



Stability of Caputa– Hadmmard Fractional Differential Nonlinear Control System with Delay Riemann –Katugampola

Ahmed Sami Sleibi¹, Sameer Qasim Hasan²

^{1,2}Department of Mathematics, College of Education, Al- Mustansiriyah University, IRAQ

*Corresponding Author: Ahmed Sami Sleibi

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ABSTRACT: This research examines the finite-time stability of a specific category of neural networks with fractional order. Using modified Gronwall inequality and estimations of Mittag–Leffler functions, we provide adequate requirements to guarantee the finite-time stability of neural models with Caputo fractional derivatives. In addition, we have also established insights about the asymptotic stability of fractional-order neural models, this research examines the Finite-time stability of a specific fractional-order neural network using the "generalized Grunwald inequality". This study introduces the concept of finite-time stability for neural models with fractional derivatives. By utilizing the generalized Gronwall inequality and estimates of Mittag-Leffler functions, sufficient conditions are derived to guarantee the finite-time stability of these models. The study also establishes results regarding the asymptotical stability of fractional-order neural models, our results suggest that our conclusions are more precise than the current literature on stability requirements, and we provide examples demonstrating the proposed theory's significance

Keywords: neural networks, fractional-order, finite-time stability, Gronwall inequality



1. INTRODUCTION

Between Leibniz and L'Hospital in 1695, which became an extension of the conventional integer order differentiation and integral analysis. Given the complexity and lack of application background, it drew very little interest. It has recently been shown to be a valuable tool for modeling various engineering, physics, and economics phenomena, including heat conduction, viscoelastic systems, electromagnetic waves, dielectric polarization, biology, and finance[1],[2],[3]. Researchers from various fields are getting increasingly involved, and it is becoming one of the main topics. The fractional-order derivative is an effective instrument for analyzing memory and heredity. Artificial neural networks have employed fractional calculus in recent years because of its limitless memory. This ability to compute may support effective information processing, stimulus anticipation, and frequency-independent rhythmic neural firing phase changes. The oculomotor integrator, which translates eye velocity into eye position instructions, may be of fractional order, according to a proposal in Ref [4].

It proved that higher levels of approximation were obtained when neural network approximation was performed at the fractional level[5]. Furthermore, fractional-order recurrent neural networks are important in parameter estimation. Therefore, studying fractional-order neural networks is essential, and adding memory terms (a fractional derivative or integral operator) to neural network models is an important step forward.

It is generally accepted that the dynamical characteristics of neural models, such as asymptotic and Lyapunov stability, are critical to applications in the real world. Neural network stability thus became one of the most researched topics and has been thoroughly examined. Several stability discoveries have been made in the last decades for integer-order neural networks; see Refs for a list of these studies.[6],[7],[8] and the references therein. Several essential and wondrous discoveries about the stability of fractional-order artificial neural networks have recently been made. For instance, Ref. [9] examined the stability and multi-stability, bifurcations, and chaos of fractional-order neural networks of the Hopfield type.

The topic of stability behaviors in noninteger-order cellular neural networks was covered. An energy-like function was used [10] to look at the stability of a suggested fractional-order Caputa Hadmmard neural model

$$\begin{cases} {}^{CH}D_t^{\alpha,\mu}x(t) = A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}D_t^{\alpha,\mu}x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}D_t^{\alpha,\mu}x(t - \tau) \right) \right) + B_2(t)u(t) \\ \varphi(t) = \end{cases}, t \in [0, T] \\ , t \in [-\tau, 0] \quad (1)$$

Where ${}^{CH}D_t^{\alpha,\mu}$ denotes the Caputo –Hadmard Fractional Derivatives of order $\alpha, \mu > 0$ ${}^{RK}D_t^{\alpha,\mu}$ denotes the Riemann–Katugampola derivative of order C be aconstant, $\alpha > 0$, $\mu > 0$ and $A_1(\cdot), A_2(\cdot), B_1(\cdot), B_2(\cdot)$ and $x(\cdot) \in R^n$ for $t \in [-\infty, T]$, where $0 < a < b < \infty$, ${}^{RH}D_t^{\alpha,\mu}$ denotes the Riemann–Hadamard Fractional derivative of order $0 < \alpha < 1, \mu$ be a positive value . $F(\cdot, \cdot, \cdot): R^n \rightarrow R^n$, $G(\cdot, \cdot, \cdot): R^n \rightarrow R^n$, $B_{n \times m}$ is a control matrix and $u(t): [0, T] \rightarrow R^m$ is a control function. Finely the function $x(t)$ is a nonlocal function defined on $[-\tau, 0]$, $\tau \in (-\infty, 0)$.

Numerical simulations were used to examine the intricate dynamical characteristics of a proposed fractional-order four-cell cellular neural network [11]. -stability and -synchronization for fractional-order neural networks were studied by Yu et al. in [12] .

Once the initial states of a neural network complete a particular limitation and the states do not exceed a specific limit within predetermined time intervals, the network is said to be finite-time stable. Real neural networks always function across limited time intervals, compared to the infinite periods required by classical Lyapunov stability notions . In addition, the asymptotic behavior is the primary goal of traditional Lyapunov stability. In reality, applications often require keeping the states inside a set of parameters for some time.

In some cases, the system's finite-time behavior is of greater importance than its asymptotic behavior. Thus, finite-time stability was introduced, addressing systems with predetermined boundaries and limited time intervals [13], since then, much research has been done on limited spatial stability; see, for instance, Refs.[14],[15],[16] .

The paper offered sufficient criteria for the finite-time stability of linear fractional-order systems. This research examines the finite-time stability of cellular neural networks with fractional order. The finite-time stability is demonstrated by using the generalized Gronwall inequality and using estimations of Mittag-Leffler functions. In addition, conclusions about the asymptotic stability are also found, the main achievement of this study is the development of test criteria for determining the finite-time stability of fractional networks. These conditions will be valuable in guiding the development and execution of fractional networks .

Based on our extensive research, there is a need for more information on this particular matter. A significant number of studies are being conducted on the dynamic behaviors of complex networks. Includes studies referenced in

The remaining content is organized in the following manner. Section 2 presents essential concepts and lemmas and provides the fractional-order neural networks. The necessary conditions for guaranteeing the stability of the model within a limited duration are derived in Section 3. A specific numerical illustration is provided in Section 4. in Section 5, we have conclusion of the paper.

2. PRELIMINARIES

This section talks through certain definitions and lemmas relating to fractional calculus.

