y \in (A_{s.p.c})^{\circ}$.. Therefore, $V \subseteq (A_{s.p.c})^{\circ}$.and hence, $(A_{s.p.c})^{\circ}$. is the largest sp-c-open set contained in A.

Proposition 3.18: Let X be a topological space and $A \subseteq X$. Then, A is sp-c-open set if and only if $(A_{s.p.c})^{\circ}. = A$.
Proof: Let A be sp-c-open set, then for any $x \in A, x \in (A_{s.p.c})^{\circ}$ and $A \subseteq (A_{s.p.c})^{\circ}$. But by Proposition.3.17, we have $(A_{s.p.c})^{\circ} \subseteq A$ and hence, $(A_{s.p.c})^{\circ} = A$.
Conversely, if $(A_{s.p.c})^{\circ} = A$, then by Proposition 3.17, $(A_{s.p.c})^{\circ}$ is sp-c-open and so, A is sp-c-open set.
Remark3.19: This can be demonstrated by the following example: If $(F_{s.p.c})^{\circ} = (K_{s.p.c})^{\circ}$, it does not necessarily mean that $F = K$.

Example3.20: Let $X = \{a, b, c\}$ and $T = \{\emptyset, X, \{a\}\}$, then if $F = \{a\}$ and $K = \{b\}$, we have $(F_{s.p.c})^{\circ} = (K_{s.p.c})^{\circ} = \emptyset$ but $F \neq K$.

Theorem3.21: Let A and B be two subsets of a topological space(X, T) ,then:
1. $(A_{s.p.c})^{\circ} \subseteq A$.
2. $(\emptyset_{s.p.c})^{\circ} = \emptyset$ and $(X_{s.p.c})^{\circ} = X$.
3. If $A \subseteq B$, then $(A_{s.p.c})^{\circ} \subseteq (B_{s.p.c})^{\circ}$.
4. $(((A_{s.p.c})^{\circ})_{s.p.c})^{\circ} = (A_{s.p.c})^{\circ}$.
5. If $A \cap B = \emptyset$, then $(A_{s.p.c})^{\circ} \cap (B_{s.p.c})^{\circ} = \emptyset$.
6. $(A_{s.p.c})^{\circ} \cup (B_{s.p.c})^{\circ} \subseteq ((A \cup B)_{s.p.c})^{\circ}$.
7. $((A \cap B)_{s.p.c})^{\circ} \subseteq (A_{s.p.c})^{\circ} \cap (B_{s.p.c})^{\circ}$.
8. $((A^c)_{s.p.c})^{\circ} \subseteq ((A_{s.p.c})^{\circ})^c$

9. $\left((A - B)_{s.p.c}\right)^{\circ} \subseteq \left(A_{s.p.c}\right)^{\circ} - \left(B_{s.p.c}\right)^{\circ}$

Proof: (1) From Proposition.3.17.

(2) From Definition 3.15.

(3) Let $A \subseteq B$ and $x \in (A_{s.p.c})^{\circ}$, then there exists sp-c-open set V such that $x \in V \subseteq A$. But $A \subseteq B$ which implies, $x \in V \subseteq B$. Hence, $x \in (B_{s.p.c})^{\circ}$ and therefore, $(A_{s.p.c})^{\circ} \subseteq (B_{s.p.c})^{\circ}$.

(4) Since $(A_{s.p.c})^{\circ} \in SP - C - O(X,T)$ .By Proposition3.18, then $(((A_{s.p.c})^{\circ})_{s.p.c})^{\circ} = (A_{s.p.c})^{\circ}$.

(5) If$(A_{s.p.c})^{\circ} \cap (B_{s.p.c})^{\circ} \neq \emptyset$, then there is $x \in (A_{s.p.c})^{\circ} \cap (B_{s.p.c})^{\circ}$. So there exist sp-c-open sets U and V such that $x \in U \subseteq A$ and $x \in V \subseteq B$ which implies, $x \in U \cap V \subseteq U \subseteq A$ and $x \in U \cap V \subseteq V \subseteq B$. Hence, $x \in A \cap B$ and therefore, $A \cap B \neq \emptyset$.

