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Solving Some types of Linear Systems of Partial Differential Equations by Using Albazy Altememe Transformation

Tuqa Haider Kareem[1]*, Ali Hassan Mohammed[2]

[1,2]Mathematics Department, Education for Girls Faculty, Kufa University, Najaf, IRAQ

*Corresponding Author: Tuqa Haider Kareem

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ABSTRACT: In this work we will use a novel approach to solving linear partial differential equations (PDEs) utilizing the transformation ("Albazy Altememe Integral Transform"), which is defined by the integral:

$$\text{"}HA[f(x)]=\frac{(-1)^n}{n!}\int_0^1 (\ln x)^n\ f(x)\,dx;\ n\in z^+.\text{"} \qquad \dots (1)$$

Also presenting the characteristics, theorems, and transformations of some functions such as (constant functions, logarithm functions and others. finding the way to use it in solving PDEs
.

Keywords: partial differential equations, integral transformation, Albazy Altememe Integral Transform, linear systems, partial derivatives, ordinary differential equations

1. INTRODUCTION

A lot of integral transformations have appeared in recent years, the researcher Ali Hassan Mohammad has encountered integral changes, such as the Al-Zughair transformation [3] and the AL-Tememe transformation [2] in addition to Batoor Al-Tememe and Batoor Al-Zaghair's changes, as well as the growth of Al-Zughair [4] and Al-Zughair metamorphosis [5], Kuffi Al-Zughair and Kuffi Al-Tememe [6], as well as the transformation we'll employ in this work, Albazy Altememe Integral Transform. All these works are used to solve many types of ordinary and PDEs, beside integral equations.

This research will present Albazy Altememe Integral Transform [7] which appeared at 2023 A.D. as a way for solving some types of PDEs, this transformation formulated (1). We will make changes in the parameters of above formula to use it in our work and it will be:

$$\mathcal{H}\AA[u(x,\ln t)]=\frac{(-1)^n}{n!}\int_0^1 (\ln t)^n\ u(x,\ln t)\,dt\,;\ n\in z^+$$

*Corresponding author: taqih.almoussawi@student.uokufa.edu.iq

Proposition 2.1

The functions $u_1(x, \ln t)$ and $u_2(x, \ln t)$ are defined where $t \in (0,1]$, and A , B are constants, then:

$$\mathcal{H}\text{Å}[A\, u_1(x, \ln t) \pm B\ u_2(x, \ln t)] = A\, \mathcal{H}\text{Å}\,[u_1(x, \ln t)\,] \pm B\, \mathcal{H}\text{Å}\,[u_2(x, \ln t)\,]\,,$$

Proof:

$$\mathcal{H}\text{Å}[A\, u_1(x, \ln t) \pm B\, u_2(x, \ln t)\,] = \frac{(-1)^n}{n!}[\int_0^1 (A\,(\ln t)^n\, u_1(x, \ln t) \pm B\,(\ln t)^n\, u_2(x, \ln t))dt],$$

$$= \frac{(-1)^n}{n!}[\int_0^1 A\,(\ln t)^n\ u_1(x, \ln t)dt \pm \int_0^1 B\,(\ln t)^n\, u_2(x, \ln t)dt]\,,$$

$$= A\frac{(-1)^n}{n!}\int_0^1 (\ln t)^n\ u_1(x, \ln t)dt \pm B\frac{(-1)^n}{n!}\int_0^1 (\ln t)^n\ u_2(x, \ln t)dt\ ,$$

$$= A\, \mathcal{H}\text{Å}\,[u_1(x, \ln t)\,] \pm B\, \mathcal{H}\text{Å}\,[u_2(x, \ln t)\,]\,,$$

Theorem 2.2 [7]

Let $f(x)$ be a function, our conversion for some basic functions are listed in the table below:

$f(x)$	$\mathcal{H}\text{Å}[f(x)] = \frac{(-1)^n}{n!}\int_0^1 (\ln x)^n\ f(x)dx;\ n \in\ z^+$	
1	1	
$(\ln t)$	$-(n+1)$	
$(\ln t)^{-1}$	$-\frac{1}{n}$	
$(\ln t)^a$	$\frac{(-1)^{n+a}}{n!}(n+a)!$	$a \in\ z^+$
$(\ln t)^{-a}$	$\frac{(-1)^{n-a}}{n!}(n-a)!$	$a \in\ z^+$
$\sinh \ln \ln t$	$\frac{-(n+1)}{2} + \frac{1}{2n}$	
$\cosh \ln \ln t$	$\frac{-(n+1)}{2} - \frac{1}{2n}$	
$\sin\mathcal{H}\text{Å} \ln \ln t$	$\frac{(-1)^a}{2n!}(n+a)! - \frac{(-1)^{-a}}{2n!}(n-a)!$	$a \in\ z^+$
$\cos\mathcal{H}\text{Å} \ln \ln t$	$\frac{(-1)^a}{2n!}(n+a)! + \frac{(-1)^{-a}}{2n!}(n-a)!$	$a \in\ z^+$
t	$\frac{1}{2^{n+1}}$	
t^2	$\frac{1}{3^{n+1}}$	
t^a	$\frac{1}{(a+1)^{n+1}}$	$a \in\ z^+$
$t^{\frac{1}{a}}$	$\frac{(a)^{n+1}}{(a+1)^{n+1}}$	$a \in\ z^+$
$t^{\frac{a}{b}}$	$\frac{(b)^{n+1}}{(a+b)^{n+1}}$	$a\&b \in z^+$

3. MAIN RESULTS

In this section, the linear systems of first-order PDEs with variable coefficients—the Albazy Altememe equation—will be solved using the Albazy Altememe transformation.

The linear systems of PDEs-of 1^{st} order with variable coefficients, will be considered as following:

$$(\ln t, \text{x̧})\ \frac{u_{1_t}(\ln t, \text{x̧})}{t} = a_{11}\frac{u_1(\ln t, \text{x̧})}{t} + a_{12}\frac{u_2(\ln t, \text{x̧})}{t} + f_1(\ln t, \text{x̧})\ ,$$

$$(\ln t, x)\ \frac{u_{2_t}(\ln t, x)}{t} = a_{21}\ \frac{u_1(\ln t, x)}{t} + a_{22}\frac{u_2(\ln t, x)}{t} + f_2(\ln t, x) \qquad ...(3.1)$$

where a_{11}, a_{12}, a_{21} and a_{22} are constants, $u_{1_t}(\ln t, x), u_{2_t}(\ln t, x)$ are the first partial derivatives of the functions $u_1(\ln t, x)$ and $u_2(\ln t, x)$ respectively, such that $u_1(\ln t, x)$ and $u_2(\ln t, x)$ are continuous functions on $(0,1]$ and it is known what the ($\mathscr{H}$Å) of f_1 and f_2 are known . After simplifying, we take ($\mathscr{H}$Å) to both sides of equation (3.1) in order to solve the systems (3.1).

$U_1 = \mathcal{H}Å[\frac{u_1(\ln t, x)}{t}], U_2 = \mathcal{H}Å[\frac{u_2(\ln t, x)}{t}], F_1 = \mathcal{H}Å[f_1(\ln t, x)],$

$F_2 = \mathcal{H}Å[f_2(\ln t, x)]$ in equations (3.1) we get :

$$-(\rho + 1)\mathcal{H}Å[\frac{u_1(\ln t, x)}{t}] = a_{11}U_1 + a_{22}U_2 + F_1 \ ,$$

$$-(\rho + 1)\mathcal{H}Å[\frac{u_2(\ln t, x)}{t}] = a_{21}U_1 + a_{22}U_2 + F_2 \ \cdot$$

So,

$$-\big((\rho + 1) + a_{11}\big)U_1 - a_{22}U_2 = F_1 \qquad ...(3.2)$$

$$-a_{21}U_1 - \big((\rho + 1) + a_{22}\big)U_2 = F_2 \qquad ...(3.3)$$

Through the multiplication of eq. (3.2) by a_{21} and eq. (3.3) by $\big((\rho + 1) + a_{11}\big)$ we obtain:

$$-a_{21}\big((\rho + 1) + a_{11}\big)U_1 - a_{21}a_{22}U_2 = a_{21}F_1 \ ,$$

$$-a_{21}\big((\rho + 1) + a_{11}\big)U_1 - \big((\rho + 1) + a_{11}\big)\big((\rho + 1) + a_{22}\big)U_2 = \big((\rho + 1) + a_{11}\big)F_2 \ \cdot$$

Thus, if we subtract the two equations above, we obtain:

$$-a_{21}a_{22}U_2 + \big((\rho + 1) + a_{11}\big)\big((\rho + 1) + a_{22}\big)U_2 = a_{21}F_1 - \big((\rho + 1) + a_{11}\big)F_2 \ ,$$

So,

$$U_2 = \frac{a_{21}F_1 - ((\rho + 1) + a_{11})F_2}{[-a_{21}a_{22} + \big((\rho + 1) + a_{22}\big)\big((\rho + 1) + a_{11}\big)]}; \qquad ...(3.4)$$

Such that the denominator not equal to zero .

