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Analysis and Solutions for First-Order Multi- Integro-Differential Impulsive Equations

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ABSTRACT: In this paper the existence and uniqueness of first order multi-integro –multi-impulsive equation has been presented in details and explained their approach depended on some inequalities and estimations and some special functions as estimators also, all these provided for interesting results that will be presented in all. The problem formulation was presented as a first time with their suitable extension formulation with some conditions depended on provided first order multi-integro –multi-impulsive problem. The nonlinear analytic of impulsive differential ordinary equations and definition of generalized β-Ulam- Hyers -Rassias stable are used as a basis to establish technical of proving as well as a fixed-point theorem have been used for existence and stability with some interesting estimators for this type of stability to grantee the trajectory to be stable as well as the impulsive analytic and their extension of the proposal first order multi-integro –multi impulsive problem are presented in this issue and given how all these concepts work together. The perturbed impulsive part is presented in this problem as a first time. Also, some illustrative examples have been presented in details to explain how is the results satisfies and true.

Keywords: β-Ulam- Hyers -Rassias, Stability, Existence, Uniqueness, fixed point theorem, multi-differential

1. INTRODUCTION

Impulsive systems with continuous evolution which modelled by some ordinary differential equations with state jumps also impulses, [11]. The nonlinear boundary value problems combine with scientific and engineering problems such as second-order have been studied extensively, [1]. Impulsive systems modeled as a process that combine the behavior of continuous and discontinuous. Many applications of impulsive systems such as logistics, robotics, population dynamics, etc. The basis of mathematical theory of impulsive systems as well as the existence and stability of solutions as fundamental results, [15], [24].

The impulsive differential equation and the impulsive integro diffential equation are interesting in modeling of many applications of engendering and physics problems such that the difference of the differential between some discreet points only satisfied of one side and the limits of two sides are not equal, so this needs some technique to find the solution to study other properties of boundedness and existence and uniqueness and stability. The classes of the impulsive integro differential equations are solve it with some necessary and sufficient condition suitable discusses the modeling equations involving nonlinear functions.

The important subject is a stability of nonlinear system and that why studied by many researchers of applied mathematics since it is applied in many a branch of scientific applications with some experiments of suitable input function such as control function, see [13,14,23,32]. Also, the perturbed of system need it a study with different necessary and sufficient conditions as without perturbed.

Also, we mention to the studies on the stability in [5,8,17,25], so our aim how to model the natural phenomena to a linear and nonlinear control systems and interest on minimize the errors, see [14]. The stabilization depended the perturbed coming from the control feedback which take a major role in many real-life control problems.

There is a nonlinear relation between input and output constraint of the feedback control negative or positive to be the system perfect see, [18,20,23,25].

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The method of stability and stabilization dependent on some necessary and sufficient conditions established on the system under studied and these points are a fixed point for the solutions from effect it of the feedback closed control system and perturbed parts of control and output matrices, as well as for another approach of stability is a Lyapunov's rule for nonlinear system, see [4,8,9,21,26,27,33,37]. Problem mitigation of nonlinear vibrations, for approximate stability have been used parallel compensator by using Lassalle's [7] of closed loop.

Sometimes the phenomena appeared in many different real lives bur a big area in physics, medicine, population dynamics the study of impulsive functional which the processes are depended on sort time that is perturbed during evaluation of the solution state of differential equations and appeared as a discretely which comparison with the total of the output of processes and phenomena under studied was simulated to a mathematical model

The important branch of qualitative analysis of ordinary and partial differential equations is the stability theory and impulsive stability theory that which combine continuous of modules of ordinary differential equations with instantaneous state jumps or resets which referred to as impulsive.

Akhmetov and Zafer [3] studied the second method of Lyapunov for impulsive differential equations to be stable. The nonlinear impulsive Stochastic systems with dwell time condition in [10]. well as stability and stabilization of some fractions systems with delay function studied from Lazarevic in [16]. As for nonlinear system with delay impulsive have been presented in [16]. LIU and et.al. in [20], interested on analysis stability of some of impulsive control systems and with time delay interested from LIU in [19]. Obloza [21,22] were among many approaches of studding the behaviors of H-U-stable one of stability types for interesting differential equations. [36], There are many authors studied with some problems a H-U-stable and H-U-R-stable as criteria for guarantee this issue. In [29,30], the application of impulsive differential equations in different fields such as engineering and natural sciences. There are different classes of impulsive differential equations with first order reported in- [28,35]. Also, system of integro-differential equations with their stability explained in [22] with details information. Some of papers studied the solutions by using fixed point theorems such as [2], also, the nonlocal boundary value of impulsive integro-differential equations as a system studied in [34], interesting existence result, and the concept of Ulam stability have studied in [6]. Ulam's-type stabilities for class of bounded variable delays of first-order impulsive differential equations on compact interval, [31].

The purpose of this paper the first order multi-integro –multi-impulsive equation have been studied in details. The space of pointwise continuous bounded functions with divided intervals is used for sub solution depended on the intervals. Also, the behaviors of stable existence solution and their extension have been presented in details as proving of interesting results with illustrative example.

2. BASIC DEFINITIONS AND CONCEPTS:

Consider the following first order multi-integro –multi-impulsive equation

$$\begin{cases} \frac{d}{dt}\left(x(t)+\int_0^t k\big(t,x(s)\big)f_1(s,\tau)\,d\tau\right)=f\big(t,x(t)\big)+\int_0^t k_1\big(t,x(s)\big)\,ds\int_0^t k_2(t,s)\,ds \\ \qquad ,t\in(s_k,t_{k+1}],k=0,1,\ldots,m \\ x(t)=g_k\big(t,x(t_k^+)\big), \qquad t\in(t_k,s_k]\ ,k=1,\ldots,m, \\ x(0)=x_0\in\mathbb{R} \end{cases} \tag{1}$$

Where $0=t_0=s_0<t_1\le s_1\le t_2<\ldots<t_m\le s_m<t_{m+1}=T$

$f,k,k_1:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$, $k_2,f_1:[0,T]\times[0,T]\longrightarrow\mathbb{R}$ are a continuous function and $g_k:[t_k,s_k]\times\mathbb{R}\longrightarrow\mathbb{R}$ *is a* continuous for all $k=1,\ldots,m$. Let $0<\beta\le 1$, $\psi\ge 0$ and $\varphi\in PC(I,\mathbb{R}_+)$ is nondecreasing.

If x satisfies $x(0)=x_0$, $x(t)=g_k\big(t,x(t_k^+)\big)$, $t\in(t_k,s_k)$,$k=1,\ldots,m$. and $x(t)=x(0)-\int_0^t k\big(t,x(s)\big)f_1(s,t)\,ds-\int_0^t\int_0^t k_t\big(t,x(s)\big)f_1(s,\tau)\,d\tau\,d\sigma+\int_0^t f\big(t,x(t)\big)\,ds+\int_0^t\int_0^t k_1\big(t,x(s)\big)\,ds\int_0^t k_2(t,s)\,ds\,d\sigma$, $t\in[0,t_1]$

$x(t)=g_k\big(t,x(t_k^+)\big)-\int_{s_k}^t k\big(t,x(s)\big)f_1(s,t)\,ds-\int_{s_k}^t\int_{s_k}^t k_t\big(t,x(s)\big)f_1(s,\tau)\,d\tau\,d\sigma+\int_{s_k}^t f\big(t,x(t)\big)\,ds+\int_{s_k}^t\int_{s_k}^t k_1\big(t,x(s)\big)\,ds\int_{s_k}^t k_2(t,s)\,ds\,d\sigma$, $t\in(s_k,t_{k+1}]$,$k=0,1,\ldots,m$.

Now we need to apply fixed point approach for stability of impulsive multi-integro- differential perturbed with integral function nonlinear equation (2).

Consider the following inequality integral equation:

$$\begin{cases} \left| \begin{array}{l} y'(t)-\int_0^t k\big(t,y(s)\big)f_1(s,t)\,ds-\int_0^t\int_0^t k_t\big(t,y(s)\big)f_1(s,\tau)\,d\tau\,d\sigma \\ \quad+\int_0^t f\big(t,y(t)\big)\,ds+\int_0^t\int_0^t k_1\big(t,y(s)\big)\,ds\int_0^t k_2(t,s)\,ds\,d\sigma \end{array} \right| \\ \qquad \le\varphi\big(m(t)\big),\ t\in(s_k,t_{k+1}]\ ,k=0,1,\ldots,m, \\ \left|y(t)-g_k\big(t,y(t_k^+)\big)\right|\le\ \psi \qquad ,\ t\in(t_k,s_k]\ ,k=1,\ldots,m. \end{cases} \tag{3}$$

Definition 2.1.
The β-Ulam- Hyers -Rassias stable for equation (2) satisfied if $c_{f,\beta,g_i,\varphi} > 0$ therefore the solution $y \in PC(I,\mathbb{R}) \cap_{k=0}^{m} C^1((s_k,t_{k+1}],\mathbb{R})$ of the inequality (3) there exists a solution $x \in PC(I,\mathbb{R}) \cap_{k=0}^{m} C^1((s_k,t_{k+1}],\mathbb{R})$ of the equation (2) with
$|y(t) - x(t)|^\beta \le c_{f,\beta,g_k,\varphi}(\psi^\beta + \varphi^\beta(m(t)))$, $t \in I$.