2.1 Definition , [17]

Let $\alpha > 0, \mu > 0$, $x \in L^1([a, b], R)$, $0 < a < b < \infty$.The left and right Riemann–Katugampola fractional derivatives (R-K FDs) of order α is defined by

$${}^{RK}D_t^{\alpha,\mu}x(t) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{\tau^{\mu-1}}{(t^\mu-\tau^\mu)^\alpha}x(\tau)d\tau$$

$${}^{RK}D_t^{\alpha,\mu}x(t) = \frac{-\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_t^b \frac{\tau^{\mu-1}}{(\tau^\mu-t^\mu)^\alpha}x(\tau)d\tau.$$

2.2 Definition,[18]

Let $\alpha > 0, \mu > 0, x \in L^1([a, b]), R, 0 < a < b < \infty$. The left and right Riemann–Katugampola fractional integrals (R-KFIs) of order α is defined by

$${}^{RK}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{\alpha-1}x(\tau)\frac{d\tau}{\tau^{1-\mu}}$$

$${}^{RK}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{\tau^\mu-t^\mu}{\mu}\right)^{\alpha-1}x(\tau)\frac{d\tau}{\tau^{1-\mu}}.$$

2.3 Definition, [17]

Let $\alpha \in (0,1), \mu > 0, [a, b] \in R, 0 < a < b < \infty$, The left and right Caputo –Katugampola fractional derivatives (C–KFDs) are defined respectively by

$${}^{CK}D_t^{\alpha,\mu}x(t) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{\tau^{\mu-1}}{(t^\mu-\tau^\mu)^\alpha}(x(\tau)-x(b))d\tau,$$

$${}^{CK}D_t^{\alpha,\mu}x(t) = \frac{-\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_t^b \frac{\tau^{\mu-1}}{(\tau^\mu-t^\mu)^\alpha}(x(\tau)-x(b))d\tau.$$

2.4 Definition,[19]

Let $\alpha > 0$ be a real number with $n = [\alpha] + 1$. The left and right RH-FDs of order $\alpha > 0$ defined by

$${}^{RH}D_t^{\alpha,\mu}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1}\frac{x(\tau)}{\tau}d\tau \quad \tau > a$$

$${}^{RH}D_t^{\alpha,\mu}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(-t\frac{d}{dt}\right)^n\int_t^b \left(\log\frac{t}{\tau}\right)^{n-\alpha-1}\frac{x(\tau)}{\tau}d\tau \quad \tau < b$$

$${}^{RH}D_t^{n,\mu}x(t) = \left(t\frac{d}{dt}\right)^n x(t) \text{ and } {}^{RH}D_t^{n,\mu}x(t) = \left(-t\frac{d}{dt}\right)^n x(t) \text{ where } \alpha = n$$

2.5 Definition,[19]:

Let a, b be two real

numbers with $0 < a < b$. The left and right RH-FIs of order $\alpha > 0$ for function $f: [a, b] \rightarrow R$ is defined by

$${}^{RH}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{x(\tau)}{\tau}d\tau \quad \tau > a$$

$${}^{RH}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{x(\tau)}{\tau}d\tau \quad \tau < b$$

2.6 Definition,[19]:

Let $\alpha > 0$ be a real number with $n = [\alpha] + 1$. The left and right CH-FDs of order $\alpha > 0$ defined by

$${}^{CH}D_t^{\alpha,\mu}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1}\frac{x(\tau)-x(a)}{\tau}d\tau \quad \tau > a$$

$${}^{CH}D_t^{\alpha,\mu}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(-t\frac{d}{dt}\right)^n\int_t^b \left(\log\frac{t}{\tau}\right)^{n-\alpha-1}\frac{x(\tau)-x(a)}{\tau}d\tau \quad \tau < b$$

$${}^{CH}D_t^{n,\mu}x(t) = \left(t\frac{d}{dt}\right)^n x(t) \text{ and } {}^{CH}D_t^{n,\mu}x(t) = \left(-t\frac{d}{dt}\right)^n x(t) \text{ where } \alpha = n$$

2.7 Definition ,[19]:

Let a, b be two real numbers with $0 < a < b$. The left and right CH-FIs of order $\alpha > 0$ for function $f: [a, b] \rightarrow R$ is defined by

$${}^{CH}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{x(\tau)-x(a)}{\tau}d\tau \quad \tau > a$$

$${}^{CH}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{x(\tau)-x(a)}{\tau}d\tau \quad \tau < b$$

2.1 Lemma ,[20]:

Let $\alpha > 0, g_1(t)$ and $g_2(t)$ are nonnegative, nondecreasing function locally integrable on $[0, T)$ (some $T \leq +\infty$), $g(t) \leq M$, and suppose that $x(t)$ is nonnegative and locally integrable on $[0, T)$ with

$$x(t) \leq g_1(t) + g_2(t){}_a^R D_t^{-\alpha}x(t)$$

on this interval. Then $x(t) \leq g_1(t)E_\alpha(g_2(t)t^\alpha), t \in [0, T)$,

where E_α is the Mittag-Leffler function defined by $E_\alpha(v) = \sum_{\ell=0}^{\infty} \frac{v^\ell}{\Gamma(\ell\alpha+1)}$

2.2 Lemma, [20]:

Assume $0 < g_2 \leq g_1, 0 \leq \alpha \in (0,1]$. Then $g_1^\alpha - g_2^\alpha \leq (g_1 - g)^\alpha$

3.PROBLEM FORMULATION FOR PROPOSAL SYSTEM

In this section, we look at the existence, uniqueness and stability theorems for α -order nonlinear differential equations.

3.3 Lemma

The relation between Caputo–Hadamard fractional integrals and Caputo–Hadamard fractional derivatives have the following formulation

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = f(t)$$

Proof

By definition (2.6) and (2.7)

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = {}^{CH}_a D_t^{0, \mu} (f(t)) = f(t) \text{ or}$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(t \frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} \frac{x(\tau) - x(a)}{\tau} d\tau d\tau$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\left(t \frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} \frac{x(\tau) - x(a)}{\tau} d\tau \right) d\tau$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = \left(\frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-1} \frac{tx(\tau)}{\tau} d\tau$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = x(t)$$

3.4 Lemma, [21][6]:

Let f be a continuous function on a rectangle $R = [a, b] \times [c, d]$, then $\int \int f(x, y) dx dy$ have the following $\int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$.

3.5 Lemma

Let $\alpha > 0, \beta > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$ and let $\mu \in \mathbb{R}$ and $c \in \mathbb{R}$ be such that $\mu \geq c$. Then for $x \in X_c^p(a, b)$. Then ${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{-\beta, \mu} x \right) = {}^{CH}_a D_t^{-\alpha-\beta, \mu} x(t)$

Proof:

By using definition (2.6), (2.7), we have that

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{-\beta, \mu} x(\tau) \right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\int_a^\tau \left(\log \frac{\tau}{s} \right)^{\beta-1} \frac{x(s) - x(a)}{s} ds \right) \frac{x(\tau) - x(a)}{\tau} d\tau$$

Now we have that,

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{-\beta, \mu} x(\tau) \right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\int_\tau^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\log \frac{\tau}{s} \right)^{\beta-1} \frac{x(s) - x(a)}{s} ds \right) \frac{1}{\tau} d\tau$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{-\beta, \mu} x(\tau) \right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\int_\tau^t \left(\log \frac{t}{s} \right)^{\alpha+\beta-2} \frac{x(s) - x(a)}{s} ds \right) \frac{1}{\tau} d\tau$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{-\beta, \mu} x(\tau) \right) = \frac{1}{(\alpha + \beta - 1)\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\left(\log \frac{t}{\tau} \right)^{\alpha+\beta-1} \frac{x(\tau) - x(a)}{\tau} \right) d\tau$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{-\beta, \mu} x(\tau) \right) = \frac{1}{\Gamma(\alpha + \beta)} \int_a^t \left(\left(\log \frac{t}{\tau} \right)^{\alpha+\beta-1} \frac{x(\tau) - x(a)}{\tau} \right) d\tau$$

$${}^{CH}_a D_t^{-\alpha-\beta, \mu} x(\tau)$$

3.6 Lemma, [22]: [17]

If $f(t) \in C^n[0, +\infty]$ and $n - 1 < \alpha < n \in \mathbb{Z}^+$,

1. ${}^{RK}_a D_t^{\alpha, \mu} \left({}^{RK}_a D_t^{-\alpha, \mu} f(t) \right) = f(t)$
2. ${}^{CK}_a D_t^{\alpha, \mu} f(t) = {}^{RK}_a D_t^{\alpha, \mu} f(t) - \frac{\mu^\alpha f(a)}{\Gamma(1-\alpha)} (t^\mu - a^\mu)^{-\alpha}$
- 3.