Using a similar approach, we discover U_1.
By applying eq. (3.2) by $\big((\rho + 1) + a_{22}\big)$ and eq. (3.3) by a_{22} obtain:

$$-\big((\rho + 1) + a_{11}\big)\big((\rho + 1) + a_{22}\big)U_1 - a_{22}\big((\rho + 1) + a_{22}\big)U_2 = \big((\rho + 1) + a_{22}\big)F_1 \ ,$$

$$\pm a_{21}a_{22}U_1 \pm a_{22}\big((\rho + 1) + a_{22}\big)U_2 = \mp a_{22}F_2 \ \cdot$$

Thus, if we subtract the two equations above, we obtain:

$$-\big((\rho + 1) + a_{11}\big)\big((\rho + 1) + a_{22}\big)U_1 + a_{21}a_{22}U_1 = \big((\rho + 1) + a_{22}\big)F_1 - a_{22}F_2 \ \cdot$$

So,

$$U_1 = \frac{((\rho + 1) + a_{22})F_1 - a_{22}F_2}{-\big((\rho + 1) + a_{11}\big)\big((\rho + 1) + a_{22}\big) + a_{21}a_{22}}; \qquad ...(3.5)$$

Such that the denominator not equal to zero.

From equations (3.4) and (3.5), we obtain $(\mathcal{H}Å)^{-1}$

$$u_1 = \mathcal{H}Å^{-1}[\frac{((\rho + 1) + a_{22})F_1 - a_{22}F_2}{-\big((\rho + 1) + a_{11}\big)\big((\rho + 1) + a_{22}\big) + a_{21}a_{22}}] \ , ...(3.6\ a)$$

$$u_2 = \mathcal{H}\mathring{A}^{-1}\left[\frac{a_{21}F_1 - ((\rho+1)+a_{11})F_2}{-a_{21}a_{22} + \big((\rho+1)+a_{22}\big)\big((\rho+1)+a_{11}\big)}\right] \quad \text{... (3.6 b)}$$

Eq. (3.6) represents the solution of system (3.1).

3.1 Table of Albazy Altememe Integral Transform to the partial derivatives

$u(\ln t, x)$	$\mathcal{H}\mathring{A}[u(\ln t, x)] = \frac{(-1)^\rho}{\rho!}\int_0^1 (\ln t)^\rho\ u(\ln t, x)\, dt = v(x,\rho) \quad ;\ \rho \in z^+$	
$(\ln t)\ \frac{u_t(\ln t, x)}{t}$	$-(\rho+1)\ v(x,\rho)$	$t \in (0,1]$
$\frac{(\ln t)^2}{t} u_{tt}(\ln t, x)$	$(\rho+2)(\rho+1)\ v(x,\rho)$	$t \in (0,1]$
$\frac{(\ln t)^3}{t} u_{ttt}(\ln t, x)$	$-(\rho+3)(\rho+2)(\rho+1)\ v(x,\rho)$	$t \in (0,1]$
$\frac{(\ln t)^4}{t} u^{iv}(\ln t, x)$	$(\rho+4)(\rho+3)(\rho+2)(\rho+1)\ v(x,\rho)$	$t \in (0,1]$
$\frac{(\ln t)}{t} u_{xt}(\ln t, x)$	$-(\rho+1)\frac{dv}{dx}$	$t \in (0,1]$
$\frac{(\ln t)^2}{t} u_{xtt}(\ln t, x)$	$(\rho+2)(\rho+1)\frac{dv}{dx}$	$t \in (0,1]$
$\frac{(\ln t)}{t}\ u_{xxt}(\ln t, x)$	$-(\rho+1)\ \frac{d^2v}{dx^2}$	$t \in (0,1]$
$\frac{(\ln t)^2}{t}\ U_{xxtt}(\ln t, x)$	$(\rho+2)(\rho+1)\ \frac{d^2v}{dx^2}$	$t \in (0,1]$

EXAMPLE 1

In order to solve the PDEs first-order linear system:

$$\frac{\ln t}{t} u_{1_t} + 2\frac{\ln t}{t} u_{2_t} = \ln t + 1\,, \qquad \text{... (3.7)}$$

$$\frac{\ln t}{t} u_{1_t} - 2\frac{\ln t}{t} u_{2_t} = \ln t - 1\,, \qquad \text{... (3.8)}$$

The referred conversion is applied to both sides of above equation,

we arrive at:

$\mathcal{H}\mathring{A}(\frac{\ln t}{t}u_{1_t}) + 2\mathcal{H}\mathring{A}(\frac{\ln t}{t}u_{2_t}) = \mathcal{H}\mathring{A}(\ln t) + \mathcal{H}\mathring{A}(1)$,