(6) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by Part (3), $(A_{s.p.c})^{\circ} \subseteq ((A \cup B)_{s.p.c})^{\circ}$ and $(B_{s.p.c})^{\circ} \subseteq ((A \cup B)_{s.p.c})^{\circ}$. Hence, $(A_{s.p.c})^{\circ} \cup (B_{s.p.c})^{\circ} \subseteq ((A \cup B)_{s.p.c})^{\circ}$.

(7) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Part (3), $((A \cap B)_{s.p.c})^{\circ} \subseteq (A_{s.p.c})^{\circ}$ and $((A \cap B)_{s.p.c})^{\circ} \subseteq (B_{s.p.c})^{\circ}$. Hence, $((A \cap B)_{s.p.c})^{\circ} \subseteq (A_{s.p.c})^{\circ} \cap (B_{s.p.c})^{\circ}$.

(8) Let $z \in ((A^c)_{s.p.c})^{\circ}$. According to,$(A_{s.p.c})^{\circ} \subseteq A$ .Therefore, $z \in A^c \Rightarrow z \notin A \Rightarrow z \notin (A_{s.p.c})^{\circ} \Rightarrow z \in \left(\left(A_{s.p.c}\right)^{\circ}\right)^c$. Hence,$((A^c)_{s.p.c})^{\circ} \subseteq \left(\left(A_{s.p.c}\right)^{\circ}\right)^c$

(9) So, $((A - B)_{s.p.c})^{\circ} = ((A \cap B^c)_{s.p.c})^{\circ}$

$= (A_{s.p.c})^{\circ} \cap ((B^c)_{s.p.c})^{\circ} \subseteq \left(A_{s.p.c}\right)^{\circ} \cap ((B_{s.p.c})^{\circ})^c = \left(A_{s.p.c}\right)^{\circ} - (B_{s.p.c})^{\circ}$.Hence ,

$((A - B)_{s.p.c})^{\circ} \subseteq \left(A_{s.p.c}\right)^{\circ} - (B_{s.p.c})^{\circ}$

Theorem 3.22: Let (X, T)be a topological space, and $A \subseteq X$. Then the following statements hold.

(1) $(\overline{A_{s.p.c}})^c = ((A^c)_{s.p.c})^{\circ}$.

(2) $((A_{s.p.c})^{\circ})^c = \overline{(A^c)_{s.p.c}}$.

(3) $\overline{A_{s.p.c}} = [((A^c)_{s.p.c})^{\circ}]^c$.

(4) $(A_{s.p.c})^{\circ} = [\overline{(A^c)_{s.p.c}}]^c$.

Proof: Only we prove the first one and the other parts can be proved similarly.

$$[\overline{A_{s.p.c}}]^c = [\cap \{K : K \in SP - C - C(X,T) \text{ and } A \subseteq K\}]^c .$$
$$= [\cup \{K^c : K^c \in SP - C - O(X,T) \text{ set and } K^c \subseteq A^c\}].$$
$$= [\cup \{B : B \in SP - C - O(X,T) \text{ and } B \subseteq A^c\}].$$
$$= ((A^c)_{s.p.c})^{\circ} .$$

Definition3.23: Let A be a subset of a topological space (X, T). A point $x \in X$ is said to be semi pre-c-limit point of A if for each sp-c-open subset G of X containing x, $G \cap A\backslash\{x\} \neq \emptyset$.
The set of all sp-c-limit points of A is called the smei pre-c-derived set of A and its denoted by sp-c-D(A) or $A'_{s.p.c}$.

Theorem3.24: Let (X T) be a topological space, A and B be subsets of the space X with the sp-c-derived sets $(A_{s.p.c})'$and $(B_{s.p.c})'$ respectively. Then the following properties hold:

(1) $(\emptyset_{s.p.c})' = \emptyset$.

(2) If $x \in \left(A_{s.p.c}\right)'$, then $x \in \left((A\backslash\{x\})_{s.p.c}\right)'$

(3) If $A \subseteq B$, then $(A_{s.p.c})' \subseteq (B_{s.p.c})'$.
(4) $(A_{s.p.c})' \cup (B_{s.p.c})' \subseteq ((A \cup B)_{s.p.c})'$.
(5) $((A \cap B)_{s.p.c})' \subseteq (A_{s.p.c})' \cap (B_{s.p.c})'$.