3.7 Lemma

The relation between Caputo–Hadamard fractional integrals and Caputo–Hadamard fractional derivatives have the following formulation

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = f(t)$$

Proof

By definitions (2.6) (2.7) we have get

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = {}^{CH}_a D_t^{0, \mu} (f(t)) = f(t) \text{ or}$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(t \frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} \frac{x(\tau) - x(a)}{\tau} d\tau d\tau$$

$$\begin{aligned}
 {}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) &= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\left(t \frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} \frac{x(\tau)-x(a)}{\tau} d\tau \right) d\tau \\
 {}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) &= \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\left(t \frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} \frac{x(\tau)}{\tau} d\tau \right) d\tau \\
 {}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) &= \int_a^t \left(\left(t \frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-1} \frac{x(\tau)}{\tau} d\tau \right) d\tau \\
 {}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) &= \left(\frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-1} \frac{tx(\tau)}{\tau} d\tau \\
 {}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} f(t) \right) &= x(t)
 \end{aligned}$$

3.8 Lemma

A nonnegative function locally integrable $v(t)$ on $0 \leq t \leq T$ (some $T \leq +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t \leq T$ $g(t) \leq M$ (constant), $\alpha > 0$, and suppose $v(t)$ is nonnegative and locally integrable on $0 \leq t \leq T$ with

$$\begin{aligned}
 u(t) &= (v(t) - v(a)) + g(t) \int_a^t \left(\log \frac{t}{s} \right)^{\alpha-1} (v(s) - v(a)) ds. \text{ Then} \\
 u(t) &\leq v(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)\Gamma((k+1)\alpha)} \left(\log \frac{t}{s} \right)^{n\alpha-1} (v(s) - v(a)) \right] ds \quad \text{or} \quad u(t) \leq (v(t) - v(a)) E_{\alpha} \left(g(t) \left(\log \frac{t}{a} \right)^{\alpha} \right)
 \end{aligned}$$

proof

Let $B\varphi(t) = \frac{g(t)}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{x(\tau)-x(a)}{\tau} d\tau, t \geq 0$. Then $u(t) \leq (v(t) - v(a)) \left(\log \frac{t}{a} \right) + B(u(t) - u(a))$ imply

$$u(t) \leq \sum_{k=0}^{n-1} B^k v(t) + B^n(u(t) - u(a))$$

$$\begin{aligned}
 \text{Now to prove } B^n(u(t) - u(a)) &\leq \int_a^t \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log \frac{t}{\tau} \right)^{n\alpha-1} (u(\tau) - u(a)) \frac{d\tau}{\tau} \\
 B^n u(t) &\rightarrow 0 \text{ as } n \rightarrow +\infty \quad \forall t \in [0, T)
 \end{aligned} \tag{2}$$

We know this relation (2) is true for $n = 1$. Assume that it is true for some $n = k$. If $n = k + 1$, then the induction hypothesis implies

$$B^{k+1}u(t) = B(B^k) \leq \frac{g(t)}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left[\int_a^s \frac{(g(s)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log \frac{s}{\tau} \right)^{k\alpha-1} (u(\tau) - u(a)) d\tau \right] \frac{ds}{s}$$

Since $g(t)$ is nondecreasing, it follows that

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{\alpha-1} \left[\int_a^s \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log \frac{s}{\tau} \right)^{k\alpha-1} u(\tau) d\tau \right] \frac{ds}{s^{1-\mu}}$$

By interchanging the order of integration, we have

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\int_{\tau}^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log \frac{t}{s} \right)^{\alpha-1} \left(\log \frac{s}{\tau} \right)^{k\alpha-1} ds \right] (u(\tau) - u(a)) \frac{d\tau}{\tau}$$

Where $z \in [0, T]$, β is a beta function

$$\begin{aligned}
 B^{k+1}u(t) &\leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\int_{\tau}^t \frac{(\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\left(\log \frac{t}{\tau} \right) (1-z) \right)^{\alpha-1} z \left(\log \frac{t}{\tau} \right)^{k\alpha-1} ds \right] (u(\tau) - u(a)) \frac{d\tau}{\tau} \\
 B^{k+1}u(t) &\leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\int_{\tau}^t \frac{(\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\log \frac{t}{\tau} \right)^{k\alpha-1} (1-z)^{\alpha} z^{k\alpha-1} dz \right] (u(\tau) - u(a)) \frac{d\tau}{\tau} \\
 B^{k+1}u(t) &\leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log \frac{t}{s} \right)^{(k+1)\alpha-1} \frac{\Gamma(\alpha)\Gamma(n\alpha)}{\Gamma((k+1)\alpha)} \right] (u(s) - u(a)) \frac{ds}{s} \\
 B^{k+1}u(t) &\leq \int_a^t \left[\frac{(g(t)\Gamma(\alpha))^{k+1}}{\Gamma(\alpha)\Gamma((k+1)\alpha)} \left(\log \frac{t}{s} \right)^{(k+1)\alpha-1} \right] (u(s) - u(a)) \frac{ds}{s}
 \end{aligned}$$

$$\text{Since } B^n u(t) \leq \int_a^t \left[\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)\Gamma((k+1)\alpha)} \left(\log \frac{t}{s} \right)^{n\alpha-1} \right] (u(s) - u(a)) ds \rightarrow 0, n \rightarrow \infty \quad \forall t \in [0, T)$$

$$u(t) \leq (v(t) - v(a)) \left(\log \frac{t}{a} \right) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)\Gamma(n\alpha)} \left(\log \frac{t}{s} \right)^{n\alpha-1} (v(t) - v(a)) \right] \frac{ds}{s}, \quad 0 \leq t \leq T$$

$$u(t) \leq (v(t) - v(a)) \left(\log \frac{t}{a} \right) \left(1 + \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)\Gamma(n\alpha)} \int_a^t \left[\left(\log \frac{t}{s} \right)^{n\alpha-1} \right] \frac{ds}{s} \right)$$

$$u(t) \leq v(t) - v(a) \left(\log \frac{t}{a} \right) \left(\frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(g(t)\Gamma(\alpha) \left(\log \frac{t}{a} \right)^{\alpha})^n}{\Gamma(n\alpha+1)} \right)$$

$$u(t) \leq (v(t) - v(a)) \left(\log \frac{t}{a} \right) E_{\alpha} \left(g(t) \left(\log \frac{t}{a} \right)^{\alpha} \right).$$

3.9 Lemma

Let $\alpha > 0, \mu > 0$, and $x(t) \in L^1([a, b], R)$, $0 < a < b < \infty$ then $\|({}^{RK}_a D_t^{\alpha, \mu} x(t) - ({}^{RK}_a D_t^{\alpha, \mu} y(t)))\| \leq C_1(t) \|x(t) - y(t)\|$ where $C_1(t) = \frac{\mu^{\alpha}}{\Gamma(1-\alpha)} \left((t^{\mu} - a^{\mu}) \right)$

Proof:

From Definition (2.4), we have that

$$\begin{aligned} {}^{RH}_a D_t^{\alpha, \mu} x(t) &= \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt} \right)^n \int_a^t \left(\log \frac{t}{\tau} \right)^{n-\alpha-1} \frac{x(\tau)}{\tau} d\tau \\ \left(({}^{RH}_a D_t^{\alpha, \mu} x(t) - ({}^{RH}_a D_t^{\alpha, \mu} y(t))) \right) &= \frac{1}{\Gamma(1-\alpha)} \left(t \frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} x(\tau) \frac{d\tau}{\tau} - \frac{1}{\Gamma(1-\alpha)} \left(t \frac{d}{dt} \right) \int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} y(\tau) \frac{d\tau}{\tau} \\ &= \frac{1}{\Gamma(1-\alpha)} \left(t \frac{d}{dt} \right) \left[\int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} (x(\tau) - y(\tau)) \frac{d\tau}{\tau} \right] \end{aligned}$$

Therefore

$$\begin{aligned} \|({}^{RH}_a D_t^{\alpha, \mu} x(t) - ({}^{RH}_a D_t^{\alpha, \mu} y(t)))\| &\leq \frac{1}{\Gamma(1-\alpha)} \left(t \frac{d}{dt} \right) \left[\int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} \|x(\tau) - y(\tau)\| \frac{d\tau}{\tau} \right] \\ \|({}^{RH}_a D_t^{\alpha, \mu} x(t) - ({}^{RH}_a D_t^{\alpha, \mu} y(t)))\| &\leq \|x(t) - y(t)\| \frac{1}{\Gamma(1-\alpha)} \left(t \frac{d}{dt} \right) \left[\int_a^t \left(\log \frac{t}{\tau} \right)^{-\alpha} \frac{d\tau}{\tau} \right] \\ \|({}^{RH}_a D_t^{\alpha, \mu} x(t) - ({}^{RH}_a D_t^{\alpha, \mu} y(t)))\| &\leq \|x(t) - y(t)\| \frac{-1}{t\Gamma(1-\alpha)} \left(\left(\log \frac{t}{a} \right)^{\alpha} \right) \\ \text{set } C_1(t) &= \frac{-1}{t\Gamma(1-\alpha)} \left(\left(\log \frac{t}{a} \right)^{\alpha} \right), \text{ we get} \\ \|({}^{RH}_a D_t^{\alpha, \mu} x(t) - ({}^{RH}_a D_t^{\alpha, \mu} y(t)))\| &\leq C_1(t) \|x(t) - y(t)\| \end{aligned} \quad (3)$$

3.10 Lemma

Let $\alpha > 0, \mu > 0$, and $x(t) \in L^1([a, b], R)$, $0 < a < b < \infty$
 $\|({}^{RK}_a D_t^{-\alpha, \mu} x(t) - ({}^{RK}_a D_t^{-\alpha, \mu} y(t)))\| \leq C_2(t) \|x(t) - y(t)\|$ where $C_2(t) = \frac{1}{\mu\alpha\Gamma(\alpha)} \left(\frac{t^{\mu} - a^{\mu}}{\mu} \right)$

Proof:

From Definition (2.2), we have that

$$\begin{aligned} {}^{RK}_a D_t^{-\alpha, \mu} x(t) - {}^{RK}_a D_t^{-\alpha, \mu} y(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^{\mu} - \tau^{\mu}}{\mu} \right)^{\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\mu}} - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^{\mu} - \tau^{\mu}}{\mu} \right)^{\alpha-1} y(\tau) \frac{d\tau}{\tau^{1-\mu}} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^{\mu} - \tau^{\mu}}{\mu} \right)^{\alpha-1} (x(\tau) - y(\tau)) \frac{d\tau}{\tau^{1-\mu}} \end{aligned}$$

Therefore

$$\begin{aligned} \|({}^{RK}_a D_t^{-\alpha, \mu} x(t) - ({}^{RK}_a D_t^{-\alpha, \mu} y(t)))\| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^{\mu} - \tau^{\mu}}{\mu} \right)^{\alpha-1} \|x(\tau) - y(\tau)\| \frac{d\tau}{\tau^{1-\mu}} \\ \|({}^{RK}_a D_t^{-\alpha, \mu} x(t) - ({}^{RK}_a D_t^{-\alpha, \mu} y(t)))\| &\leq \|x(t) - y(t)\| \frac{-1}{\mu\alpha\Gamma(\alpha)} \left(-\frac{t^{\mu} - a^{\mu}}{\mu} \right) \\ \text{set } C_2(t) &= \frac{1}{\mu\alpha\Gamma(\alpha)} \left(\frac{t^{\mu} - a^{\mu}}{\mu} \right) \text{ we get} \\ \|({}^{RK}_a D_t^{-\alpha, \mu} x(t) - ({}^{RK}_a D_t^{-\alpha, \mu} y(t)))\| &\leq C_2(t) \|x(t) - y(t)\| \end{aligned} \quad (4)$$

3.3 Theorem

Let $x: [-\tau, T] \rightarrow R^m$ be continuous differential function, $\tau > 0$ then $x(t)$ is a solution of the Caputo –Katugampola fractional order nonlinear differential control nonlocal system (1.1), if and only if

$$x(t) = \begin{cases} {}^{CH}_a D_t^{-\alpha, \mu} \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) \right) + B_2(t)u(t) \right) \\ \varphi(t) & \text{for } -\tau \leq t \leq 0 \end{cases} \quad (5)$$

Proof

For $-\tau \leq t \leq 0$, we have the solution is $x(t) = \varphi(t)$, now from lemma (3.7), we have that

$$, \quad x(t) = {}^{CH}_a D_t^{-\alpha, \mu} \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) \right) + B_2(t)u(t) \right) \quad 0 \leq t \leq T, \text{ implies that}$$

$$, \quad {}^{CH}_a D_t^{\alpha, \mu} (x(t)) = {}^{CH}_a D_t^{\alpha, \mu} \left({}^{CH}_a D_t^{-\alpha, \mu} \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) \right) + B_2(t)u(t) \right) \right) \text{ using lemma (3.7), we obtain}$$

$${}^{CH}_a D_t^{\alpha, \mu} (x(t)) = \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) \right) + B_2(t)u(t) \right)$$

The other side of proving which given by:

$${}^{CH}_a D_t^{\alpha, \mu} x(t) = A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) \right) + B_2(t)u(t)$$

$${}^{CH}_a D_t^{-\alpha, \mu} \left({}^{CH}_a D_t^{\alpha, \mu} x(t) \right) = {}^{CH}_a D_t^{-\alpha, \mu} \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) \right) + B_2(t)u(t) \right), \text{ therefore}$$

$$x(t) = {}^{CH}_a D_t^{-\alpha, \mu} \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) \right) + B_2(t)u(t) \right),$$

3.4 Theorem

Consider the following Caputo –Katugampola fractional order nonlinear differential control nonlocal system and ${}^{CH}_a D_t^{\alpha, \mu} x(t) = A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) \right) + B_2(t)u(t)$ with $(f(t, 0, 0) = [0, \dots, 0]^T)$ has a unique continuous solution