$\mathcal{H}\mathring{A}(\frac{\ln t}{t}u_{1_t}) - 2\mathcal{H}\mathring{A}(\frac{\ln t}{t}u_{2_t}) = \mathcal{H}\mathring{A}(\ln t) - \mathcal{H}\mathring{A}(1)$,

So,

$-(\rho + 1)\,\mathcal{H}\mathring{A}(\frac{u_1}{t}) - 2(\rho + 1)\,\mathcal{H}\mathring{A}(\frac{u_2}{t}) = -(\rho + 1) + 1$,

$-(\rho + 1)\,\mathcal{H}\mathring{A}(\frac{u_1}{t}) + 2(\rho + 1)\,\mathcal{H}\mathring{A}(\frac{u_2}{t}) = -(\rho + 1) - 1$,

So, from first equation , we get:

$$-\mathcal{H}\mathring{A}(\frac{u_1}{t}) - 2\mathcal{H}\mathring{A}(\frac{u_2}{t}) = \frac{-\rho}{(\rho + 1)}, \qquad \dots (3.9)$$

And, from second equation , we get :

$$-\mathcal{H}\mathring{A}(\frac{u_1}{t}) + 2\mathcal{H}\mathring{A}(\frac{u_2}{t}) = \frac{-(\rho + 2)}{(\rho + 1)}, \qquad \dots (3.10)$$

By addition the last two equations , we get:

$-2\mathcal{H}\mathring{A}(\frac{u_1}{t}) = -2$,

$\mathcal{H}\mathring{A}(\frac{u_1}{t}) = 1$,

The inverse of the referred conversion is applied to both sides of above

equation, we get :

$\left(\mathcal{H}\mathring{A}\right)^{-1}\mathcal{H}\mathring{A}(\frac{u_1}{t}) = \left(\mathcal{H}\mathring{A}\right)^{-1}(1)$,

$\frac{u_1}{t} = 1$, $\qquad \therefore\ u_1 = t$,

And so, by multiplying equation (3.9)by $(-1)$and by simple calculations ,

we obtained:

$\mathcal{H}\mathring{A}(\frac{u_1}{t}) + 2\mathcal{H}\mathring{A}(\frac{u_2}{t}) = \frac{\rho}{(\rho + 1)}$,

$-\mathcal{H}\mathring{A}(\frac{u_1}{t}) + 2\mathcal{H}\mathring{A}(\frac{u_2}{t}) = \frac{-(\rho + 2)}{(\rho + 1)}$,

By addition of the last two equations , we obtain :

$\frac{-4}{(\rho + 1)}\mathcal{H}\mathring{A}(\frac{u_2}{t}) = \frac{\rho}{(\rho + 1)} + \frac{-(\rho + 2)}{(\rho + 1)}$,

$\frac{-4}{(\rho + 1)}\mathcal{H}\mathring{A}(\frac{u_2}{t}) = \frac{-2}{(\rho + 1)}$, $\qquad \mathcal{H}\mathring{A}(\frac{u_2}{t}) = \frac{-2}{(\rho + 1)}\,\frac{(\rho + 1)}{-4} = \frac{1}{2}$,

By taking $\left(\mathcal{H}\mathring{A}^{-1}\right)$ to both sides we obtain:

$(\mathcal{H}\mathring{A}^{-1})\mathcal{H}\mathring{A}(\frac{u_2}{t}) = (\mathcal{H}\mathring{A}^{-1})(\frac{1}{2})$,

$\frac{u_2}{t} = \frac{1}{2}$, $\qquad \therefore u_2 = \frac{t}{2}$,

EXAMPLE 2

The 1^s lineman PDEs must be solved as follows:

$$\ln t\,\frac{u_{1_t}}{t}+\ln t\,\frac{u_{1_{tx}}}{t}-\ln t\,\frac{u_{2_{tx}}}{t}=\ln t\ , \qquad \dots(3.11)$$

$$\ln t\,\frac{u_{1_t}}{t}+\ln t\,\frac{u_{1_{tx}}}{t}+\ln t\,\frac{u_{2_{tx}}}{t}=(\ln t)^2\ , \qquad \dots(3.12)$$