Remark 2.2.
A function $y \in PC(I,\mathbb{R}) \cap_{i=0}^{m} C^1((s_k,t_{k+1}],\mathbb{R})$ is a solution of inequality (3) if and only if there is $G \cap_{k=0}^{m} C^1((s_k,t_{k+1}],\mathbb{R})$ and $g \cap_{k=1}^{m} C^1((t_k,s_k],\mathbb{R})$ such that

i. $|G(t)| \le \varphi(m(t))$, $t \in \cup_{k=0}^{m}(s_k,t_{k+1}]$ *and* $|g(t)| \le \psi$, $t \in \cup_{k=0}^{m}(t_k,s_k]$;
ii. $y'(t) = -\int_0^t k(t,y(s)) f_1(s,t)\, ds - \int_0^t \int_0^t k_t(t,y(s)) f_1(s,\tau)\, d\tau\, d\sigma + \int_0^t f(t,y(t))\, ds + \int_0^t \int_0^t k_1(t,y(s))\, ds \int_0^t k_2(t,s)\, ds\, d\sigma + G(t)$, $t \in (s_k,t_{k+1}]$, $k = 0,1,\dots,m$;
iii. $y(t) = g_k(t,y(t_k^+)) + g(t)$, $t \in (t_k,s_k]$, $k = 1,\dots,m$.

Remark 2.3.
If $y \in PC(I,\mathbb{R}) \cap_{k=0}^{m} C^1((s_k,t_{k+1}],\mathbb{R})$ is a solution of the inequality (3) then y is a solution of the following impulsive multi-integro- differential perturbed with integral function nonlinear equation

$$
\begin{cases}
\left|y(t) - g_k(t,y(t_k^+))\right| \le \psi, \quad t \in (t_k,s_k] \ ,k = 1,\dots m, \\
\left| y(t) - y(0) - \int_0^t k(t,y(s)) f_1(s,t)\, ds - \int_0^t \int_0^t k_t(t,y(s)) f_1(s,\tau)\, d\tau\, d\sigma + \int_0^t f(t,y(t))\, ds + \int_0^t \int_0^t k_1(t,y(s))\, ds \int_0^t k_2(t,y(s))\, ds\, d\sigma \right| \\
\qquad \le \int_0^t \varphi\left(\int_0^t m(\tau)\, d\tau\right) ds \quad ,t \in [0,t_1] \\
\left| y(t) - g_k(t,y(t_k^+)) - \int_{s_k}^t k(t,y(s)) f_1(s,t)\, ds - \int_{s_k}^t \int_{s_k}^t k_t(t,y(s)) f_1(s,\tau)\, d\tau\, d\sigma + \int_{s_k}^t f(t,y(t))\, ds + \int_{s_k}^t \int_{s_k}^t k_1(t,y(s))\, ds \int_{s_k}^t k_2(t,s)\, ds\, d\sigma \right| \\
\qquad \le \psi + \int_{s_k}^t \varphi\left(\int_{s_k}^t m(\tau)\, d\tau\right) ds \ , t \in (s_k,t_{k+1}] \ , \ k = 0,1,\dots,m.
\end{cases}
\tag{4}
$$

The extended of generalized β-Ulam- Hyers -Rassias stability of the equation

$$
\begin{cases}
x'(t) = e^{a(t)} x(t) - \int_0^t k(t,x(s)) f_1(s,t)\, ds - \int_0^t \int_0^t k_t(t,x(s)) f_1(s,\tau)\, d\tau\, d\sigma \\
\qquad + \int_0^t f(t,x(t))\, ds + \int_0^t \int_0^t k_1(t,x(s))\, ds \int_0^t k_2(t,s)\, ds\, d\sigma \\
\qquad ,t \in (s_k,t_{k+1}], \ = 0,1,\dots,m \ , \lambda > 0 \\
x(t) = g_k(t,x(t_k^+)), t \in (t_k,s_k] \ , k = 1,2 \dots ,m.
\end{cases}
\tag{5}
$$

Definition 2.4.
The generalized β-Ulam- Hyers -Rassias stable for equation (5) if there exists $c_{f,\beta,g_k,\varphi} > 0$ then the solution as follows:
$y \in PC(I,\mathbb{R}) \cap_{k=0}^{m} C^1((s_k,t_{k+1}],\mathbb{R})$ of the inequality

$$
\begin{cases}
\left| y'(t) - e^{a(t)} y(t) + \int_0^t k(t,y(s)) f_1(s,t)\, ds + \int_0^t \int_0^t k_t(t,y(s)) f_1(s,\tau)\, d\tau\, d\sigma - \int_0^t f(t,y(t))\, ds - \int_0^t \int_0^t k_1(t,y(s))\, ds \int_0^t k_2(t,s)\, ds\, d\sigma \right| \\
\qquad \le \varphi(m(t)), t \in (s_k,t_{k+1}], \ = 0,1,\dots,m \ , \lambda > 0 \\
\left| y(t) - g_k(t,y(t_k^+)) \right| \le \psi \quad , \ t \in (t_k,s_k] \ , k = 1,2 \dots ,m,
\end{cases}
\tag{6}
$$

there exists a solution $x \in PC(I,\mathbb{R}) \cap_{k=0}^{m} C^1((s_k,t_{k+1}],\mathbb{R})$ of the equation (5) with
$|y(t) - x(t)|^\beta \le c_{f,\beta,g_k,\varphi}\left(\psi^\beta + \varphi^\beta(m(t))\right)$, $t \in I$.

Remark 2.5.

If $y \in PC(I,\mathbb{R}) \cap_{k=0}^{m} C^1\left((s_k,t_{k+1}],\mathbb{R}\right)$ is a solution of the inequality (6) ,then y is a solution of the following

$$\begin{cases} \left|y(t) - g_k\left(t,y(t_k^+)\right)\right| \leq \psi, \ t \in (t_k,s_k] \ ,k = 1,\ldots,m \\ \left| \begin{matrix} y(t) - e^{a(t)}y(0) \\ -e^{a(t-s)} \begin{bmatrix} -\int_0^t k\left(t,y(s)\right)f_1(s,t)\,ds - \int_0^t \int_0^t k_t\left(t,y(s)\right)f_1(s,\tau)\,d\tau\,d\sigma \\ +\int_0^t f\left(t,y(t)\right)ds + \int_0^t \int_0^t k_1\left(t,y(s)\right)ds \int_0^t k_2(t,s)\,ds\,d\sigma \end{bmatrix} \end{matrix} \right| \\ \qquad \leq \int_0^t e^{a(t-s)}\varphi\left(\int_0^t m(\tau)\,d\tau\right)ds \ ,t \in [0,t_1] \\ \left| \begin{matrix} y(t) - e^{a(t-s_k)}g_k\left(t,x(t_k^+)\right) \\ -e^{a(t-s)} \left| \begin{matrix} -\int_{s_k}^t k\left(t,y(s)\right)f_1(s,t)\,ds - \int_{s_k}^t \int_{s_k}^t k_t\left(t,y(s)\right)f_1(s,\tau)\,d\tau\,d\sigma \\ +\int_{s_k}^t f\left(t,y(t)\right)ds + \int_{s_k}^t \int_{s_k}^t k_1\left(t,y(s)\right)ds \int_{s_k}^t k_2(t,s)\,ds\,d\sigma \end{matrix} \right| \end{matrix} \right| \\ \leq e^{a(t-s_k)}\psi + \int_{s_k}^t e^{a(t-s)}\varphi\left(\int_{s_k}^t m(\tau)\,d\tau\right)ds \ , t \in (s_k,t_{k+1}]\,,\ k = 0,1,\ldots,m. \end{cases} \tag{7}$$

3. PROBLEM FORMULATION

Consider the following first order multi-integro –multi-impulsive equation

$$\begin{cases} \frac{d}{dt}\left(x(t) + \int_0^t k\left(t,x(s)\right)f_1(s,\tau)\,d\tau\right) = f\left(t,x(t)\right) + \int_0^t k_1\left(t,x(s)\right)ds \int_0^t k_2(t,s)\,ds \\ \qquad\qquad\qquad\qquad , t \in (s_k,t_{k+1}], k = 0,1,\ldots,m \\ x(t) = g_k\left(t,x(t_k^+)\right) \qquad\qquad , t \in (t_k,s_k] \ ,k = 1,\ldots,m, \end{cases} \tag{8}$$

Where $0=t_0=s_0 < t_1 \leq s_1 \leq t_2 <\ldots< t_m \leq s_m < t_{m+1}= T$
$f,k,k_1 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $k_2,f_1 : [0,T] \times [0,T] \longrightarrow \mathbb{R}$ are a continuous function and $g_k : [t_k,s_k] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous for all $k = 1,\ldots,m$. Let $0< \beta \leq 1$, $\psi \geq 0$ and $\varphi \in PC(I,\mathbb{R}_+)$ is nondecreasing.