Proof

$$x(t) - y(t) = {}^{RH}_a D_t^{-\alpha, \mu} (-C(x(t) - y(t)) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) - F \left({}^{RH}_a D_t^{\alpha, \mu} y(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t - \tau) \right) - G \left({}^{RK}_a D_t^{\alpha, \mu} y(t - \tau) \right) \right))$$

$$\|x(t) - y(t)\| \leq \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(-C(x(\tau) - y(\tau)) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(\tau) \right) - F \left({}^{RH}_a D_t^{\alpha, \mu} y(\tau) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(\tau - s) \right) - G \left({}^{RK}_a D_t^{\alpha, \mu} y(\tau - s) \right) \right) - (x(a) + y(a)) \right) \frac{d\tau}{\tau} \right\|$$

$$\|e(t)\| \leq C(\|e(t) - e(a)\|) \left\| \frac{1}{\Gamma(\alpha)} \frac{\left(\log \frac{t}{\tau} \right)^\alpha}{\alpha} \right\| + \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} e(t) \right) \right) + B_1(s) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} e(t - s) \right) \right) \right) \frac{e(\tau) - e(a)}{\tau} d\tau \right\|$$

$$\|e(t)\| \leq C(\|e(t) - e(a)\|) \left\| \frac{1}{\alpha \Gamma(\alpha)} \left(\log \frac{t}{a} \right)^\alpha \right\| + \left\| A_2(t) \left(L_1 C_1(t)(e(t)) \right) + B_1(t) \left(L_2 G \left(\sup_{\vartheta \in [-\tau, 0]} e(t + \vartheta) \right) \right) \right\| \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{e(\tau) - e(a)}{\tau} d\tau \right\|$$

$$e(t) = \sup_{t \in [-\tau, 0]} e(t), g(t) = C(\|e(t) - e(a)\|) \left\| \frac{1}{\alpha \Gamma(\alpha)} \left(\log \frac{t}{a} \right)^\alpha \right\| + \left\| A_2(t) \left(L_1 C_1(t)(e(t)) \right) + B_1(t) \left(L_2 G \left(\sup_{\vartheta \in [-\tau, 0]} e(t + \vartheta) \right) \right) \right\|$$

$$\|e(t)\| \leq 0 + g(t) \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\sup_{\vartheta \in [-\tau, 0]} e(t + \vartheta) \right) \frac{e(\tau) - e(a)}{\tau} d\tau \right\|$$

$$\|e(t)\| \leq 0E_\alpha \left(g(t) \left(\log \frac{t}{\tau} \right)^\alpha \right)$$

$$\|e(t)\| \leq 0$$

Then system (1) has a unique continuous solution.

3.1 Stability of the Caputa–Katugampola and Riemann–Katugampola Fractional order nonlinear differential control nonlocal system with maximal interval (0,T]

3.1.5 Theorem

The solution to the Caputo–Hadamard fractional derivatives Riemann–Katugampola and Hadmmard fractional order nonlinear differential control nonlocal system (1) is $x(t) \in R^n$ Then the following inequalities holds:

$$\|x(t)\| \leq (\delta(t) - \delta(a))E_\alpha \left(\omega(t) \left(\log \frac{t}{a} \right)^\alpha \right) \quad (6)$$

Proof

Since $x(t)$ has the following formulation,

$$x(t) = \begin{cases} {}^{CH}_a D_t^{-\alpha, \mu} \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t-\tau) \right) \right) + B_2(t)u(t) \right) & \text{Thus,} \\ \varphi & t \in [-\tau, 0] \end{cases}$$

for $0 \leq t \leq T$, we have

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\frac{\left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t-\tau) \right) \right) + B_2(t)u(t) \right) - x(a)}{\tau} \right) d\tau \\ \|x(t)\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left(\frac{\left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t-\tau) \right) \right) + B_2(t)u(t) \right) - x(a)}{\tau} \right) d\tau \right\| \\ \|x(t)\| &= \frac{1}{\Gamma(\alpha)} \left\| \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) - F(0) \right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t-\tau) \right) - G(0) \right) + \right. \right. \\ &\quad \left. \left. B_2(t)u(t) \right) - x(a) \right\| \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{d\tau}{\tau} \\ \|x(t)\| &\leq \left(\|A_1(t)\| - C\|x(t)\| + A_2(t) \left\| F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) - F(0) \right\| + B_1(t) \left\| G \left({}^{RK}_a D_t^{\alpha, \mu} x(t-\tau) \right) \right\| + \right. \\ &\quad \left. \|B_2(t)u(t)\| - \|x(a)\| \right) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{d\tau}{\tau} \\ \|x(t)\| &\leq (\|A_1(t)\| - C\|x(t)\| + A_2(t) \|L_1(C_1(t)x(t))\| + B_1(t) \|L_2(C_2(t)x(t-\tau))\| + \|B_2(t)u(t)\| - \|x(a)\|) \\ &\quad \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{d\tau}{\tau} \end{aligned}$$

Where $x(t) = \sup_{\vartheta \in [-\tau, 0]} x(t + \vartheta)$

$$\|x(t)\| \leq (C\|x(t)\| + A_2(t) \|L_1(C_1(t)x(t))\| + B_1(t) \|L_2(C_2(t)x(t))\| - \|x(a)\|) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{d\tau}{\tau} + (\|A_1(t)\| + A_2(t) + \|B_2(t)u(t)\|) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{d\tau}{\tau}$$

$$\|x(t)\| \leq (C + A_2(t) \|L_1(C_1(t))\| + B_1(t) \|L_2(C_2(t))\|) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} x(t) - x(a) \frac{d\tau}{\tau} + \frac{1}{\alpha\Gamma(\alpha)} \left(\log \frac{t}{a} \right)^\alpha (\|A_1(t)\| + A_2(t) + \|B_2(t)u(t)\|)$$

Set $\delta(t) = \frac{1}{\alpha\Gamma(\alpha)} \left(\log \frac{t}{a} \right)^\alpha (\|A_1(t)\| + A_2(t) + \|B_2(t)u(t)\|)$, we have that

$\omega(t) = (C + A_2(t) \|L_1(C_1(t))\| + B_1(t) \|L_2(C_2(t))\|)$, implies that

$$\|x(t)\| \leq \delta(t) + \omega(t) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \left\| \sup_{\vartheta \in [-\tau, 0]} x(t + \vartheta) - x(a) \right\| \frac{d\tau}{\tau}$$

$$\|x(t)\| \leq \delta(t) + \omega(t) \left({}^{CH}_a D_t^{-\alpha, \mu} \left\| \sup_{\vartheta \in [-\tau, 0]} x(t + \vartheta) - x(a) \right\| \right) \text{ By using lemma (3.8)}$$

$$\|x(t)\| \leq (\delta(t) - \delta(a))E_\alpha \left(\omega(t) \left(\log \frac{t}{a} \right)^\alpha \right).$$

3.2 Stability of the Caputa– Hadmmard and Riemann –Katugampola and Hadmmard Fractional order nonlinear differential control nonlocal system with by using step method.