$; u(-\infty,x)=u_t(-\infty,x)=0, u_1(\ln t,0)=1, u_2(\ln t,0)=0$

The referred conversion is applied to both sides of above equation,

we arrive at:

$$\mathcal{H}\mathring{A}(\ln t\,\frac{u_{1_t}}{t}+\mathcal{H}\mathring{A}(\ln t\,\frac{u_{1_{tx}}}{t})-\mathcal{H}\mathring{A}(\ln t\,\frac{u_{2_{tx}}}{t})=\mathcal{H}\mathring{A}(\ln t)\,,$$

$$\mathcal{H}\mathring{A}(\ln t\,\frac{u_{1_t}}{t})+\mathcal{H}\mathring{A}(\ln t\,\frac{u_{1_{tx}}}{t})+\mathcal{H}\mathring{A}(\ln t\,\frac{u_{2_{tx}}}{t})=\mathcal{H}\mathring{A}((\ln t)^2)\,,$$

$$-(\rho+1)\mathcal{H}\mathring{A}(\frac{u_1}{t})-(\rho+1)\frac{dv_1}{dx}+(\rho+1)\frac{dv_2}{dx}=-(\rho+1)\,,$$

$$-(\rho+1)\mathcal{H}\mathring{A}(\frac{u_1}{t})-(\rho+1)\frac{dv_1}{dx}-(\rho+1)\frac{dv_2}{dx}=(\rho+2)(\rho+1)\,,$$

So, from the first equation, we get:

$$-V_1-\frac{dv_1}{dx}+\frac{dv_2}{dx}=-1\,, \qquad \dots(3.13)$$

And from the second equation, we get:

$$-V_1-\frac{dv_1}{dx}-\frac{dv_2}{dx}=(\rho+2)\,, \qquad \dots(3.14)$$

By addition of the last two equations together, we obtain:

$$-2V_1-2\frac{dv_1}{dx}=(\rho+2-1)\,,$$

$$-2V_1-2\frac{dv_1}{dx}=(\rho+1)\,,$$

$$\frac{dv_1}{dx}+V_1=\frac{-(\rho+1)}{2}\ ,$$

$$\frac{dv_1}{dx}e^X+V_1e^X=\frac{-(\rho+1)}{2}e^X\,,$$

$$V_1e^X=\frac{-(\rho+1)}{2}e^X+\varphi_1(t)\,,$$

$$V_1=\frac{-(\rho+1)}{2}+e^{-X}\varphi_1(t)\,,$$

$$\mathcal{H}\mathring{A}(\frac{u_1}{t})=\frac{-(\rho+1)}{2}+e^{-X}\varphi_1(t)\,,$$

The inverse of the referred conversion is applied to both sides of above equation, we get

$$(\mathcal{H}\mathring{A})^{-1}\mathcal{H}\mathring{A}(\frac{u_1}{t})=(\mathcal{H}\mathring{A})^{-1}(\frac{-(\rho+1)}{2}+e^{-X}\varphi_1(t))\,,$$

$$\frac{u_1}{t}=\frac{\ln t}{2}+e^{-X}\varphi_3(t)\,, \qquad \varphi_3(t)=(\mathcal{H}\mathring{A})^{-1}(\varphi_1(t))\,,$$

$$u_1=\frac{t\ln t}{2}+te^{-X}\varphi_3(t)\,, \qquad \text{since } u_1(\ln t,0)=1\,,$$

$$1=\frac{t\ln t}{2}+te^{0}\varphi_3(t)\ , \qquad t\varphi_3(t)=1-\frac{t\ln t}{2}\ ,$$

$\varphi_3(t) = \frac{2 - t\ln t}{2t}$, $\varphi_3(t) = \frac{1}{t} - \frac{\ln t}{2}$,

$u_1 = \frac{t\ln t}{2} + te^{-X}\left(\frac{1}{t} - \frac{\ln t}{2}\right)$, $\therefore u_1 = \frac{t\ln t}{2} + e^{-X} - \frac{t\ln t}{2}e^{-X}$,

$u_1 = \frac{t\ln t}{2}(1 - e^{-X}) + e^{-X}$,

Also, from equation (3.13) and equation (3.14) we can get:

$V_1 + \frac{dv_1}{dx} - \frac{dv_2}{dx} = 1$,

$-V_1 - \frac{dv_1}{dx} - \frac{dv_2}{dx} = (\rho + 2)$,

By addition the last two equations together, we get:

$-2\frac{dv_2}{dx} = (\rho + 3)$, $\frac{dv_2}{dx} = \frac{-(\rho + 3)}{2}$,

$2\int dv_2 = -\int (\rho + 3)\, dx$, $2v_2 = -(\rho + 3)x + c$,

$v_2 = \frac{-(\rho + 3)}{2}x + \frac{c}{2}$, $\mathcal{H}Å(\frac{u_2}{t}) = \frac{-(\rho + 3)}{2}x + \frac{c}{2}$,

The inverse of the referred conversion is applied to both sides of above equation, we get:

$(\mathcal{H}Å)^{-1}\mathcal{H}Å(\frac{u_2}{t}) = (\mathcal{H}Å)^{-1}(\frac{-(\rho + 3)}{2}x + \frac{c}{2})$,

$\frac{u_2}{t} = (\frac{\ln t - 2}{2})x + \frac{c_1}{2}$, $u_2 = (\frac{t\ln t - 2t}{2})x + \frac{t}{2}c_1$,

$c_1 = (\mathcal{H}Å)^{-1}(c)$, Since $u_2(\ln t, 0) = 0$,

$0 = (\frac{t\ln t - 2t}{2})(0) + \frac{t}{2}c_1$, $\frac{t}{2}c_1 = 0$, $c_1 = 0$,

$\therefore u_2 = (\frac{t\ln t - 2t}{2})x$,

EXAMPLE 3

The1^s linear PDEs must be solved as follows:

$$\ln t\,\frac{u_{1_t}}{t} - \ln t\,\frac{u_{1_{tx}}}{t} + \ln t\,\frac{u_{2_{tx}}}{t} = \ln t(\cosh(\ln\ln t + \sinh\ln\ln t) \quad ,\qquad \ldots (3.15)$$

$$\ln t\,\frac{u_{1_t}}{t} - \ln t\,\frac{u_{1_{tx}}}{t} - \ln t\,\frac{u_{2_{tx}}}{t} = 0 \quad ,\qquad \ldots (3.16)$$

; $u(-\infty, x) = u_t(-\infty, x) = 0, u_1(\ln t, 0) = 1, u_2(\ln t, 0) = t\ln t$,

The referred conversion is applied to both sides of above equation,

we arrive at:

$\mathcal{H}Å(\ln t\,\frac{u_{1_t}}{t}) - \mathcal{H}Å(\ln t\,\frac{u_{1_{tx}}}{t}) + \mathcal{H}Å(\ln t\,\frac{u_{2_{tx}}}{t}) = \mathcal{H}Å\,(\ln t(\cosh\ \ln\ln t + \sinh\ln\ln t))$,

$\mathcal{H}Å(\ln t\,\frac{u_{1_t}}{t}) - \mathcal{H}Å(\ln t\,\frac{u_{1_{tx}}}{t}) - \mathcal{H}Å(\ln t\,\frac{u_{2_{tx}}}{t}) = HA(0)$,

So,

$$-(\rho + 1)\,V_1 + (\rho + 1)\frac{dv_1}{dx} - (\rho + 1)\frac{dv_2}{dx} = \frac{1}{2\rho!}(\rho + 2)! + \frac{1}{2\rho!}(\rho!)$$

$$+\frac{1}{2\rho!}(\rho + 2) - \frac{1}{2\rho!}(\rho!) ,$$

$-(\rho+1)\,V_1 + (\rho+1)\frac{dv_1}{dx} + (\rho+1)\frac{dv_2}{dx} = 0\,,$

So,

$-(\rho+1)\,V_1 + (\rho+1)\frac{dv_1}{dx} - (\rho+1)\frac{dv_2}{dx} = (\rho+2)(\rho+1)\,,$

$-(\rho+1)\,V_1 + (\rho+1)\frac{dv_1}{dx} + (\rho+1)\frac{dv_2}{dx} = 0\,,$

So, from the first equation, we get:

$-V_1 + \frac{dv_1}{dx} - \frac{dv_2}{dx} = (\rho+2)\,, \qquad \ldots(3.17)$

And from the second equation, we get:

$-V_1 + \frac{dv_1}{dx} + \frac{dv_2}{dx} = 0\,, \qquad \ldots(3.18)$

By adding of the last two equations together, we obtain:

$-2\,V_1 + 2\,\frac{dv_1}{dx} = (\rho+2)\,, \qquad -2\,V_1 + 2\,\frac{dv_1}{dx} = (\rho+2)\,,$

$\frac{dv_1}{dx} - V_1 = \frac{(\rho+2)}{2}\,,$

$e^{-x}\frac{dv_1}{dx} - e^{-x}\,V_1 = e^{-x}\,\frac{(\rho+2)}{2}\,,$

$V_1\,e^{-x} = \frac{-(\rho+2)}{2}\,e^{-x} + \varphi_1(t)\,,$

$V_1 = \frac{-(\rho+2)}{2} + e^{x}\,\varphi_1(t)\,, \qquad \mathcal{H}\mathring{A}(\frac{u_1}{t}) = \frac{-(\rho+2)}{2} + e^{x}\,\varphi_1(t)\,,$

The inverse of the referred conversion is applied to both sides of above equation, we get:

$(\mathcal{H}\mathring{A})^{-1}\mathcal{H}\mathring{A}(\frac{u_1}{t}) = (\mathcal{H}\mathring{A})^{-1}(\frac{-(\rho+2)}{2} + e^{x}\,\varphi_1(t))\,,$

$\frac{u_1}{t} = \frac{\ln t - 1}{2} + e^{x}\,\varphi_3(t)\,, \qquad \varphi_3(t) = (\mathcal{H}\mathring{A})^{-1}(\varphi_1(t))\,,$

$u_1 = \frac{t\ln t - t}{2} + t\,e^{x}\,\varphi_3(t)\,, \qquad 1 = \frac{t\ln t - t}{2} + t\,e^{x}\,\varphi_3(t)\,,$

$1 - \frac{t\ln t + t}{2} = t\,e^{0}\,\varphi_3(t)\,, \qquad t\,\varphi_3(t) = \frac{2 - t\ln t + t}{2}\,,$

$\varphi_3(t) = \frac{1}{t} - \frac{\ln t}{2} + \frac{1}{2}\,,$

$u_1 = \frac{t\ln t - t}{2} + t\,e^{x}(\frac{1}{t} - \frac{\ln t}{2} + \frac{1}{2})\,,$

$\therefore u_1 = \frac{t\ln t - t}{2} + e^{x} - \frac{t\ln t}{2}e^{x} + \frac{t}{2}e^{x}\,,$

By multiplying equation (3.17) by (-1), we obtain:

$V_1 - \frac{dv_1}{dx} + \frac{dv_2}{dx} = -(\rho+2)\,, \qquad -V_1 + \frac{dv_1}{dx} + \frac{dv_2}{dx} = 0,$

By adding of the last two equations together, we obtain:

$2\frac{dv_2}{dx} = -(\rho+2)\,, \qquad 2\int dv_2 = -\int(\rho+2)dx\,,$

$2V_2 = -(\rho+2)\,x + c\,, \qquad V_2 = \frac{-(\rho+2)}{2}\,x + \frac{c}{2}\,,$

$$\mathcal{H}Å(\frac{u_2}{t}) = \frac{-(\rho+2)}{2}\, x + \frac{c}{2} \;,$$

The inverse of the referred conversion is applied to both sides of above equation, we get:

$$(\mathcal{H}Å)^{-1}\, \mathcal{H}Å(\frac{u_2}{t}) = (\mathcal{H}Å)^{-1}\, (\frac{-(\rho+2)}{2}\, x + \frac{c}{2}) \;,$$

$$\frac{u_2}{t} = \frac{\ln t - 1}{2}\, x + \frac{c_1}{2} \quad , \quad u_2 = (\frac{t\ln t - t}{2})x + t\,\frac{c_1}{2} \quad ; c_1 = (\mathcal{H}Å)^{-1}\,(c) \,,$$

$$t\ln t = (\frac{t\ln t - t}{2})(0) + t\,\frac{c_1}{2} \quad , \quad t\ln t = \frac{t}{2}c_1 \quad , \quad c_1 = 2\ln t,$$

$$\therefore u_2 = (\frac{t\ln t - t}{2})\, x + \frac{t}{2}(2\ln t) \,,$$

EXAMPLE 4

The 1^s linear PDEs must be solved as follows:

$$\ln t\, \frac{u_{1xt}}{t} + \ln t\, \frac{u_{2xt}}{t} = (\ln t)^3 + 3(\ln t)^2 + \ln t \quad , \qquad \dots (3.19)$$

$$\frac{u_1}{t} - \frac{u_2}{t} = x(\ln t - (\ln t)^2) \quad , \qquad \dots (3.20)$$

$$u(-\infty, x) = u_t(-\infty, x) = 0, u_1(\ln t, 0) = u_2(\ln t, 0) = 0 \,,$$

The referred conversion is applied to both sides of above equation,

we arrive at:

$$\mathcal{H}Å(\ln t\, \frac{u_{1xt}}{t}) + \mathcal{H}Å(\ln t\, \frac{u_{2xt}}{t}) = HA((\ln t)^3) + 3\mathcal{H}Å((\ln t)^2) + \mathcal{H}Å(\ln t) \;,$$

$$\mathcal{H}Å\left(\frac{u_1}{t}\right) - \mathcal{H}Å\left(\frac{u_2}{t}\right) = \mathcal{H}Å(x(\ln t - (\ln t)^2)) \,,$$