From (8) ,we have that

$$\frac{d}{dt}x(t) + \frac{d}{dt}\int_0^t k\left(t,x(t)\right)f_1(s,\tau)\,d\tau = f\left(t,x(t)\right) + \int_0^t k_1\left(t,x(s)\right)ds \int_0^t k_2(t,s)\,ds$$

$$x' + k\left(t,x(s)\right)f_1(s,t) + \int_0^t k_t\left(t,x(s)\right)f_1(s,\tau)\,d\tau = f\left(t,x(t)\right) + \int_0^t k_1\left(t,x(s)\right)ds \int_0^t k_2(t,s)\,ds$$

By Leibnitz

$$x(t) - x(0) + \int_0^t k\left(t,x(s)\right)f_1(s,t)\,ds + \int_0^t \int_0^t k_t\left(t,x(s)\right)f_1(s,\tau)\,d\tau\,d\sigma = \int_0^t f\left(t,x(t)\right)ds + \int_0^t \int_0^t k_1\left(t,x(s)\right)ds \int_0^t k_2(t,s)\,ds\,d\sigma$$

$$x(t) = x(0) - \int_0^t k\left(t,x(s)\right)f_1(s,t)\,ds - \int_0^t \int_0^t k_t\left(t,x(s)\right)f_1(s,\tau)\,d\tau\,d\sigma + \int_0^t f\left(t,x(t)\right)ds + \int_0^t \int_0^t k_1\left(t,x(s)\right)ds \int_0^t k_2(t,s)\,ds\,d\sigma$$

THEOREM 3.6.

Suppose the condition as follows:

(1) $k,k_1,f \in C(I \times \mathbb{R};\mathbb{R})$

(2) $|k(t,x_1) - k(t,x_2)| \leq L_k|x_1 - x_2|$, $\left|\int_0^t \int_0^t f_1(s,\tau)\,d\tau\,d\sigma\right| \leq t^2M$,

, $|k_t(t,x_1) - k_t(t,x_2)| \leq L_{k_t}|x_1 - x_2|$,

, $|k_1(t,x_1) - k_1(t,x_2)| \leq L_{k_1}|x_1 - x_2|$

, $\left|\int_0^t k_2(t,s)\,ds\right| \leq \int_0^t |k_2(t,s)|\,ds \leq \int_0^t M\,ds \leq Mt$

And $|f(t,x_1) - f(t,x_2)| \leq L_f|x_1 - x_2|$

For some $L_k, L_{k_t} L_{k_1}, L_f > 0$ and for each $t \in I$,

(3) $g_k \in C([t_k,s_k] \times \mathbb{R}; \mathbb{R})$ and for k= 1,…,m a positive constant L_{g_k} satisfy

$|g_k(t,x_1) - g_k(t,x_2)| \le L_k|x_1 - x_2|$, for $t \in [t_k,s_k]$ and all $x_1,x_2 \in \mathbb{R}$.

(4) A nondecreasing function $\varphi \in C(I,\mathbb{R}_+)$. Then

$\int_0^t \varphi\left(\int_0^t m(\tau)\, d\tau\right) ds \le c_\varphi \varphi(m(t))$, for each $t \in I$,

And

$\int_0^t \int_0^t \hat{\varphi}\left(\int_0^t m(s,\tau)\, ds\right) d\tau d\sigma \le \hat{c}_\varphi \hat{\varphi}(m(t,t))$, for each $t \in I$, and $c_\varphi > 0$.

Let $\hat{c}_\varphi \hat{\varphi}(\widetilde{m}(t,t)) \le c_\varphi \varphi(m(t))$

If y satisfing (3) the unique function as follows:

$$y_\circ(t) = \begin{cases} x(0) - \int_0^t k(t,y_\circ(s)) f_1(s,t)\, ds - \int_0^t \int_0^t k_t(t,y_\circ(s)) f_1(s,\tau)\, d\tau\, d\sigma \\ + \int_0^t f(t,y_\circ(t))\, ds + \int_0^t \int_0^t k_1(t,y_\circ(s)) ds \int_0^t k_2(t,s)\, ds d\tau,\ t \in [0,t_1], \\ g_k(t,y_\circ(t_k^+)),\ t \in (t_k,s_k], \qquad k = 1,\ldots,m, \\ g_k(s_k,y_\circ(t_k^+)) - \int_{s_k}^t k(t,y_\circ(s)) f_1(s,t)\, ds - \int_{s_k}^t \int_{s_k}^t k_t(t,y_\circ(s)) f_1(s,\tau)\, d\tau\, d\sigma \\ + \int_{s_k}^t f(t,y_\circ(t))\, ds + \int_{s_k}^t \int_{s_k}^t k_1(t,y_\circ(s)) ds \int_{s_k}^t k_2(t,s)\, ds\, d\tau,\ t \in (s_k,t_{k+1}], k = 1,\ldots,m, \end{cases} \tag{9}$$

and $|y(t) - y_\circ(t)|^\beta \le \frac{(1+c_\varphi^\beta)(\varphi^\beta(m(t))+\psi^\beta)}{1-\rho}$ (10)

where $\rho := max\left\{L_{g_k}^\beta + \left(L_k^\beta (t_1 M)^\beta + L_{k_k}^\beta (t_1^2 M)^\beta + L_f^\beta + L_{k_1}^\beta (t_1^2 M)^\beta\right) c_\varphi^\beta | k = 1,\ldots,m\right\} < 1.$ (11)

Proof.

Let $\Gamma: x \to x$ an operator defined by

$$(\Gamma x)(t) = \begin{cases} x(0) - \int_0^t k(t,x(s)) f_1(s,t)\, ds - \int_0^t \int_0^t k_t(t,x(s)) f_1(s,\tau)\, d\tau\, d\sigma \\ + \int_0^t f(t,x(t))\, ds + \int_0^t \int_0^t k_1(t,x(s)) ds \int_0^t k_2(t,s)\, ds d\tau, t \in [0,t_1], \\ g_k(t,x(t_k^+)),\ t \in (t_k,s_k], \qquad k = 1,\ldots,m, \\ g_k(s_k,x(t_k^+)) - \int_{s_k}^t k(t,x(s)) f_1(s,t)\, ds - \int_{s_k}^t \int_{s_k}^t k_t(t,x(s)) f_1(s,\tau)\, d\tau\, d\sigma \\ + \int_{s_k}^t f(t,x(t))\, ds + \int_{s_k}^t \int_{s_k}^t k_1(t,x(s)) ds \int_0^t k_2(t,s)\, ds\, d\tau, t \in (s_k,t_{k+1}], k = 1,\ldots,m, \end{cases} \tag{12}$$

where $x \in X$ and $t \in [0,T]$.

The operator $\Gamma \in C(I \times R,R)$,since $k_1, f, k : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $k_2, f_1 : [0,T] \times [0,T] \longrightarrow \mathbb{R}$,for any $\hat{g},\hat{h} \in x$; and $C_1,C_2 \in [0,\infty]$, we have that

$$|\hat{g}(t) - \hat{h}(t)|^\beta \le \begin{cases} C_1 \varphi^\beta(m(t)), t \in (s_k,t_{k+1}], k = 0,1,\ldots,m, \\ C_2 \psi^\beta, t \in (t_k,s_k], k = 1,\ldots,m, \end{cases} \tag{13}$$

It is easy to see that (13) is equivalent to

$$|\hat{g}(t) - \hat{h}(t)| \le \begin{cases} C_1^{\frac{1}{\beta}} \varphi(m(t)), t \in (s_k,t_{k+1}], k = 0,1,\ldots,m, \\ C_2^{\frac{1}{\beta}} \psi, \quad t \in (t_k,s_k], k = 1,\ldots,m \end{cases} \tag{14}$$