Theorem (3.2.6)

Assume that Caputo –Katugampola fractional order nonlinear differential control nonlocal system (1) satisfy the following conditions lemma (3.4). Then the solution of (1) is finite-time stable, if the following conditions is satisfied:

$$\delta_T(\tau)E_\alpha\sigma_0(T)\left(\frac{T^\mu}{\mu}\right)^\alpha \leq \varepsilon \text{ where}$$

$$\omega_T(\tau) = \delta_1(T) + \sigma_1(T)[\|\varphi\|_c \left(\log \frac{\tau}{a}\right)^\alpha] + \sigma_1(T)[\sum_{j=1}^n \omega_j(\tau)E_\alpha\sigma_0(j\tau) \left(\log \frac{j\tau}{a}\right)^\alpha] \frac{\left(\log \frac{\tau}{a}\right)^\alpha}{\Gamma(1+\alpha)} +$$

$$\left[\omega_{n+1}(\tau)E_\alpha\sigma_0((n+1)\tau) \left(\log \frac{(n+1)\tau}{a}\right)^\alpha\right] \frac{\left(\log \frac{\tau}{(n+1)\tau}\right)^\alpha}{\Gamma(1+\alpha)}.$$

$$\omega((i+1)\tau) = \delta_1((i+1)\tau) + \sigma_1((i+1)\tau) \left[\|\varphi\|_c \left(\log \frac{\tau}{a}\right)^\alpha\right] + \sigma_1((i+1)\tau)[\sum_{j=1}^i \sigma_{j+1}(\tau)E_\alpha\sigma_0(j\tau)$$

$$\left(\log \frac{j\tau}{a}\right)^\alpha] \frac{\left(\log \frac{(i+1)\tau}{a}\right)^\alpha}{\Gamma(1+\alpha)}.$$

$$\delta_1(t) = \omega_1(t) = (\delta_1(t) + \sigma_1\|\varphi\|_c)$$

Proof

From formula (5)

$$x(t) = {}^{CH}_a D_t^{-\alpha,\mu}(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha,\mu} x(t)\right)\right) + B_1(t) \left(G \left({}^{RK}_a D_t^{\alpha,\mu} x(t - \tau)\right)\right) + B_2(t)u(t)).$$

$$x(t) = {}^{CH}_a D_t^{-\alpha,\mu} \left(A_1(t) - Cx(t) + A_2(t) \left(F \left({}^{RH}_a D_t^{\alpha,\mu} x(t)\right)\right) + \right) {}^{CH}_a D_t^{-\alpha,\mu}(A_1(t) + B_1(t)$$

$$\left(G \left({}^{RK}_a D_t^{\alpha,\mu} x(t - \tau)\right)\right) + B_2(t)u(t))$$

$$\|x(t)\| \leq {}^{CH}_a D_t^{-\alpha,\mu}(\|A_1(t)\| + L_2\|B_1(t)\| \|C_2(t)x(t - \tau)\| + \|B_2(t)u(t)\| - C\|x(t)\| + L_1\|A_2(t)\|$$

$$\|(C_1(t)x(t))\|)$$

$$\|x(t)\| \leq {}^{CH}_a D_t^{-\alpha,\mu}(\|A_1(t)\| + L_2\|B_1(t)\| \|C_2(t)\|\varphi\|_c) + \|B_2(t)u(t)\| - C\|x(t)\| + L_1\|A_2(t)\|$$

$$\|(C_1(t)x(t))\|)$$

$$\|x(t)\| \leq {}^{CH}_a D_t^{-\alpha,\mu}(\delta_1(t) - C + L_1\|A_2(t)(C_1(t))\|) \text{ where } \sigma_0 = L_1\|A_2(t)(C_1(t))\| - C$$

$$\delta_1(t) = {}^{CH}_a D_t^{-\alpha,\mu}(\|A_1(t)\| + \|B_2(t)u(t)\|), \sigma_1 = L_2\|B_1(t)\| \|C_2(t)\|$$

$$\|x(t)\| \leq \delta_1(t) + \sigma_1\|\varphi\|_c + {}^{CH}_a D_t^{-\alpha,\mu}(L_1\|A_2(t)(C_1(t))\| - C)(\|x(t)\|)$$

$$\|x(t)\| \leq \delta_1(t) + \sigma_1\|\varphi\|_c + \sigma_0(t) {}^{CH}_a D_t^{-\alpha,\mu}(\|x(t)\|)$$

Using the Lemma (3.8) for $t \in [0, \tau]$, we have that

$$\|x(t)\| \leq \delta_1(t) + \sigma_1\|\varphi\|_c + \sigma_0(t) {}^{CH}_a D_t^{-\alpha,\mu} x(t) \text{ for } t \in [0, \tau], \text{ we have that}$$

$$\|x(t)\| \leq (\delta_1(t) + \sigma_1\|\varphi\|_c) E_\alpha(\sigma_0(\tau) \left(\log \frac{t}{a}\right)^\alpha)$$

$$\omega_1(t) = (\delta_1(t) + \sigma_1\|\varphi\|_c)$$

$$\|x(t)\| \leq \omega_1(\tau) E_\alpha(\sigma_0(\tau) \left(\log \frac{\tau}{a}\right)^\alpha)$$

For $t \in (i\tau, (i+1)\tau], 1 \leq i \leq n$, we have

$$\|x(t)\| \leq {}^{CH}_a D_t^{-\alpha,\mu}(\|A_1(t)\| + L_2\|B_1(t)\| \|C_2(t)x(t - \tau)\| + \|B_2(t)u(t)\| - C\|x(t)\| + L_1\|A_2(t)\|$$

$$\|(C_1(t)x(t))\|)$$

$$\|x(t)\| \leq \delta_1(t) + \sigma_0(t) {}^{CH}_a D_t^{-\alpha,\mu} x(t) + \sigma_1(t) \left[\frac{1}{\Gamma(\alpha)} \int_a^\tau \left(\log \frac{t}{s}\right)^{\alpha-1} \|x(s - \tau) - x(a)\| \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_\tau^{2\tau} \left(\log \frac{t}{s}\right)^{\alpha-1} \|x(s - \tau) -$$

$$x(a)\| \frac{ds}{s} + \dots + \frac{1}{\Gamma(\alpha)} \int_{i\tau}^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|x(s - \tau) - x(a)\| \frac{ds}{s} \right]$$

$$\|x(t)\| \leq \delta_1(t) + \sigma_0(t) {}^{CH}_a D_t^{-\alpha,\mu} x(t) + \sigma_1(t) \left[\left(\sigma_2(\tau) E_\alpha(\sigma_0(\tau) \left(\log \frac{\tau}{a}\right)^\alpha)\right) \left[\frac{\left(\log \frac{t}{a}\right)^\alpha - \left(\log \frac{\tau}{a}\right)^\alpha}{\Gamma(1+\alpha)}\right] \right]$$

$$+ \left[\sigma_2(\tau) E_\alpha(\sigma_0(\tau) \left(\log \frac{\tau}{a}\right)^\alpha)\right] \left[\frac{\left(\log \frac{t}{\tau}\right)^\alpha - \left(\log \frac{\tau}{2\tau}\right)^\alpha}{\Gamma(1+\alpha)}\right] + \dots + \left[\sigma_{i+1}(\tau) E_\alpha(\sigma_0(i\tau) \left(\log \frac{i\tau}{a}\right)^\alpha)\right] \left[\frac{\left(\log \frac{t}{i\tau}\right)^\alpha - \left(\log \frac{i\tau}{i\tau}\right)^\alpha}{\Gamma(1+\alpha)}\right]$$

$$\|x(t)\| \leq \delta_1((i+1)\tau) + \sigma_1((i+1)\tau) [\|\varphi\|_c \left(\log \frac{\tau}{a}\right)^\alpha] + \sigma_1((i+1)\tau) [\sum_{j=1}^i \sigma_{j+1}(\tau)E_\alpha\sigma_0(j\tau)$$