So,

$$-(\rho+1)\frac{dv_1}{dx} - (\rho+1)\frac{dv_2}{dx} = -(\rho+3)(\rho+2)(\rho+1) + 3(\rho+2)(\rho+1) - (\rho+1)$$

$$\mathcal{H}Å\left(\frac{u_1}{t}\right) - \mathcal{H}Å\left(\frac{u_2}{t}\right) = x\big(-(\rho+1) - (\rho+2)(\rho+1)\big) \,,$$

So, from the first equation, we get:

$$\frac{dv_1}{dx} + \frac{dv_2}{dx} = (\rho+3)(\rho+2) - 3(\rho+2) + 1 \;, \qquad \dots (3.21)$$

And, from the second equation, we get:

$$v_1 - v_2 = x\big(-(\rho+1) - (\rho+2)(\rho+1)\big) \;,$$

By differentiating the last equation, we get:

$$\frac{dv_1}{dx} - \frac{dv_2}{dx} = -(\rho+1) - (\rho+2)(\rho+1) \,, \qquad \dots (3.22)$$

By adding of the last two equations (3.21) and (3.22) together, we obtain:

$$2\frac{dv_1}{dx} = (\rho+3)(\rho+2) - 3(\rho+2) + 1 - (\rho+1) - (\rho+2)(\rho+1),$$

$$2\frac{dv_1}{dx} = (\rho+2)\rho - (\rho+1)(\rho+3) + 1 \quad = \rho^2 + 2\rho - \rho^2 - 4\rho - 3 + 1 = -2\rho - 2 \;,$$

$$\frac{dv_1}{dx} = -(\rho+1) \,, \qquad \int dv_1 = -\int (\rho+1)\, dx \;,$$

$$v_1 = -(\rho+1)\, x + c_1 \,, \qquad HA\left(\frac{u_1}{t}\right) = -(\rho+1)\, x + c_1 \,,$$

The inverse of the referred conversion is applied to both sides of above equation, we get:

$(\mathcal{H}\text{Å})^{-1}\mathcal{H}\text{Å}\left(\frac{u_1}{t}\right) = (\mathcal{H}\text{Å})^{-1}(-(\rho+1)\,x + c_1)\,,$

$\frac{u_1}{t} = x\,\ln t + c_3 \quad , c_3 = (\mathcal{H}\text{Å})^{-1}(c_1)\,, \qquad u_1 = x\,t\ln t + t\,c_3,$

Since $u_1(\ln t, 0) = 0 \quad , \quad c_3 = 0\,,$

$\therefore u_1 = x\,t\ln t\,,$

And so , by multiplying equation (3.22) by (-1) and by simple calculations, we obtain:

$$-\frac{dv_1}{dx} + \frac{dv_2}{dx} = (\rho+1) + (\rho+2)(\rho+1)\,, \qquad \dots(3.23)$$

$$\frac{dv_1}{dx} + \frac{dv_2}{dx} = (\rho+3)(\rho+2) - 3(\rho+2) + 1\,, \qquad \dots(3.24)$$

By adding of the last two equations (3.23) and (3.24) together, we obtain:

$2\frac{dv_2}{dx} = (\rho+3)(\rho+2) - 3(\rho+2) + 1 + (\rho+1) + (\rho+2)(\rho+1)\,,$

$= \rho^2 + 5\rho + 6 - 3\rho - 6 + 1 + \rho + 1 + \rho^2 + 3\rho + 2\,,$

$= 2\rho^2 + 6\rho + 4\,,$

$\frac{dv_2}{dx} = \rho^2 + 3\rho + 2 = (\rho+2)(\rho+1)\,,$

$\int dv_2 = \int (\rho+2)(\rho+1)\,dx\,, \qquad v_2 = (\rho+2)(\rho+1)\,x + c_2\,,$

$\text{HA}\left(\frac{u_2}{t}\right) = (\rho+2)(\rho+1)\,x + c_2\,,$

The inverse of the referred conversion is applied to both sides of above equation, we get:

$(\mathcal{H}\text{Å})^{-1}\mathcal{H}\text{Å}\left(\frac{u_2}{t}\right) = (\mathcal{H}\text{Å})^{-1}\left((\rho+2)(\rho+1)\,x + c_2\right)\,,$

$\frac{u_2}{t} = x(\ln t)^2 + c_4\,, \quad c_4 = (\mathcal{H}\text{Å})^{-1}(c_2), \qquad u_2 = x\,t(\ln t)^2 + t\,c_4\,,$

Since $u_2(\ln t, 0) = 0\,, \quad c_4 = 0\,,$

$\therefore u_2 = x\,t(\ln t)^2.$

4. CONCLUSIONS

The transform subject of the work has been used to find a solution for linear systems of PDEs with variable coefficient from the first order with some initial conditions

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