By the definition of Γ in (12) , (2),(3) and (14) we have the following

Case 1. $t \in [0,t_1]$, we get that

$$\left|(\Gamma\hat{g})(t)-(\Gamma\hat{h})(t)\right|^{\beta}$$
$$=\Bigg|x(0)-\int_0^t k(t,\hat{g}(s))f_1(s,t)\,ds-\int_0^t\int_0^t k_t(t,\hat{g}(s))f_1(s,\tau)\,d\tau\,d\sigma$$
$$+\int_0^t f(t,\hat{g}(t))\,ds+\int_0^t\int_0^t k_1(t,\hat{g}(s))\,ds\int_0^t k_2(t,s)\,ds\,d\tau-x(0)+\int_0^t k\left(t,\hat{h}(s)\right)f_1(s,t)\,ds$$
$$+\int_0^t\int_0^t k_t\left(t,\hat{h}(s)\right)f_1(s,\tau)\,d\tau\,d\sigma-\int_0^t f\left(t,\hat{h}(t)\right)ds-\int_0^t\int_0^t k_1\left(t,\hat{h}(s)\right)ds\int_0^t k_2(t,s)\,ds\,d\tau\Bigg|^{\beta}$$
$$\leq\Bigg(\left|\int_0^t\left(k\left(t,\hat{h}(s)\right)-k(t,\hat{g}(s))\right)f_1(s,t)\,ds\right|+\left|\int_0^t\int_0^t\left(k_t\left(t,\hat{h}(s)\right)-k_t(t,\hat{g}(s))\right)f_1(s,\tau)\,d\tau\,d\sigma\right|$$
$$+\left|\int_0^t\left(f(t,\hat{g}(t))-f\left(t,\hat{h}(t)\right)\right)ds\right|+\left|\int_0^t k_2(t,s)\,ds\right|\left|\int_0^t\int_0^t\left(k_1(t,\hat{g}(s))-k_1\left(t,\hat{h}(s)\right)\right)ds\,d\tau\right|\Bigg)^{\beta}$$
$$\leq\Bigg((t_1M)L_k\int_0^t\left|\hat{h}(s)-\hat{g}(s)\right|ds+(t_1^2M)L_{k_t}\int_0^t\int_0^t\left|\hat{h}(s)-\hat{g}(s)\right|d\tau\,d\sigma+L_f\int_0^t\left|\hat{g}(s)-\hat{h}(s)\right|ds$$
$$+(t_1M)L_{k_1}\int_0^t\int_0^t\left|\hat{g}(s)-\hat{h}(s)\right|ds\,d\tau\Bigg)^{\beta}$$
$$\leq L_k^{\beta}(t_1M)^{\beta}\left[\int_0^t\left|\hat{h}(s)-\hat{g}(s)\right|ds\right]^{\beta}+L_{k_t}^{\beta}(t_1^2M)^{\beta}\left[\int_0^t\int_0^t\left|\hat{h}(s)-\hat{g}(s)\right|d\tau\,d\sigma\right]^{\beta}+L_f^{\beta}\left[\int_0^t\left|\hat{g}(s)-\hat{h}(s)\right|ds\right]^{\beta}+L_{k_1}^{\beta}(t_1^2M)^{\beta}\left[\int_0^t\int_0^t\left|\hat{g}(s)-\hat{h}(s)\right|ds\,d\tau\right]^{\beta}$$
$$\leq L_k^{\beta}(t_1M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_0^t\varphi\left(\int_0^t m(\tau)\,d\tau\right)ds\right]^{\beta}+$$
$$L_{k_t}^{\beta}(t_1^2M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_0^t\int_0^t\hat{\varphi}\left(\int_0^t m(s,\tau)\,ds\right)d\tau d\sigma\right]^{\beta}+L_f^{\beta}\left[C_1^{\frac{1}{\beta}}\int_0^t\varphi\left(\int_0^t m(\tau)\,d\tau\right)ds\right]^{\beta}+$$
$$L_{k_1}^{\beta}(t_1^2M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_0^t\int_0^t\hat{\varphi}\left(\int_0^t m(s,\tau)\,ds\right)d\tau d\sigma\right]^{\beta}$$
$$\leq L_k^{\beta}(t_1M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))+L_{k_t}^{\beta}(t_1^2M)^{\beta}C_1\hat{c}_{\varphi}^{\beta}\hat{\varphi}(m(t,t))+L_f^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))+L_{k_1}^{\beta}(t_1^2M)^{\beta}C_1\hat{c}_{\varphi}^{\beta}\hat{\varphi}(m(t,t))$$
$$\leq L_k^{\beta}(t_1M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))+L_{k_t}^{\beta}(t_1^2M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))+L_f^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))+L_{k_1}^{\beta}(t_1^2M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))$$
$$\leq\left(L_k^{\beta}(t_1M)^{\beta}+L_{k_t}^{\beta}(t_1^2M)^{\beta}+L_f^{\beta}+L_{k_1}^{\beta}(t_1^2M)^{\beta}\right)C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))$$

Case 2. *For* $t\in(t_k,s_k]$

$$\left|(\Gamma\hat{g})(t)-(\Gamma\hat{h})(t)\right|^{\beta}=\left|g_k(t,\hat{g}(t_k^+))-g_k\left(t,\hat{h}(t_k^+)\right)\right|^{\beta}$$
$$\leq\left(L_{g_k}\left|\hat{g}(t_k^+)-\hat{h}(t_k^+)\right|\right)^{\beta}$$
$$\leq L_{g_k}^{\beta}\left|\hat{g}(t_k^+)-\hat{h}(t_k^+)\right|^{\beta}$$
$$\leq L_{g_k}^{\beta}\left(C_2^{\frac{1}{\beta}}\psi\right)^{\beta}$$
$$\leq L_{g_k}^{\beta}C_2\psi^{\beta}$$

Case 3. For $t\in(s_k,t_{k+1}]$, $\left|(\Gamma\hat{g})(t)-(\Gamma\hat{h})(t)\right|^{\beta}=\Big|g_k(s_k,\hat{g}(t_k^+))-\int_{s_k}^t k(t,\hat{g}(s))f_1(s,t)\,ds\,ds-\int_{s_k}^t\int_{s_k}^t k_t(t,\hat{g}(s))f_1(s,\tau)\,d\tau\,d\sigma+\int_{s_k}^t f(t,\hat{g}(t))\,ds+\int_{s_k}^t\int_{s_k}^t k_1(t,\hat{g}(s))\,ds\int_{s_k}^t k_2(t,s)\,ds\,d\tau-g_k\left(s_k,\hat{h}(t_k^+)\right)+\int_{s_k}^t k\left(t,\hat{h}(s)\right)f_1(s,t)\,ds\,ds+\int_{s_k}^t\int_{s_k}^t k_t\left(t,\hat{h}(s)\right)f_1(s,\tau)\,d\tau\,d\sigma-\int_{s_k}^t f\left(t,\hat{h}(t)\right)ds\,ds-\int_{s_k}^t\int_{s_k}^t k_1\left(t,\hat{h}(s)\right)ds\int_{s_k}^t k_2(t,s)\,ds\,d\tau\Big|^{\beta}$

$$\leq\Bigg(\left|g_k(s_k,\hat{g}(t_k^+))-g_k\left(s_k,\hat{h}(t_k^+)\right)\right|+\left|\int_{s_k}^t\left(k\left(t,\hat{h}(s)\right)-k(t,\hat{g}(s))\right)f_1(s,t)\,ds\right|$$
$$+\left|\int_{s_k}^t\int_{s_k}^t\left(k_t\left(t,\hat{h}(s)\right)-k_t(t,\hat{g}(s))\right)f_1(s,\tau)\,d\tau\,d\sigma\right|+\left|\int_{s_k}^t\left(f(t,\hat{g}(t))-f\left(t,\hat{h}(t)\right)\right)ds\right|$$
$$+\left|\int_{s_k}^t k_2(t,s)\,ds\right|\left|\int_{s_k}^t\int_{s_k}^t\left(k_1(t,\hat{g}(s))-k_1\left(t,\hat{h}(s)\right)\right)ds\,d\tau\right|\Bigg)^{\beta}$$

$$\leq \Big(L_{g_k}\big|\hat{g}(t_k^+) - \hat{h}(t_k^+)\big| + (t_1M)L_k\int_{s_k}^{t}\big|\hat{h}(s) - \hat{g}(s)\big|\,ds + (t_1^2M)L_{k_t}\int_{s_k}^{t}\int_{s_k}^{t}\big|\hat{h}(s) - \hat{g}(s)\big|\,ds + L_f\int_{s_k}^{t}\big|\hat{g}(s) - \hat{h}(s)\big|\,ds + (t_1M)L_{k_1}\int_{s_k}^{t}\int_{s_k}^{t}\big|\hat{g}(s) - \hat{h}(s)\big|\,ds\,d\tau\Big)^{\beta}$$

$$\leq L_{g_k}^{\beta}\big|\hat{g}(t_k^+) - \hat{h}(t_k^+)\big|^{\beta} + L_k^{\beta}(t_1M)^{\beta}\left[\int_{s_k}^{t}\big|\hat{h}(s) - \hat{g}(s)\big|\,ds\right]^{\beta} + L_{k_t}^{\beta}(t_1^2M)^{\beta}\left[\int_{s_k}^{t}\int_{s_k}^{t}\big|\hat{h}(s) - \hat{g}(s)\big|\,ds\right]^{\beta} + L_f^{\beta}\left[\int_{s_k}^{t}\big|\hat{g}(s) - \hat{h}(s)\big|\,ds\right]^{\beta} + L_{k_1}^{\beta}(t_1^2M)^{\beta}\left[\int_{s_k}^{t}\int_{s_k}^{t}\big|\hat{g}(s) - \hat{h}(s)\big|\,ds\,d\tau\right]^{\beta}$$