$$\begin{aligned}
 & \left(\log \frac{j\tau}{a} \right)^\alpha \left[\frac{\left(\log \frac{(i+1)\tau}{a} \right)^\alpha}{\Gamma(1+\alpha)} + \sigma_0((i+1)\tau) {}^{CH}_a D_t^{-\alpha, \mu} x(t) \right. \\
 & \left. \omega((i+1)\tau) = \delta_1((i+1)\tau) + \sigma_1((i+1)\tau) \left[\|\varphi\|_c \left(\log \frac{\tau}{a} \right)^\alpha \right] + \sigma_1((i+1)\tau) \left[\sum_{j=1}^i \sigma_{j+1}(\tau) E_\alpha \sigma_0(j\tau) \right. \right. \\
 & \left. \left. \left(\log \frac{j\tau}{a} \right)^\alpha \right] \frac{\left(\log \frac{(i+1)\tau}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right. \\
 & \left. \|x(t)\| \leq \omega((i+1)\tau) + \sigma_0((i+1)\tau) {}^{CH}_a D_t^{-\alpha, \mu} x(t) \right. \\
 & \text{Using lemma (3.8), we obtain that for } t \in (i\tau, (i+1)\tau], \text{ implies that} \\
 & \|x(t)\| \leq \omega_{i+1}(\tau) E_\alpha \sigma_0((i+1)\tau) \left(\log \frac{\tau}{a} \right)^\alpha \\
 & \|x(t)\| \leq \omega_{i+1}(\tau) E_\alpha \sigma_0((i+1)\tau) \left(\log \frac{(i+1)\tau}{a} \right)^\alpha \\
 & \text{Finally, for } t \in ((n+1)\tau, T], 1 \leq i \leq n, \text{ we get that} \\
 & \|x(t)\| \leq {}^{CH}_a D_t^{-\alpha, \mu} (\|A_1(t)\| + L_2 \|B_1(t)\| \|C_2(t)x(t-\tau)\| + \|B_2(t)u(t)\| - C\|x(t)\| + L_1 \|A_2(t)\| \\
 & \|C_1(t)x(t)\|) \\
 & \|x(t)\| \leq {}^{CH}_a D_t^{-\alpha, \mu} (\|A_1(t)\| + \|B_2(t)u(t)\| + L_2 \|B_1(t)\| \|C_2(t)x(t-\tau)\| - C\|x(t)\| + L_1 \|A_2(t)\| \\
 & \|C_1(t)x(t)\|) \\
 & \sigma_0(t) = L_1 \|A_2(t)(C_1(t))\| - C, \sigma_1(t) = L_2 \|B_1(t)\| \|C_2(t)\|, \delta_1(t) = {}^{CH}_a D_t^{-\alpha, \mu} (\|A_1(t)\| + \|B_2(t)u(t)\|) \\
 & \|x(t)\| \leq \delta_1(t) + \sigma_0(t) {}^{CH}_a D_t^{-\alpha, \mu} x(t) + \sigma_1(t) {}^{CH}_a D_t^{-\alpha, \mu} \|x(t-\tau)\| \\
 & + [\omega_1(\tau) E_\alpha \sigma_0(\tau) \left(\log \frac{\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{t}{a} \right)^\alpha - \left(\log \frac{\tau}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right] + \dots + [\omega_n(\tau) E_\alpha \sigma_0(n\tau) \left(\log \frac{n\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{t}{a} \right)^\alpha - \left(\log \frac{n\tau}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right] + \\
 & [\omega_{n+1}(\tau) E_\alpha \sigma_0((n+1)\tau) \left(\log \frac{(n+1)\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{t}{a} \right)^\alpha - \left(\log \frac{(n+1)\tau}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right] \\
 & \|x(t)\| \leq \delta_1(t) + \sigma_0(t) {}^{CH}_a D_t^{-\alpha, \mu} x(t) + \sigma_1(t) \left[\|\vartheta\|_c \left(\log \frac{\tau}{a} \right)^\alpha \right] + [\omega_1(\tau) E_\alpha \sigma_0(\tau) \left(\log \frac{\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{t}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right] + \\
 & \dots + [\omega_n(\tau) E_\alpha \sigma_0(n\tau) \left(\log \frac{n\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{t}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right] + [\omega_{n+1}(\tau) E_\alpha \sigma_0((n+1)\tau) \left(\log \frac{(n+1)\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{t}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right] \\
 & \|x(t)\| \leq \delta_1((n+1)\tau) + \sigma_1((n+1)\tau) \left[\|\varphi\|_c \left(\log \frac{\tau}{a} \right)^\alpha \right] + \sigma_1((n+1)\tau) \left[\sum_{j=1}^n \omega_j(\tau) E_\alpha \sigma_0(j\tau) \right. \\
 & \left. \left(\log \frac{j\tau}{a} \right)^\alpha \right] \frac{\left(\log \frac{(n+1)\tau}{a} \right)^\alpha}{\Gamma(1+\alpha)} + [\omega_{n+1}(\tau) E_\alpha \sigma_0((n+1)\tau) \left(\log \frac{(n+1)\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{T}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right] + \sigma_0((n+1)\tau) {}^{CH}_a D_t^{-\alpha, \mu} x(t) \\
 & \text{Where} \\
 & \omega((n+1)\tau) = \delta_1(T) + \sigma_1(T) \left[\|\varphi\|_c \left(\log \frac{\tau}{a} \right)^\alpha \right] + \sigma_1(T) \left[\sum_{j=1}^n \omega_j(\tau) E_\alpha \sigma_0(j\tau) \left(\log \frac{j\tau}{a} \right)^\alpha \right] \frac{\left(\log \frac{T}{a} \right)^\alpha}{\Gamma(1+\alpha)} + \\
 & [\omega_{n+1}(\tau) E_\alpha \sigma_0((n+1)\tau) \left(\log \frac{(n+1)\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{T}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right] \\
 & \|x(t)\| \leq \omega((i+1)\tau) + \sigma_0((i+1)\tau) {}^{CH}_a D_t^{-\alpha, \mu} x(t) \\
 & \text{Using lemma (3.8), we obtain that for } t \in ((n+1)\tau, T], \text{ implies that} \\
 & \|x(t)\| \leq \omega_T(\tau) E_\alpha \sigma_0(T) \left(\log \frac{T}{a} \right)^\alpha
 \end{aligned}$$

3.1 Remark

The Caputo–Hadamard fractional order nonlinear differential control nonlocal system (1) is finite-time stable if satisfy the following condition $\omega_T(\tau) E_\alpha \sigma_0(T) \left(\log \frac{T}{a} \right)^\alpha \leq \varepsilon$, where $\omega((n+1)\tau) = \delta_1(T) + \sigma_1(T) \left[\|\varphi\|_c \left(\log \frac{\tau}{a} \right)^\alpha \right] + \sigma_1(T) \left[\sum_{j=1}^n \omega_j(\tau) E_\alpha \sigma_0(j\tau) \left(\log \frac{j\tau}{a} \right)^\alpha \right] \frac{\left(\log \frac{T}{a} \right)^\alpha}{\Gamma(1+\alpha)} + [\omega_{n+1}(\tau) E_\alpha \sigma_0((n+1)\tau) \left(\log \frac{(n+1)\tau}{a} \right)^\alpha] \left[\frac{\left(\log \frac{T}{a} \right)^\alpha}{\Gamma(1+\alpha)} \right]$. If $T \in (0, \tau]$.

4. ILLUSTRATIVE EXAMPLE

In this section, we provide an example to illustrate the finite time stability of the Caputo–Katugampola fractional order nonlinear differential control nonlocal system (1).