Proof:(1) Let $x \in (\emptyset_{s.p.c})'$ and G be any sp-c-open set containing x. Then $G \cap \emptyset\backslash\{x\} = \emptyset$ which is a contradiction.
(2) Let $x \in (A_{s.p.c})'$Then for any sp-c-open set G containing x we have, $G \cap A\backslash\{x\} \neq \emptyset$. That is, for any sp-c-open set G containing x, $G \cap A$ contain points other than x and so the intersection of G and $A\backslash\{x\}$ contain points and hence $x \in \left((A\backslash\{x\})_{s.p.c}\right)'$.
(3) Let $x \in (A_{s.p.c})'$. Then for any sp-c-open set G containing x, $G \cap A\backslash\{x\} \neq \emptyset$. But $A \subseteq B$ and so, $G \cap B\backslash\{x\} \neq \emptyset$. Hence, $x \in (B_{s.p.c})'$.
(4) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by Part (3), $(A_{s.p.c})' \subseteq ((A \cup B)_{s.p.c})'$ and $(B_{s.p.c})' \subseteq ((A \cup B)_{s.p.c})'$. Hence, $(A_{s.p.c})' \cup (B_{s.p.c})' \subseteq ((A \cup B)_{s.p.c})'$.
(5) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Part (3), $((A \cap B)_{s.p.c})' \subseteq (A_{s.p.c})'$ and $((A \cap B)_{s.p.c})' \subseteq (B_{s.p.c})'$. Hence, $((A \cap B)_{s.p.c})' \subseteq (A_{s.p.c})' \cap (B_{s.p.c})'$.

Definition3.25: Let (X, T) be a topological space for any subset $A \subseteq X$, is called the semi pre-c-frontier of A if $(A_{s.p.c})^f = [\overline{A_{s.p.c}} - (A_{s.p.c})^\circ]$ and denoted by sp-c-F (A) or $(A_{s.p.c})^f$.

Lemma3.26: Let (X, T) be a topological space and $F \subseteq X$, then $F \in SP - C - C(X, T)$ if and only if $F^f_{s.p.c} \subseteq F$.

Proof: Assume that $F \in SP - C - C(X, T)$. Then, $F^f_{s.p.c} = \overline{F_{s.p.c}} \backslash (F_{s.p.c})^\circ = F\backslash(F_{s.p.c})^\circ \subseteq F$.
Conversely, suppose $F^f_{s.p.c} \subseteq F$. Then, $\overline{F_{s.p.c}}\ (F_{s.p.c})^\circ \subseteq F$, implying $\overline{F_{s.p.c}} \subseteq F$.Therefore, $F = \overline{F_{s.p.c}}$ confirming that $F \in SP - C - C(X, T)$

Proposition 3.27: Let (X, T) be a topological space for any $A \subseteq X$. The following statements hold:

1. $\overline{A_{s.p.c}} = (A_{s.p.c})^\circ \cup (A_{s.p.c})^f$.
2. $(A_{s.p.c})^\circ \cap (A_{s.p.c})^f = \emptyset$.
3. $(A_{s.p.c})^f = \overline{A_{s.p.c}} \cap \overline{(A^c)_{s.p.c}}$).
4. $(F_{s.p.c})^\circ = F \backslash F^f_{s.p.c}$.
5. $(F^f_{s.p.c})^f_{s.p.c} \subseteq F^f_{s.p.c}$.

Proof: (1) $(A_{s.p.c})^\circ \cup (A_{s.p.c})^f = (A_{s.p.c})^\circ \cup [\overline{A_{s.p.c}} - (A_{s.p.c})^\circ]$

$$= [(A_{s.p.c})^\circ \cup \overline{A_{s.p.c}}] \cap [(A_{s.p.c})^\circ \cup ((A^c)_{s.p.c})^\circ]$$
$$= \overline{A_{s.p.c}} \cap X$$
$$= \overline{A_{s.p.c}}.$$

(2) $(A_{s.p.c})^\circ \cap (A_{s.p.c})^f = (A_{s.p.c})^\circ \cap [\overline{A_{s.p.c}} - (A_{s.p.c})^\circ]$

$$= (A_{s.p.c})^\circ \cap [\overline{A_{s.p.c}} \cap ((A^c)_{s.p.c})^\circ]$$
$$= \emptyset .$$

(3) $(A_{s.p.c})^{f} = [\overline{A_{s.p.c}} - (A_{s.p.c})^{\circ}]$
$= \overline{A_{s.p.c}} \cap ((A^{c})_{s.p.c})^{\circ}$
$= \overline{A_{s.p.c}} \cap \overline{(A^{c})_{s.p.c}}$.