$$\leq L_{g_k}^{\beta}\left(C_2^{\frac{1}{\beta}}\psi\right)^{\beta} + L_k^{\beta}(t_1M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_{s_k}^{t}\varphi\left(\int_{s_k}^{t}m(\tau)\,d\tau\right)ds\right]^{\beta} +$$

$$L_{k_t}^{\beta}(t_1^2M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_{s_k}^{t}\int_{s_k}^{t}\hat{\varphi}\left(\int_{s_k}^{t}m(s,\tau)\,ds\right)d\tau d\sigma\right]^{\beta} + L_f^{\beta}\left[C_1^{\frac{1}{\beta}}\int_{s_k}^{t}\varphi\left(\int_{s_k}^{t}m(\tau)\,d\tau\right)ds\right]^{\beta} +$$

$$L_{k_1}^{\beta}(t_1^2M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_{s_k}^{t}\int_{s_k}^{t}\hat{\varphi}\left(\int_{s_k}^{t}m(s,\tau)\,ds\right)d\tau d\sigma\right]^{\beta}$$

$$\leq L_{g_k}^{\beta}C_2\psi^{\beta} + L_k^{\beta}(t_1M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}\big(m(t)\big) + L_{k_t}^{\beta}(t_1^2M)^{\beta}C_1\hat{c}_{\varphi}^{\beta}\hat{\varphi}\big(m(t,t)\big) + L_f^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}\big(m(t)\big) + L_{k_1}^{\beta}(t_1^2M)^{\beta}C_1\hat{c}_{\varphi}^{\beta}\hat{\varphi}\big(m(t,t)\big)$$

$$\leq L_{g_k}^{\beta}C_2\psi^{\beta} + L_k^{\beta}(t_1M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}\big(m(t)\big) + L_{k_t}^{\beta}(t_1^2M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}\big(m(t)\big) + L_f^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}\big(m(t)\big) + L_{k_1}^{\beta}(t_1^2M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}\big(m(t)\big)$$

$$\leq \big(L_{g_k}^{\beta} + \big(L_k^{\beta}(t_1M)^{\beta} + L_{k_t}^{\beta}(t_1^2M)^{\beta} + L_f^{\beta} + L_{k_1}^{\beta}(t_1^2M)^{\beta}\big)c_{\varphi}^{\beta}\big)(C_1 + C_2)\big(\varphi^{\beta}(m(t)) + \psi^{\beta}\big).$$

we have that,

$$\big|(\Gamma\hat{g})(t) - (\Gamma\hat{h})(t)\big|^{\beta} \leq max\left\{L_{g_k}^{\beta} + \left(L_k^{\beta}(t_1M)^{\beta} + L_{k_k}^{\beta}(t_1^2M)^{\beta} + L_f^{\beta} + L_{k_1}^{\beta}(t_1^2M)^{\beta}\right)c_{\varphi}^{\beta}|k = 1,\ldots,m\right\}(C_1 + C_2)\big(\varphi^{\beta}(m(t)) + \psi^{\beta}\big)$$ *For* $t \in I$ *. we know that,*

$$d\big(\Gamma\hat{g},\Gamma\hat{h}\big) \leq \rho(C_1 + C_2)\big(\varphi^{\beta}(m(t)) + \psi^{\beta}\big)$$

Hence ,we derive $g\big(\Gamma\hat{g},\Gamma\hat{h}\big) \leq \rho d\big(\hat{g},\hat{h}\big)$

from condition (11) and $\hat{g},\hat{h} \in X$ and from continuous property of g_0 and $\wedge g_0$ we get for $0 < G_1 < \infty$ as the following:

follows that there exists a constant $0 < \hat{G}_1 < \infty$ such that

$$y' = -k\big(t,x(s)\big)f_1(s,t) - \int_0^t k_t\big(t,x(s)\big)f_1(s,\tau)\,d\tau + f\big(t,x(t)\big) + \int_0^t k_1\big(t,x(s)\big)\,ds\int_0^t k_2(t,s)\,ds$$

$$\int_0^t y'(s)\,ds = -\int_0^t k\big(t,x(s)\big)f_1(s,t)\,ds - \int_0^t\int_0^t k_t\big(t,x(s)\big)f_1(s,\tau)\,d\tau + + \int_0^t f\big(t,x(t)\big)\,ds + \int_0^t\int_0^t k_1\big(t,x(s)\big)\,ds\int_0^t k_2(t,s)\,ds\,d\sigma$$

$$= \left|\int_0^t\left(y'(s) + k\big(t,x(s)\big)f_1(s,t) + \int_0^t k_t\big(t,x(s)\big)f_1(s,\tau)\,d\tau - f\big(t,x(t)\big) + \int_0^t k_1\big(t,x(s)\big)\,ds\int_0^t k_2(t,s)\,ds\right)ds\right|^{\beta}$$

$$\leq \left|\int_0^t\varphi\left(\int_0^t m(\tau)\,d\tau\right)\right|^{\beta} ds \leq \hat{G}_1\varphi^{\beta}\big(m(t)\big) \leq \hat{G}_1\left(\varphi^{\beta}\big(m(t) + \psi^{\beta}\big)\right)$$

then

$$|(\Gamma\hat{g}_{\circ})(t) - \hat{g}_{\circ}(t)|^{\beta} =$$

$$\left|x(0) - \int_0^t k\big(t,\hat{g}_{\circ}(s)\big)f_1(s,t)\,ds - \int_0^t\int_0^t k_t\big(t,\hat{g}_{\circ}(s)\big)f_1(s,\tau)\,d\tau + \int_0^t f\big(t,\hat{g}_{\circ}(t)\big)\,ds + \int_0^t\int_0^t k_1\big(t,\hat{g}_{\circ}(s)\big)\,ds\int_0^t k_2(t,s)\,ds\,d\sigma - \hat{g}_{\circ}(t)\right|^{\beta}$$

$$\leq \hat{G}_1\varphi^{\beta}\big(m(t)\big) \leq \hat{G}_1\left(\varphi^{\beta}\big(m(t) + \psi^{\beta}\big)\right)$$

There exists a constant $0 < \hat{G}_2 < \infty$ such that

$$|(\Gamma\hat{g}_{\circ})(t) - \hat{g}_{\circ}(t)|^{\beta} = \big|g_k\big(t,\hat{g}_{\circ}(t_k^+)\big) - \hat{g}_{\circ}(t)\big|^{\beta}$$

$$\leq \hat{G}_2\psi^{\beta} \leq \hat{G}_2\big(\varphi^{\beta}\big(m(t)\big) + \psi^{\beta}\big), t \in (t_k,s_k],\ k = 1,\ldots,m.$$

There exists a constant $0 < \hat{G}_3 < \infty$ such that

$$|(\Gamma\hat{g}_\circ)(t)-\hat{g}_\circ(t)|^\beta$$
$$=\left|g_k\big(t,\hat{g}_\circ(t_k^+)\big)-\int_{s_k}^{t}k\big(t,\hat{g}_\circ(s)\big)f_1(s,t)\,ds-\int_{s_k}^{t}\int_{s_k}^{t}k_t\big(t,\hat{g}_\circ(s)\big)f_1(s,\tau)\,d\tau+\int_{s_k}^{t}f\big(t,\hat{g}_\circ(t)\big)\,ds\right.$$
$$\left.+\int_{s_k}^{t}\int_{s_k}^{t}k_1\big(t,\hat{g}_\circ(s)\big)\,ds\int_{s_k}^{t}k_2(t,s)\,ds\,d\sigma-\hat{g}_\circ(t)\right|^\beta$$
$\leq \hat{G}_3\big(\varphi^\beta\big(m(t)\big)+\psi^\beta\big)$, $t\in(s_k,t_{k+1}]$, $k=1,\dots,m$.
from continuous function $y_\circ:I\longrightarrow\mathbb{R}$ such that $\Gamma^n g_\circ\longrightarrow y_\circ$, $\Gamma\, y_\circ=y_\circ$ then $y_\circ$ satisfies equation (9) for $t\in I$.
from equation (4) and condition (4) and (9) is the property have been needed in this prove, we have a unique continuous function as follows:

$$d(y,\Gamma y)\leq 1+c_\varphi^\beta. \tag{15}$$

Thus, we derive

$d(y,y_\circ)\leq\frac{d(\Gamma y,y)}{1-\rho}\leq\frac{1+c_\varphi^\beta}{1-\rho}$,that means (10) is true for $t\in I$.

Example 3.7.