4.1 Example

Consider the following the Caputo –Hadamard fractional order nonlinear differential control nonlocal system (1.1),

$${}^{CH}_a D_t^{\alpha, \mu} x(t) = \begin{pmatrix} -0.1 & 0.05 \\ 0.05 & 0.1 \end{pmatrix} x(t) + \begin{pmatrix} -0.7 \\ -0.6 \end{pmatrix} x(t) + \begin{pmatrix} -0.1 \\ 0.05 \end{pmatrix} \left(F \left({}^{RH}_a D_t^{\alpha, \mu} x(t) \right) \right) + \begin{pmatrix} 0.05 \\ 0.1 \end{pmatrix} \left(G \left({}^{RK}_a D_t^{\alpha, \mu} x(t-0.1) \right) \right) + \begin{pmatrix} -0.1 \\ 0.2 \end{pmatrix} f(x(t-0.1)) + \begin{pmatrix} -0.1 & 0.2 \\ 0.2 & 0.1 \end{pmatrix} u(t).$$

$$f(x_j(t)) = f_j(x_j(t)) = g_j(x_j(t)) = 0.5(|x+1| - |x-1|), j = 1, 2, u(t) = 1$$

$$A_1 = \begin{pmatrix} -0.1 & 0.05 \\ 0.05 & 0.1 \end{pmatrix}, B_1 = \begin{pmatrix} -0.1 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}, A_2 = \begin{pmatrix} 0.3 & 0.02 \\ 0.02 & 0.1 \end{pmatrix}, B_2 = \begin{pmatrix} -0.4 & 0.2 \\ 0.2 & -0.4 \end{pmatrix}$$

$$C = 0.7, L_j = G_j = 1, \|A_1(t)\| = 0.1118, B_1(t) = 0.2236, \|A_2(t)\| = 0.3020, B_2(t) = 0.4200$$

By theorem 3.2.5 $(\delta(t) - \delta(a))E_\alpha \left(\omega(t) \left(\log \frac{t}{a} \right)^\alpha \right)$ we get

Table1. The value of ε , for $a = 0.1, \alpha = 0.1$

t	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = \alpha$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$
0.2	0.1343	0.0833	0.0675	0.0602	0.0561	0.0536	0.0519	0.0507	0.0499
0.4	0.4913	0.2711	0.2019	0.1696	0.1516	0.1403	0.1327	0.1272	0.1232
0.6	0.8181	0.4395	0.3201	0.2641	0.2328	0.2133	0.2001	0.1908	0.1838
0.8	1.1078	0.5894	0.4255	0.3488	0.3059	0.2792	0.2615	0.2490	0.2398
1	1.3676	0.7248	0.5214	0.4263	0.3733	0.3406	0.3192	0.3043	0.2936

Table 2. The value of ε , for $a = 0.1, \alpha = 0.5$

t	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = \alpha$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$
0.2	3.1601	1.9219	1.5372	1.3612	1.2655	1.2076	1.1700	1.1440	1.1251
0.4	8.2470	4.4626	3.2727	2.7216	2.4185	2.2340	2.1132	2.0297	1.9689
0.6	12.1112	6.3846	4.5750	3.7331	3.2690	2.9868	2.8034	2.6779	2.5882
0.8	15.2442	7.9618	5.6545	4.5794	3.9874	3.6294	3.3994	3.2452	3.1383
1	17.8985	9.3147	6.5915	5.3224	4.6256	4.2075	3.9430	3.7704	3.6558

Table 3. The value of ε , for $a = 0.1, \alpha = 0.9$

t	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = \alpha$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$
0.2	3.2368	1.9980	1.6130	1.4377	1.3437	1.2880	1.2527	1.2290	1.2122
0.4	11.0164	6.0190	4.4458	3.7206	3.3269	3.0927	2.9445	2.8461	2.7779
0.6	17.8665	9.4856	6.8324	5.6030	4.9343	4.5379	4.2903	4.1302	4.0238
0.8	23.8278	12.5140	8.9216	7.2535	6.3472	5.8139	5.4869	5.2829	5.1561
1	29.1097	15.2177	10.7988	8.7460	7.6338	6.9856	6.5968	6.3652	6.2344

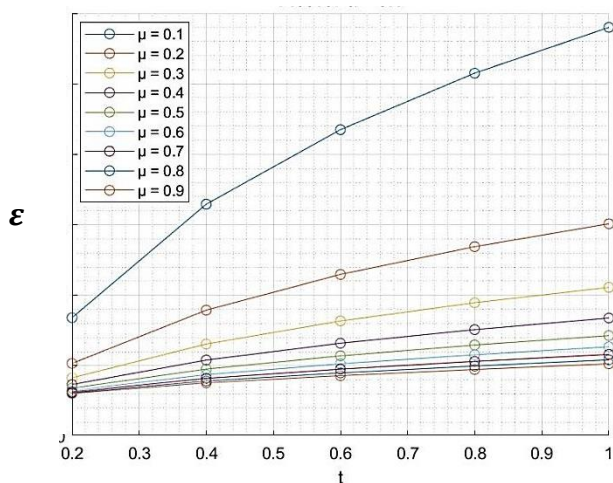


Figure 1. of table1.

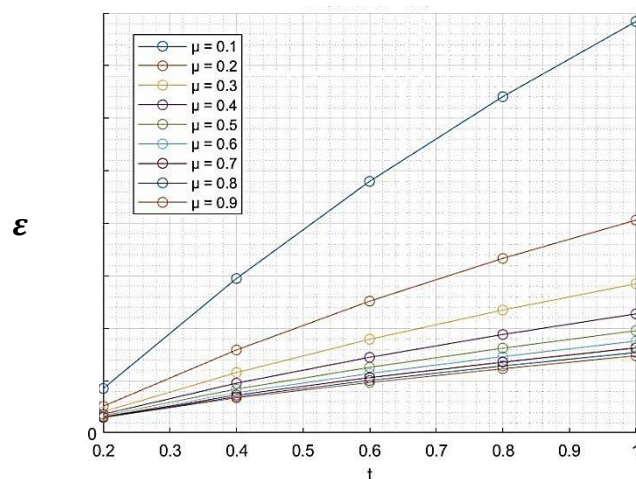


Figure 2. of table 2.

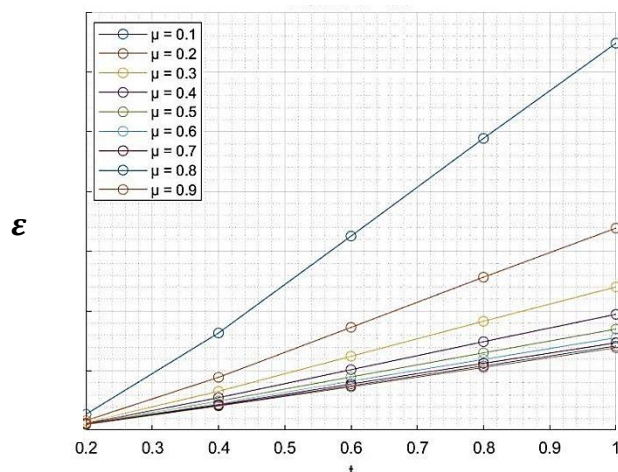


Figure 3. of table3.

5. CONCLOTION

1. The Caputa Caputa– Hadmmard Fractional order nonlinear differential control nonlocal system (1) was complex for studding since the fractional derivative types has expression are difficult constriction.
2. The uniqueness, existence, Necessary, and sufficient condition depended on Generalized Gronwall Inequality of the Riemann – Caputa fractional derivative was presented first time and make good role in stability of the system.
3. The stability of finite time for our presented system was depended on maximal interval or on step size of maximal interval to obtain the guarantee estimation of epsilon

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