(4) Utilizing the definition of the sp-c-frontier of F and fundamental properties property of set theory, we find that $F\backslash F^{f}_{s.p.c} = F\backslash\left(\overline{F_{s.p.c}}\backslash\left(F_{s.p.c}\right)^{\circ}\right) = \left(F\backslash\overline{F_{s.p.c}}\right) \cup \left(F \cap \overline{F_{s.p.c}} \cap \left(F_{s.p.c}\right)^{\circ}\right) = \left(F\backslash\overline{F_{s.p.c}}\right) \cup \left(F_{s.p.c}\right)^{\circ} = \emptyset \cup (F_{s.p.c})^{\circ} = (F_{s.p.c})^{\circ}$

(5) Since $F^{f}_{s.p.c} \in SP - C - C(X, T)$, by Lemma3.26, $(F^{f}_{s.p.c})^{f}_{s.p.c} \subseteq F^{f}_{s.p.c}$.

Definition3.28: The semipre − c − interior of the compliment of K is called the semi pre − c − Exterior of K and is denoted by $K^{x}_{s.p.c}$

i.e. $K^{x}_{s.p.c} = (K^{c})^{\circ}_{s.p.c}$

Theorem3.29: Let (X, T) be a topological space, K and F subset of a space X the following statements hold;

1. $K^{x} \subseteq K^{x}_{s.p.c}$
2. $K^{x}_{s.p.c} \in SP - C - O(X, T)$
3. $K^{x}_{s.p.c} = (\overline{K_{s.p.c}})^{c}$
4. $(K^{x}_{s.p.c})^{x}_{s.p.c} = (\overline{K_{s.p.c}})^{\circ}_{s.p.c}$
5. If $K \subseteq F \implies F^{x}_{s.p.c} \subseteq K^{x}_{s.p.c}$

Proof:

1. Let $x \in K^{x} \Rightarrow x \in (K^{c})^{\circ}$ Therefore G is open set such that $x \in G \subseteq K^{c}$, and also $G \in SP - c - O\ (X, T) \Rightarrow x \in G \subseteq A^{c}$ for semi pre-c- open set $G \Rightarrow x \in (K^{c})^{\circ}_{s.p.c}$. i .e $x \in K^{x}_{s.p.c}$Therefore $K^{x} \subseteq K^{x}_{s.p.c}$
2. $K^{x}_{s.p.c}$is sp-c- open set
Now, $(K^{x}_{s.p.c})^{\circ}_{s.p.c} = [(K^{c})^{x}_{s.p.c}]^{\circ}_{s.p.c} = (K^{c})^{\circ}_{s.p.c} = K^{x}_{s.p.c} \Rightarrow K^{x}_{s.p.c}$ is sp − c − open set.

3. $K^{x}_{s.p.c} = (K^{c})^{\circ}_{s.pc} = (\overline{K_{s.p.c}})^{c}$
4. $\left(K^{x}_{s.p.c}\right)^{x}_{s.p.c} = \left[\left(\overline{A_{s.p.c}}\right)^{c}\right]^{x}_{s.p.c}$ by par (3)

$$= [\left[\left(\overline{A_{s.p.c}}\right)^{c}\right]^{c}]^{\circ}_{s.p.c} = (\overline{A_{s.p.c}})^{\circ}_{s.p.c}$$

5. If $F \subseteq K$ then $F^{c} \subseteq K^{c} \Rightarrow (K^{c})^{\circ}_{s.p.c} \subseteq (F^{c})^{\circ}_{s.p.c}$ i. e $(K)^{x}_{s.p.c} \subseteq (A^{x}_{s.p.c})$

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