Consider the following impulsive multi-integro- differential perturbed with integral function nonlinear equation

$$\begin{cases}\dfrac{d}{dt}\left(x(t)+\displaystyle\int_0^t\frac{|x(s)|}{35+e^t}(s+\tau)\,d\tau\right)=\dfrac{|x(t)|}{24+e^t}\displaystyle\int_0^t\left(\frac{|x(s)|}{37+e^t}\right)\int_0^t(ts)\,ds\,,\ t\in(0,1]\\ x(t)=\dfrac{|x(1^+)|}{(34+e^{t-1})(1+|x(1^+)|)}\qquad\qquad ,\ t\in(1,2]\ ,\end{cases}$$

and

$$\begin{cases}\left|\dfrac{d}{dt}\left(y(t)+\displaystyle\int_0^t\frac{|y(s)|}{35+e^t}(s+\tau)\,d\tau\right)-\dfrac{|y(t)|}{24+e^t}\displaystyle\int_0^t\left(\frac{|y(s)|}{37+e^t}\right)\int_0^t(ts)\,ds\right|\leq\displaystyle\int_0^t e^t\,dt\\ \qquad ,\ t\in(0,1]\\ \left|y(t)-\dfrac{|y(1^+)|}{(34+e^{t-1})(1+|y(1^+)|)}\right|\leq 1\qquad\qquad ,\ t\in(1,2]\ ,\end{cases}$$

Let $\beta=\frac{1}{2}$, $T=2$, $J=[0,2]$ and $0=t_0=s_0<t_1=1\leq s_1=2$,also

$k\big(t,x(s)\big)=\frac{|x(s)|}{35+e^t}$, $f_1(s,\tau)=s+\tau$

$f\big(t,x(t)\big)=\frac{|x(t)|}{24+e^t}$, $k_1\big(t,x(s)\big)=\frac{|x(s)|}{37+e^t}$, $k_2(t,s)=ts$
and $g_k\big(t,x(t_k^+)\big)=\frac{|x(1^+)|}{(34+e^{t-1})(1+|x(1^+)|)}$
also $L_k=\frac{1}{36}$, $L_f=\frac{1}{25}$ and $L_{g_k}=\frac{1}{35}$, $L_{k_1}=\frac{1}{38}$
let $\varphi(t)=\varphi\left(\int_0^t m(\tau)\,d\tau\right)=\int_0^t e^\tau\,d\tau$ *and* $\psi=1$.
Then $\int_0^t\left(\int_0^t e^\tau\,d\tau\right)dt=\int_0^t(e^t-1)\,ds$
$\qquad\leq\int_0^t e^t\,dt=c_\varphi\varphi\big(m(t)\big)$
Therefore $c_\varphi=1$
$\rho:=max\left\{L_{g_k}^\beta+\left(L_k^\beta(t_1M)^\beta+L_{k_t}^\beta(t_1^2M)^\beta+L_f^\beta+L_{k_1}^\beta(t_1M)^\beta\right)c_\varphi^\beta|k=1,\dots,m\right\}$
$\quad=max\left\{\left(\frac{1}{35}\right)^{\frac{1}{2}}+\left(\left(\frac{1}{38}\right)^{\frac{1}{2}}(t_1^+M)^{\frac{1}{2}}+\frac{1}{36}\left(t_1^{+2}M\right)^\beta+\frac{1}{25}\right)\right\}=0.3375$
$|y(t)-y_\circ(t)|^\beta\leq\frac{\left(1+c_\varphi^\beta\right)\left(\varphi^\beta(m(t))+\psi^\beta\right)}{1-\rho}$

$\leq\frac{2\left((e^t-1)^{\frac{1}{2}}+1\right)}{1-0.3375}$, for $t\in[0,2]$

4. EXTEENSION PROBLEM FORMULATION

$$\begin{cases} \frac{d}{dt}\left(x(t)+\int_0^t k\big(t,x(s)\big)f_1(s,\tau)\,d\tau\right)=ax(t)+f\big(t,x(t)\big) \\ +\int_0^t k_1\big(t,x(s)\big)\,ds\int_0^t k_2(t,s)\,ds\,,\ t\in(t_k,s_{k+1}]\ ,k=0,1,\dots,m, \\ x(t)=g_k\big(t,x(t_k^+)\big) \qquad ,\ t\in(t_k,s_k]\ ,k=1,\dots,m, \\ x(0)=x_\circ\ \in\mathbb{R}. \end{cases} \tag{16}$$

The classical solution $x\in PC(I,\mathbb{R})\cap_{k=0}^{m} C^1\big((s_k,t_{k+1}],\mathbb{R}\big)$ defined in (16) satisfies

If x satisfies $x(0)=x_\circ$

$x(t)=g_k\big(t,x(t_k^+)\big),\ t\in(t_k,s_k]\ ,k=1,\dots,m,$

$$x(t)=e^{at}\,x(0)-\int_0^t e^{a(t-s)}k\big(t,x(s)\big)f_1(s,t)\,ds-\int_0^t\int_0^t e^{a(t-s)}k_t\big(t,x(s)\big)f_1(s,\tau)\,d\tau\,d\sigma+\int_0^t e^{a(t-s)}f\big(t,x(t)\big)\,ds + \int_0^t\int_0^t e^{a(t-s)}k_1\big(t,x(s)\big)\,ds\int_0^t k_2(t,s)\,ds\,d\sigma$$

$t\in[0,t_1]$,

$x(t)\ =e^{a(t-s_k)}g_k\big(s_k,x(s_k^+)\big)-\int_0^t e^{a(t-s)}k\big(t,x(s)\big)f_1(s,t)\,ds-\int_0^t\int_0^t e^{a(t-s)}k_t\big(t,x(s)\big)f_1(s,\tau)\,d\tau\,d\sigma+\int_0^t e^{a(t-s)}f\big(t,x(t)\big)\,ds+\int_0^t\int_0^t e^{a(t-s)}k_1\big(t,x(s)\big)\,ds\int_0^t k_2(t,s)\,ds\,d\sigma,(s_k,t_{k+1}],\ k=1,\dots,m,$

THEOREM 4.8.

From condition (1),(4) and equation (6) .Then the unique solution $y_\circ: I\longrightarrow\mathbb{R}$ such that

$$y_\circ(t)=\begin{cases} e^{at}x(0)-\int_0^t e^{a(t-s)}k\big(t,y_\circ(s)\big)f_1(s,t)\,ds\ -\int_0^t\int_0^t e^{a(t-s)}k_t\big(t,y_\circ(s)\big)f_1(s,\tau)\,d\tau\,d\sigma \\ +\int_0^t e^{a(t-s)}f\big(t,y_\circ(t)\big)\,ds+\int_0^t\int_0^t e^{a(t-s)}k_1\big(t,y_\circ(s)\big)ds\int_0^t k_2(t,s)\,dsd\tau\,,\ t\in[0,t_1] \\ g_k\big(t,y_\circ(t_k^+)\big) \qquad ,\ t\in(t_k,s_k] \\ e^{a(t-s_k)}g_k\big(s_k,y_\circ(t_k^+)\big)-\int_{s_k}^t e^{a(t-s)}k\big(t,y_\circ(s)\big)f_1(s,t)\,ds \\ -\int_{s_k}^t\int_{s_k}^t e^{a(t-s)}k_t\big(t,y_\circ(s)\big)f_1(s,\tau)\,d\tau\,d\sigma+\int_{s_k}^t e^{a(t-s)}f\big(t,y_\circ(t)\big)\,ds \\ +\int_{s_k}^t\int_{s_k}^t e^{a(t-s)}k_1\big(t,y_\circ(s)\big)ds\int_{s_k}^t k_2(t,s)\,ds\,d\tau \qquad ,t\in(s_k,t_{k+1}],k=1,\dots,m, \end{cases} \tag{17}$$

and $\qquad |y(t)-y_\circ(t)|^\beta\le\frac{e^{a\beta T}\left(1+c_\varphi^\beta\right)\left(\varphi^\beta(m(t))+\psi^\beta\right)}{1-\rho}\,,t\in I\,, \qquad (18)$

provided that

$$\rho_a=e^{a\beta T}\max\left\{\left(L_{g_k}^\beta+\left(L_k^\beta(t_1M)^\beta+L_{k_k}^\beta(t_1^2M)^\beta+L_f^\beta+L_{k_1}^\beta(t_1^2M)^\beta\right)c_\varphi^\beta\right)|k=1,\dots,m\right\}<1. \tag{19}$$

Proof.

We define an operator $\Gamma_a\!: x\longrightarrow x$ by

$$(\Gamma_a x)(t)=\begin{cases} e^{at}x(0)-\int_0^t e^{a(t-s)}k\big(t,y_\circ(s)\big)f_1(s,t)ds-\int_0^t\int_0^t e^{a(t-s)}k_t\big(t,y_\circ(s)\big)f_1(s,\tau)\,d\tau\,d\sigma \\ +\int_0^t e^{a(t-s)}f\big(t,y_\circ(t)\big)\,ds+\int_0^t\int_0^t e^{a(t-s)}k_1\big(t,y_\circ(s)\big)ds\int_0^t k_2(t,s)\,dsd\tau\ t\in[0,t_1] \\ g_k\big(t,y_\circ(t_k^+)\big) \qquad ,\quad t\in(t_k,s_k],\ k=1,\dots,m, \\ e^{a(t-s_k)}g_k\big(s_k,y_\circ(t_k^+)\big)-\int_{s_k}^t e^{a(t-s)}k\big(t,y_\circ(s)\big)f_1(s,t)\,ds \\ -\int_{s_k}^t\int_{s_k}^t e^{a(t-s)}k_t\big(t,y_\circ(s)\big)f_1(s,\tau)\,d\tau\,d\sigma+\int_{s_k}^t e^{a(t-s)}f\big(t,y_\circ(t)\big)\,ds \\ +\int_{s_k}^t\int_{s_k}^t e^{a(t-s)}k_1\big(t,y_\circ(s)\big)ds\int_{s_k}^t k_2(t,s)\,ds\,d\tau \qquad ,t\in(s_k,t_{k+1}],\ k=1,\dots,m. \end{cases} \tag{20}$$

From (1) , Γ_a is a well-defined to show that Γ_a is strictly contractive on X.

Case 1. For $t\ \in[0,t_1]$,

$$\begin{aligned}
&\left|(\Gamma\hat{g})(t)-(\Gamma\hat{h})(t)\right|^{\beta} \\
&\quad=\left|e^{at}x(0)-\int_0^t e^{a(t-s)}k(t,\hat{g}(s))f_1(s,t)\,ds-\int_0^t\int_0^t e^{a(t-s)}k_t(t,\hat{g}(s))f_1(s,\tau)\,d\tau\,d\sigma\right. \\
&\quad+\int_0^t e^{a(t-s)}f(t,\hat{g}(t))\,ds+\int_0^t\int_0^t e^{a(t-s)}k_1(t,\hat{g}(s))\,ds\int_0^t k_2(t,s)\,ds\,d\tau-e^{at}x(0) \\
&\quad+\int_0^t e^{a(t-s)}k\left(t,\hat{h}(s)\right)f_1(s,t)\,ds+\int_0^t\int_0^t e^{a(t-s)}k_t\left(t,\hat{h}(s)\right)f_1(s,\tau)\,d\tau\,d\sigma \\
&\quad\left.-\int_0^t e^{a(t-s)}f\left(t,\hat{h}(t)\right)ds-\int_0^t\int_0^t e^{a(t-s)}k_1\left(t,\hat{h}(s)\right)ds\int_0^t k_2(t,s)\,ds\,d\tau\right|^{\beta} \\
&\le\left(\left|\int_0^t e^{a(t-s)}\left(k\left(t,\hat{h}(s)\right)-k(t,\hat{g}(s))\right)f_1(s,t)\,ds\right|+\left|\int_0^t\int_0^t e^{a(t-s)}\left(k_t\left(t,\hat{h}(s)\right)-k_t(t,\hat{g}(s))\right)f_1(s,\tau)\,d\tau\,d\sigma\right|\right. \\
&\quad+\left|\int_0^t e^{a(t-s)}\left(f(t,\hat{g}(t))-f\left(t,\hat{h}(t)\right)\right)ds\right| \\
&\quad\left.+\left|\int_0^t e^{a(t-s)}k_2(t,s)\,ds\right|\left|\int_0^t\int_0^t\left(k_1(t,\hat{g}(s))-k_1\left(t,\hat{h}(s)\right)\right)ds\,d\tau\right|\right)^{\beta} \\
&\le\left(e^{a(t-s)}(t_1M)L_k\int_0^t\left|\hat{h}(s)-\hat{g}(s)\right|ds+e^{a(t-s)}(t_1^2M)L_{k_t}\int_0^t\int_0^t\left|\hat{h}(s)-\hat{g}(s)\right|ds+e^{a(t-s)}L_f\int_0^t\left|\hat{g}(s)-\hat{h}(s)\right|ds\right. \\
&\quad\left.+e^{a(t-s)}(t_1M)L_{k_1}\int_0^t\int_0^t\left|\hat{g}(s)-\hat{h}(s)\right|ds\,d\tau\right)^{\beta} \\
&\le e^{a\beta(t-s)}L_k^{\beta}(t_1M)^{\beta}\left[\int_0^t\left|\hat{h}(s)-\hat{g}(s)\right|ds\right]^{\beta}+e^{a\beta(t-s)}L_{k_t}^{\beta}(t_1^2M)^{\beta}\left[\int_0^t\int_0^t\left|\hat{h}(s)-\hat{g}(s)\right|ds\right]^{\beta}+e^{a\beta(t-s)}L_f^{\beta}\left[\int_0^t\left|\hat{g}(s)-\right.\right. \\
&\quad\left.\left.\hat{h}(s)\right|ds\right]^{\beta}+e^{a\beta(t-s)}L_{k_1}^{\beta}(t_1^2M)^{\beta}\left[\int_0^t\int_0^t\left|\hat{g}(s)-\hat{h}(s)\right|ds\,d\tau\right]^{\beta} \\
&\le e^{a\beta(t-s)}L_k^{\beta}(t_1M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_0^t\varphi\left(\int_0^t m(\tau)\,d\tau\right)ds\right]^{\beta} \\
&\quad+e^{a\beta(t-s)}L_{k_t}^{\beta}(t_1^2M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_0^t\int_0^t\hat{\varphi}\left(\int_0^t m(s,\tau)\,ds\right)d\tau d\sigma\right]^{\beta}e^{a\beta(t-s)}L_f^{\beta}\left[C_1^{\frac{1}{\beta}}\int_0^t\varphi\left(\int_0^t m(\tau)\,d\tau\right)ds\right]^{\beta} \\
&\quad+e^{a\beta(t-s)}L_{k_1}^{\beta}(t_1^2M)^{\beta}\left[C_1^{\frac{1}{\beta}}\int_0^t\int_0^t\hat{\varphi}\left(\int_0^t m(s,\tau)\,ds\right)d\tau d\sigma\right]^{\beta} \\
&\le e^{a\beta(t-s)}L_k^{\beta}(t_1M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))+e^{a\beta(t-s)}L_{k_t}^{\beta}(t_1^2M)^{\beta}C_1\hat{c}_{\varphi}^{\beta}\hat{\varphi}(m(t,t))+e^{a\beta(t-s)}L_f^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t)) \\
&\quad+e^{a\beta(t-s)}L_{k_1}^{\beta}(t_1^2M)^{\beta}C_1\hat{c}_{\varphi}^{\beta}\hat{\varphi}(m(t,t)) \\
&\le e^{a\beta(t-s)}L_k^{\beta}(t_1M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))+e^{a\beta(t-s)}(t_1^2M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))+e^{a\beta(t-s)}L_f^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t)) \\
&\quad+e^{a\beta(t-s)}L_{k_1}^{\beta}(t_1^2M)^{\beta}C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t)) \\
&\le e^{a\beta(t-s)}\left(L_k^{\beta}(t_1M)^{\beta}+L_{k_t}^{\beta}(t_1^2M)^{\beta}+L_f^{\beta}+L_{k_1}^{\beta}(t_1^2M)^{\beta}\right)C_1c_{\varphi}^{\beta}\varphi^{\beta}(m(t))
\end{aligned}$$

Case 2. For $t\in(t_k,s_k]$,

$$\begin{aligned}
&\left|(\Gamma\hat{g})(t)-(\Gamma\hat{h})(t)\right|^{\beta}=\left|g_k(t,\hat{g}(t_k^+))-g_k\left(t,\hat{h}(t_k^+)\right)\right|^{\beta} \\
&\le\left(L_{g_k}\left|\hat{g}(t_k^+)-\hat{h}(t_k^+)\right|\right)^{\beta} \\
&\le L_{g_k}^{\beta}\left|\hat{g}(t_i^+)-\hat{h}(t_i^+)\right|^{\beta} \\
&\le L_{g_k}^{\beta}\left(C_2^{\frac{1}{\beta}}\psi\right)^{\beta} \\
&\le L_{g_k}^{\beta}C_2\psi^{\beta}
\end{aligned}$$

Case 3. For $t \in (s_k, t_{k+1}]$

$$\left|(\Gamma\hat{g})(t) - (\Gamma\hat{h})(t)\right|^\beta$$

$$= \Big| e^{a(t-s_k)} g_k(s_k, \hat{g}(t_k^+)) - \int_{s_k}^t e^{a(t-s)} k(t,\hat{g}(s)) f_1(s,t)\, ds\, ds - \int_{s_k}^t \int_{s_k}^t e^{a(t-s)} k_t(t,\hat{g}(s)) f_1(s,\tau)\, d\tau\, d\sigma$$

$$+ \int_{s_k}^t e^{a(t-s)} f(t,\hat{g}(t))\, ds + \int_{s_k}^t \int_{s_k}^t e^{a(t-s)} k_1(t,\hat{g}(s))\, ds \int_{s_k}^t k_2(t,s)\, ds\, d\tau - e^{a(t-s_k)} g_k\left(s_k, \hat{h}(t_k^+)\right)$$

$$+ \int_{s_k}^t e^{a(t-s)} k\left(t,\hat{h}(s)\right) f_1(s,t)\, ds\, ds$$

$$+ \int_{s_k}^t \int_{s_k}^t e^{a(t-s)} k_t\left(t,\hat{h}(s)\right) f_1(s,\tau)\, d\tau\, d\sigma - \int_{s_k}^t e^{a(t-s)} f\left(t,\hat{h}(t)\right) ds\, ds$$

$$- \int_{s_k}^t \int_{s_k}^t e^{a(t-s)} k_1\left(t,\hat{h}(s)\right) ds \int_{s_k}^t k_2(t,s)\, ds\, d\tau \Big|^\beta$$

$$\leq \Bigg(e^{a(t-s_k)} \left| g_k(s_k, \hat{g}(t_k^+)) - g_k\left(s_k, \hat{h}(t_k^+)\right)\right| + \left|\int_{s_k}^t e^{a(t-s)} \left(k\left(t,\hat{h}(s)\right) - k(t,\hat{g}(s))\right) f_1(s,t)\, ds\right|$$

$$+ \left|\int_{s_k}^t \int_{s_k}^t e^{a(t-s)} \left(k_t\left(t,\hat{h}(s)\right) - k_t(t,\hat{g}(s))\right) f_1(s,\tau)\, d\tau\, d\sigma\right|$$

$$+ \left|\int_{s_k}^t e^{a(t-s)} \left(f(t,\hat{g}(t)) - f\left(t,\hat{h}(t)\right)\right) ds\right|$$

$$+ \left|\int_{s_k}^t e^{a(t-s)} k_2(t,s)\, ds\right| \left|\int_{s_k}^t \int_{s_k}^t \left(k_1(t,\hat{g}(s)) - k_1\left(t,\hat{h}(s)\right)\right) ds\, d\tau\right| \Bigg)^\beta$$

$$\leq \Bigg(e^{a(t-s_k)} L_{g_k} \left|\hat{g}(t_k^+) - \hat{h}(t_k^+)\right| + e^{a(t-s)} (t_1 M) L_k \int_{s_k}^t \left|\hat{h}(s) - \hat{g}(s)\right| ds$$

$$+ e^{a(t-s)} (t_1^2 M) L_{k_t} \int_{s_k}^t \int_{s_k}^t \left|\hat{h}(s) - \hat{g}(s)\right| ds + e^{a(t-s)} L_f \int_{s_k}^t \left|\hat{g}(s) - \hat{h}(s)\right| ds$$

$$+ e^{a(t-s)} (t_1 M) L_{k_1} \int_{s_k}^t \int_{s_k}^t \left|\hat{g}(s) - \hat{h}(s)\right| ds\, d\tau \Bigg)^\beta$$

$$\leq e^{a\beta(t-s_k)} L_{g_k}^\beta \left|\hat{g}(t_k^+) - \hat{h}(t_k^+)\right|^\beta + e^{a\beta(t-s)} L_k^\beta (t_1 M)^\beta \left[\int_{s_k}^t \left|\hat{h}(s) - \hat{g}(s)\right| ds\right]^\beta + e^{a\beta(t-s)} L_{k_t}^\beta (t_1^2 M)^\beta \left[\int_{s_k}^t \int_{s_k}^t \left|\hat{h}(s) - \hat{g}(s)\right| ds\right]^\beta + e^{a\beta(t-s)} L_f^\beta \left[\int_{s_k}^t \left|\hat{g}(s) - \hat{h}(s)\right| ds\right]^\beta + e^{a\beta(t-s)} L_{k_1}^\beta (t_1^2 M)^\beta \left[\int_{s_k}^t \int_{s_k}^t \left|\hat{g}(s) - \hat{h}(s)\right| ds\, d\tau\right]^\beta$$

$$\leq e^{a\beta(t-s_k)} L_{g_k}^\beta \left(C_2^{\frac{1}{\beta}} \psi\right)^\beta + e^{a\beta(t-s)} L_k^\beta (t_1 M)^\beta \left[C_1^{\frac{1}{\beta}} \int_{s_k}^t \varphi\left(\int_{s_k}^t m(\tau)\, d\tau\right) ds\right]^\beta + + e^{a\beta(t-s)} L_{k_t}^\beta (t_1^2 M)^\beta \left[C_1^{\frac{1}{\beta}} \int_{s_k}^t \int_{s_k}^t \hat{\varphi}\left(\int_{s_k}^t m(s,\tau)\, ds\right) d\tau d\sigma\right]^\beta + e^{a\beta(t-s)} L_f^\beta \left[C_1^{\frac{1}{\beta}} \int_{s_k}^t \varphi\left(\int_0^t m(\tau)\, d\tau\right) ds\right]^\beta + e^{a\beta(t-s)} L_{k_1}^\beta (t_1^2 M)^\beta \left[C_1^{\frac{1}{\beta}} \int_{s_k}^t \int_{s_k}^t \hat{\varphi}\left(\int_{s_k}^t m(s,\tau)\, ds\right) d\tau d\sigma\right]^\beta$$

$$\leq e^{a\beta(t-s_k)} L_{g_k}^\beta C_2 \psi^\beta + e^{a\beta(t-s)} L_k^\beta (t_1 M)^\beta C_1 c_\varphi^\beta \varphi^\beta(m(t)) + e^{a\beta(t-s)} L_{k_t}^\beta (t_1^2 M)^\beta C_1 \hat{c}_\varphi^\beta \hat{\varphi}(m(t,t))$$

$$+ e^{a\beta(t-s)} L_f^\beta C_1 c_\varphi^\beta \varphi^\beta(m(t)) + e^{a\beta(t-s)} L_{k_1}^\beta (t_1^2 M)^\beta C_1 \hat{c}_\varphi^\beta \hat{\varphi}(m(t,t))$$

$$\leq e^{a\beta(t-s_k)} L_{g_k}^\beta C_2 \psi^\beta + e^{a\beta(t-s)} L_k^\beta (t_1 M)^\beta C_1 c_\varphi^\beta \varphi^\beta(m(t)) + e^{a\beta(t-s)} L_{k_t}^\beta (t_1^2 M)^\beta C_1 c_\varphi^\beta \varphi^\beta(m(t)) + e^{a\beta(t-s)} L_f^\beta C_1 c_\varphi^\beta \varphi^\beta(m(t)) + e^{a\beta(t-s)} L_{k_1}^\beta (t_1^2 M)^\beta C_1 c_\varphi^\beta \varphi^\beta(m(t))$$

$$\leq \left(e^{a\beta(t-s_k)} L_{g_k}^\beta + e^{a\beta(t-s)} \left(L_k^\beta (t_1 M)^\beta + L_{k_t}^\beta (t_1^2 M)^\beta + L_f^\beta + L_{k_1}^\beta (t_1^2 M)^\beta\right) c_\varphi^\beta\right) (C_1 + C_2) \left(\varphi^\beta(m(t)) + \psi^\beta\right)$$

From condition (19) is strictly continuous and $d(\Gamma_a\hat{g}, \Gamma_a\hat{h}) \leq \rho_a d(\hat{g}, \hat{h})$From B- fixed point theorem, therefore $y_\circ : I \longrightarrow \mathbb{R}$ such that $\wedge^n g_\circ \longrightarrow y_\circ$ and $\wedge\, y_\circ = y_\circ$ the from condition (4) and equations (7) , (17) we get

$d(y, \Gamma y) \leq e^{a\beta T}\left(1 + c_\varphi^\beta\right)$.

hence, $d(y, y_\circ) \leq \frac{e^{a\beta T}\left(1 + c_\varphi^\beta\right)}{1 - \rho_a}$,that means (18) is true for $t \in I$.

5. CONCLUSION

1. The interesting results determined and computed on impulsive multi-nonlinear integral equation which coming from impulsive differential equation.
2. The results depended on special function and constant as estimation of inequalities of solutions for the impulsive multi-nonlinear integral equation.
3. The Lipchitz conditions for components of impulsive integral equation with delay function is very interesting for prove the existence and uniqueness as well as stability.
4. The fixed-point theorem and their conditions are used for existence and uniqueness a stability depended on estimators for each proposal equation.
5. The extension problem of proposal problem is under the same conditions of main results.
6. The difficulty of examples coming from the formulation of the problem and the necessary and sufficient conditions of the results.
7. The multiplication between integral equation and nonlinear equation needs a particular analytic.

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